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UNFILTRABLE HOLOMORPHIC VECTOR BUNDLES IN A CYCLIC QUOTIENT OF $C^2 \setminus \{0\}$

BY

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Abstract

Let $G := \mathbf{Z}/n\mathbf{Z}$, $n \geq 2$, act diagonally on $\mathbf{C}^2$ and set $X := \mathbf{C}^2 \setminus \{0\}/G$ (a punctured neighborhood for the normal surface singularity A_{n-1}). Here we prove the existence of a rank two holomorphic vector bundle on X without rank one subsheaves.

1. Unfiltrable Vector Bundles

Let M be a reduced and irreducible complex space and E a rank two holomorphic vector bundle on M. We will say that E is *unfiltrable* if there is no rank one torsion free sheaf L such that there is a non-zero map $L \to E$. In the case M smooth it is sufficient to use holomorphic line bundles L to test if E is unfiltrable. If M is Stein, then no rank two holomorphic vector bundle on M is unfiltrable by Theorem A of Cartan - Serre. For example of pairs (M, E) with M smooth compact complex surface and E unfiltrable, see [2] and [3]. In [1] we proved the existence of unfiltrable rank two holomorphic vector bundles on $\mathbf{C}^2 \setminus \{0\}$. If one is interested in the case in which $M = Y/S$ with Y Stein manifold and S discrete in M, then unfiltrable bundles may exist only if $\dim(Y) = 2$ by an extension theorem due to Serre ([6]).

The aim of this paper is to prove the following result.

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Theorem 1. *Let $G : \mathbf{Z}/n\mathbf{Z} \subset GL(2, \mathbf{C})$, $n \geq 2$, act diagonally on $\mathbf{C}^2$ and set $X := \mathbf{C}^2 \setminus \{0\}/G$. Then there is a rank two unfiltrable holomorphic vector bundle on X.*

Notice that X is a smooth open subset of the two-dimensional normal Stein space $\mathbf{C}^2/G$ which has a cyclic singularity of type A_{n-1} at the point of $\mathbf{C}^2/G$ which is the image of $0 \in \mathbf{C}^2$.

The case $n = 1$ (i.e. $X = \mathbf{C}^2 \setminus \{0\}$), was done in [1] using a nice result proved in [2]. Here we prove Theorem 1 (i.e. the case $n \geq 2$) reducing it to the case proved in [1].

For higher rank holomorphic vector bundles the notion of unfiltrability should be replaced with the notion of irreducibility (see [2], [3] and references therein). Here we state it in the case of singular complex spaces.

Definition 1. Let M be a reduced and irreducible complex space and E a holomorphic vector bundle on M. We will say that E is *irreducible* if there is no inclusion $i : F \to E$ with F a coherent analytic sheaf on M such that $1 \leq \text{rank}(F) < \text{rank}(E)$.

We were unable to prove the following conjecture.

Conjecture 1. For every integer $r \geq 2$ and every normal complex surface singularity there is a fundamental system $\{(Y_n, P)\}_{n \geq 1}$ of representative of it such that each $Y_n \setminus \{P\}$ admits a rank r irreducible holomorphic vector bundle.

Proof of Theorem 1. The cyclic group $G \cong \mathbf{Z}/n\mathbf{Z}$ acts as a diagonal sugroup of $GL(2, \mathbf{C})$, i.e. we fix two primitive n-roots ϵ_i, $i = 1, 2$, of unity and the generator e of G acts by the formula $e(z_1, z_2) = (\epsilon_1 z_1, \epsilon_2 z_2)$. We fix $\alpha_i \in \mathbf{C}$, $i = 1, 2$, such that $0 < |\alpha_1| \leq |\alpha_2| < 1$ and set $\lambda = 0$. With these parameters there is a unique Hopf surface M and it has a cyclic covering of order n, say $u : B \to M$, which is the primary Hopf surface constructed with the parameters α_1, α_2 and with $\lambda = 0$ ([5], Th. 32). As in [1] we assume that α_1 and α_2 are chosen so that there are no positive integers a, b such that $\alpha_1^a = \alpha_2^b$. Hence the coverings $f : \mathbf{C}^2 \setminus \{0\} \to B$ and $u \circ f : \mathbf{C}^2 \setminus \{0\} \to M$ are the universal coverings. In [1] we used the surface B to construct unfiltrable vector bundles for the trivial group. Since T has algebraic dimension zero ([5], Th. 31), M has algebraic dimension zero. Since B belongs to Kodaira's class VII_0, M belongs to Kodaira's class VII_0, i.e. $b_1(M) = 1$, $h^1(M, \mathcal{O}_M) = 1$ and $h^0(M, \omega_M) = 0$. Hence $\chi(\mathcal{O}_M) = 0$ and $\omega_M \cdot \omega_M = 0$. We fix an integer $c_2 \geq 3$ and call Z a general subset of M with $\text{card}(Z) = c_2$. Consider the general extension

$$0 \to \mathcal{O}_M \to D \to \mathcal{I}_Z \to 0. \tag{1}$$

Since $h^0(M, \omega_M) = 0$, for every subset Z' of Z with $\text{card}(Z) = \text{card}(Z) - 1$ we have $h^0(M, \mathcal{I}_{Z'} \otimes \omega_M) = 0$. Hence the Cayley - Bacharach condition is satisfied ([4]). Thus D is locally free. Hence $u^*(D)$ is locally free and fits in the exact sequence

$$0 \to \mathcal{O}_B \to u^*(D) \to Lu^{-1}(z) \to 0. \tag{2}$$

Since $Z \neq \emptyset$ and $u^{-1}(Z) \neq \emptyset$, we have $h^0(M, D) = h^0(B, u^*(D)) = 1$. Since M and B have algebraic dimension zero, they have only finitely many closed curves. Since Z is general, we see that neither D nor $u^*(D)$ contain a line bundle not contained in the trivial line subbundle arising from the exact sequences (1) and (2). We have $c_1(D) \cong \mathcal{O}_M$ and $c_1(u^*(D)) \cong \mathcal{O}_B$. With the notation of [2], p. 9, we call $m(2, \mathcal{O}_M)$ a certain integer with the property that if $c_2 \geq \max\{m(2, \mathcal{O}_M), 5\}$, there is a flat family $\{D_z\}_{z \in \triangle}$ parametrized by the unit disk $\triangle \subset \mathbf{C}$ such that $D_0 = D$ and D_z is unfiltrable (i.e. it contains no rank one subsheaf) for every $z \in \triangle \setminus \{0\}$. Consider the flat family $\{u^*(D_z)\}_{z \in \triangle}$ of holomorphic rank one vector bundles on B such that $u^*(D_0) = u^*(D)$, $c_1(u^*(D_z)) \cong \mathcal{O}_B$ and $c_2(u^*(D_z)) = nc_2$ for every $z \in \triangle$. $\square$

Claim. Each $u^*(D_z)$, $z \in \triangle \setminus \{0\}$, in unfiltrable.

Proof of the Claim. Fix any $z \in \triangle \setminus \{0\}$ and set $E_z := f^*(u^*(D_z))$. The proof of the unfiltrability of E_z for a general z given in [1], p. 202, works verbatim because it only use the unfiltrability of $u^*(D_z)$, the existence of the flat family $\{u^*(D_z)\}_{z \in \triangle}$ and that $u^*(D_0)$ fits in the exact sequence (2). $\square$

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