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INFINITE DIMENSIONAL COMPLEX ANALYSIS AS A FRAMEWORK FOR PRODUCTS OF DISTRIBUTIONS

BY

TSOY-WO MA

Abstract

Infinite dimensional complex analysis on locally convex spaces is used as the framework for products of distributions in the sense of Columbeau without the necessary prerequisites of Silva-differentiability on bornological vector spaces, or calculus on convenient vector spaces, or nonstandard analysis. However, invariance under diffeomorphisms is beyond the scope of this paper.

1. Introduction

1.1. It is well known that the products of distributions need not be distributions, e.g. [21], and that linear partial differential equations with polynomial coefficients need not have distributional solutions, e.g. [12]. Earlier work on product of distributions includes [8] based on approximation and [9] based on Fourier transform.

1.2. The idea of quotient algebras was found, e.g. in [19], and the explicit descriptions, e.g. [3], based on Silva-differentiability on bornological vector spaces [2] produce a major impact in the field. The simplification [4] abandoned later in [5] indicates that a suitable *framework of infinite dimensional differential calculus* is definitely required. Our quick response with compact equicontinuous calculus on locally convex spaces [14] could not survive among too many definitions of differentiation, e.g. [1]. Alternatively,

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nonstandard framework has been popular, e.g. in [13] and [18]. Recently, the convenient spaces [11] have gained ground, e.g. in [7]. Examples of *differentiable but discontinuous* maps in their setting can be found in [2, p.51] and [11, p.2] but this is not the case in our [14] and [15].

1.3. This paper is to promote *complex analysis on locally convex spaces as a framework* for the construction of generalized functions as in [3, Chap. 3]. However, invariance under diffeomorphisms and other aspects will be considered later.

2. Detections

2.1. A map from an open subset of a complex locally convex space into a complex locally convex space is *holomorphic* if it is (directionally) differentiable and locally bounded as recalled in [15, 2.1]. A *discontinuous* linear form is differentiable but not holomorphic because it is not locally bounded. *Functions* are scalar-valued in our convention. An *operator* on a set X is a map from X into itself. For every map f on X, we may write $< f, x > = f(x)$ for all $x \in X$. All locally convex spaces are assumed to be separated.

2.2. Let Ω be an open subset of $\mathbb{R}^\nu$. A map on Ω into a locally convex space is *smooth* if it has continuous partial derivatives of all orders. Let $\mathcal{D}(\Omega)$ be the test space of smooth complex functions with compact support contained in Ω equipped with the natural inductive topology. A holomorphic function on $\mathcal{D}(\Omega)$ into the complex plane $\mathbb{C}$ is called a *detection* on Ω. Since every continuous linear form is holomorphic, every distribution is a detection. The set of all detections on Ω is denoted by $dt(\Omega)$.

2.3. Let T be a detection on Ω. For each integer $j \in [1, \nu]$, the *partial derivative* $\partial_j T : \mathcal{D}(\Omega) \to \mathbb{C}$ is defined by

$$(\partial_j T)(\varphi) = - < DT(\varphi), \partial_j \varphi >$$

for every $\varphi \in \mathcal{D}(\Omega)$ where the (total) derivative $DT(\varphi)$ is a continuous linear form on $\mathcal{D}(\Omega)$. If T is a distribution, the new definition agrees with the old distributional derivative because $DT(\varphi) = T$. For convenience, an expression $T(\varphi)$ is identified as the map $\varphi \to T(\varphi)$ and we write $\frac{d}{d\varphi}T(\varphi) = DT(\varphi)$ similar to $\frac{d}{dt}[u(t)v(t)] = u(t)\frac{d}{dt}v(t) + v(t)\frac{d}{dt}u(t)$ in elementary calculus.

2.4. Theorem. *The partial derivative $\partial_j T$ of a detection T is a detection on Ω. Furthermore we have $\partial_i\partial_j T = \partial_j\partial_i T$. As a result, the partial derivative with respect to a multi-index $\alpha = (\alpha_1, \ldots, \alpha_\nu) \in \mathbb{N}^\nu$ is defined in the usual way by $\partial^\alpha T = \partial_1^{\alpha_1} \cdots \partial_\nu^{\alpha_\nu} T$ where $\mathbb{N}$ is the set of all integers ≥ 0.*

Proof. Since T is holomorphic, so is the function $(\varphi, \psi) \to DT(\varphi)\psi$ by [15, 2.6]. Because the continuous linear operator $\varphi \to \partial_j\varphi$ is holomorphic, the composite map $\partial_j T$ is a holomorphic function on $\mathcal{D}(\Omega)$, that is a detection on Ω. Next for every $\varphi \in \mathcal{D}(\Omega)$, we get

$$
\begin{aligned}
& (\partial_i\partial_j T)(\varphi) \\
= \ & - < D(\partial_j T)(\varphi), \partial_i\varphi > \\
= \ & \left[\frac{d}{d\varphi} < DT(\varphi), \partial_j\varphi >\right](\partial_i\varphi) \\
= \ & < D^2T(\varphi)(\partial_i\varphi), \partial_j\varphi > + \left\langle DT(\varphi), \left[\frac{d}{d\varphi}\partial_j\varphi\right](\partial_i\varphi)\right\rangle \quad \text{by [16, 10-3.6]} \\
= \ & D^2T(\varphi)(\partial_i\varphi)(\partial_j\varphi) + < DT(\varphi), \partial_j\partial_i\varphi) > \\
= \ & D^2T(\varphi)(\partial_j\varphi)(\partial_i\varphi) + < DT(\varphi), \partial_i\partial_j\varphi) > \\
= \ & (\partial_j\partial_i T)(\varphi)\,. \qquad \square
\end{aligned}
$$

2.5. Theorem. *The pointwise product ST of two detections S, T is a detection. Furthermore for every multi-index α, we have*

$$\partial^\alpha(ST) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\partial^{\alpha-\beta}S)(\partial^\beta T).$$

Proof. Since the product of two complex numbers is holomorphic, the composite map $\varphi \to [S(\varphi), T(\varphi)] \to S(\varphi)T(\varphi)$ is holomorphic in φ. Hence

ST is a detection on Ω. Observe that

$$\begin{aligned}
\partial_j(ST)(\varphi) &= -< D(ST)(\varphi), \partial_j\varphi > \\
&= -\left\langle \frac{d}{d\varphi}[S(\varphi)T(\varphi)], \partial_j\varphi \right\rangle \\
&= -< S(\varphi)DT(\varphi) + T(\varphi)DS(\varphi), \partial_j\varphi > \\
&= -S(\varphi) < DT(\varphi), \partial_j\varphi > -T(\varphi) < DS(\varphi), \partial_j\varphi > \\
&= S(\varphi)\partial_j T(\varphi) + T(\varphi)\partial_j S(\varphi) \\
&= (S\partial_j T + T\partial_j S)(\varphi).
\end{aligned}$$

Hence we obtain $\partial_j(ST) = S\partial_j T + T\partial_j S$. The general case follows by induction. □

3. Moderate Detections

3.1. A test function $\rho \in \mathcal{D}(\mathbb{R}^\nu)$ is *normalized* if $\int \rho(x)dx = 1$. For each multi-index α and $x = (x_1, \dots, x_\nu) \in \mathbb{R}^\nu$, we write $x^\alpha = x_1^{\alpha_1} \cdots x_\nu^{\alpha_\nu}$. For each integer $q \geq 0$, let A_q be the set of normalized test functions ρ such that $\int x^\alpha \rho(x)dx = 0$ for all α with $0 < |\alpha| \leq q$. Clearly A_0 is the set of all normalized test functions. The property $A_{q+1} \subset A_q$ will be used in §4.2. The condition $\int x^\alpha \rho(x)dx = 0$ will be used in §6.7 and it allows us to claim generalization of some others as in [3, 3.5.6]. The polynomials $\{x^\alpha : |\alpha| \leq q\}$ are linearly independent smooth functions and hence they form an independent subset of the distribution space $\mathcal{D}'(\mathbb{R}^\nu)$. The set A_q is nonempty by [20, p.124]. For each test function $\rho \in \mathcal{D}(\mathbb{R}^\nu)$, each $x \in \Omega$ and each $\lambda > 0$, the function $\rho_{\lambda x} : \Omega \to \mathbb{C}$ indexed by two parameters λ and x is defined by

$$\rho_{\lambda x}(y) = \frac{1}{\lambda^\nu}\, \rho\left(\frac{y-x}{\lambda}\right)$$

for all $y \in \Omega$. For the earlier role of A_q, see [17], [6], [10] and standard regularization procedure. Write $K \leq \Omega$ if K is a compact subset of Ω.

3.2. A detection T on Ω is *moderate* if $\forall\, K \leq \Omega$, $\forall\, \alpha \in \mathbb{N}^\nu$, $\exists\, n \in \mathbb{N}$, $\forall\, \rho \in A_n$, $\exists\, M > 0$, $\exists\, r > 0$, $\forall\, x \in K$, $\forall\, \lambda \in (0, r)$, we have

$$|(\ \partial^\alpha T\ )\,(\rho_{\lambda x})| \leq \frac{M}{\lambda^n}.$$

The following theorem follows immediately by routine verification.

3.3. Theorem. *The set $md(\Omega)$ of all moderate detections on Ω is an algebra under pointwise operations. Furthermore, it is invariant under partial differentiation.*

3.4. Lemma. *For each detection T on Ω and for each $\rho \in A_0$, every point $a \in \Omega$ has an open neighborhood $V \subset \Omega$ and $r > 0$ such that for every $\lambda \in (0, r)$, the following conditions hold.*

(a) $\operatorname{supp} \rho_{\lambda x} \subset \Omega$ *for every* $x \in V$.

(b) *The map* $\xi : V \to \mathcal{D}(\Omega)$ *defined by* $\xi(x) = \rho_{\lambda x}$ *is smooth with*

$$\partial^\alpha \xi(a) = \left(-\frac{1}{\lambda}\right)^{|\alpha|} (\partial^\alpha \rho)_{\lambda a}\,.$$

(c) *The function* $x \to T(\rho_{\lambda x}) : V \to \mathbb{C}$ *is smooth.*

(d) *If* $T = \partial^\alpha g$ *for some continuous function* g *on* V, *then we have*

$$T(\rho_{\lambda x}) = \left(-\frac{1}{\lambda}\right)^{|\alpha|} \int g(x + \lambda y) \partial^\alpha \rho(y) dy.$$

Proof. (a) Choose an open ball $\mathbb{B}(a, 3s) \subset \Omega$. There is $r \in (0, s)$ such that $r\|y\| < s$ for every $y \in \operatorname{supp} \rho$. Then $V = \mathbb{B}(a, s)$ is an open neighborhood of a and the closed ball $K = \overline{\mathbb{B}}(a, 2s)$ is compact subset of Ω. For each $x \in V$, we have

$$\operatorname{supp} \rho_{\lambda x} \subset x + \lambda \operatorname{supp} \rho \subset \mathbb{B}(a, s) + \mathbb{B}(0, s) \subset \mathbb{B}(a, 2s) \subset K \subset \Omega.$$

(b) By induction, it suffices to prove that the partial derivative of ξ with respect to x_1 exists and satisfies the required equation. Let $e_1, \ldots, e_\nu$ denote the standard basis for $\mathbb{R}^\nu$. For every $y \in \Omega$, we have

$$
\begin{aligned}
&\left| \frac{\xi(a+te_1)-\xi(a)}{t}\,(y) - \left(-\frac{1}{\lambda}\right)(\partial_1\rho)_{\lambda a}(y) \right| \\
= &\left| \frac{1}{t}\,\frac{1}{\lambda^\nu}\left\{\rho\left(\frac{y-a-te_1}{\lambda}\right) - \rho\left(\frac{y-a}{\lambda}\right)\right\} + \frac{1}{\lambda^{\nu+1}}(\partial_1\rho)\left(\frac{y-a}{\lambda}\right)\right| \\
\le &\ \frac{1}{\lambda^{\nu+1}}\left|\int_0^1\left\{(\partial_1\rho)\left(\frac{y-a-s\,t\,e_1}{\lambda}\right) - (\partial_1\rho)\left(\frac{y-a}{\lambda}\right)\right\}ds\right| \\
\le &\ \frac{|t|}{\lambda^{\nu+1}}\int_0^1\int_0^1\left|\partial_1^2\rho\left(\frac{y-a-\tau\,s\,t\,e_1}{\lambda}\right)\right| d\tau ds \to 0
\end{aligned}
$$

uniformly on K as $t \to 0$ in $\mathbb{R}$. Because the supports of ξ and $(\partial_1\rho)_{\lambda x}$ are contained in K, we obtain

$$
\partial_1\xi(a) = \lim_{t\to 0}\frac{\xi(a+te_1)-\xi(a)}{t} = \left(-\frac{1}{\lambda}\right)(\partial_1\rho)_{\lambda a}, \qquad \text{in} \qquad \mathcal{D}(\Omega).
$$

(c) The composite $T\xi$ of the smooth map ξ and the continuous linear form T is smooth.

(d) Suppose $T = \partial^\alpha g$ for some continuous function g on V. Then we have

$$
\begin{aligned}
T(\rho_{\lambda x}) &= \int\{\partial_z^\alpha g(z)\}\rho_{\lambda x}(z)dz \\
&= (-1)^{|\alpha|}\int g(z)\partial_z^\alpha\left\{\frac{1}{\lambda^\nu}\,\rho\left(\frac{z-x}{\lambda}\right)\right\}dz \\
&= \left(-\frac{1}{\lambda}\right)^{|\alpha|}\int g(x+\lambda y)\partial_y^\alpha\rho(y)dy\,, \quad \text{where } z = x+\lambda y.
\end{aligned}
$$

□

3.5. Lemma. *Let T be a detection on Ω and ρ be a normalized test function on $\mathbb{R}^\nu$. Then for each test function $\varphi \in \mathcal{D}(\Omega)$, there is $r > 0$ such that for every $\lambda \in (0,r)$, the integral $\int T(\rho_{\lambda x})\varphi(x)dx$ exists. Furthermore if $T = \partial^\alpha g$ for some function g continuous on a neighborhood of the support of φ, then we have*

$$
\int T(\rho_{\lambda x})\varphi(x)dx = (-1)^{|\alpha|}\iint g(x)\rho(y)\partial_x^\alpha\varphi(x-\lambda y)dxdy.
$$

Proof. Choose $r > 0$ and an open neighborhood V of the compact set supp φ such that for each $\lambda \in (0,r)$, the function $x \to T(\rho_{\lambda x})$ is smooth on V. Hence the integral $\int T(\rho_{\lambda x})\varphi(x)dx$ exists because the integrand is a continuous function with compact support. Finally for $T = \partial^\alpha g$ on V, we have

$$\begin{aligned}
&\int T(\rho_{\lambda x})\varphi(x)dx \\
&= \iint \partial_y^\alpha g(y)\rho_{\lambda x}(y)\varphi(x)dxdy \\
&= (-1)^{|\alpha|}\iint g(y)\partial_y^\alpha \rho_{\lambda x}(y)\varphi(x)dxdy \\
&= (-1)^{|\alpha|}\iint g(y)\partial_y^\alpha h(y-x)\varphi(x)dxdy \qquad \text{where } h(z) = \frac{1}{\lambda^\nu}\,\rho\left(\frac{z}{\lambda}\right) \\
&= (-1)^{|\alpha|}\int g(y)\ [(\partial^\alpha h) * \varphi]\ (y)dy \qquad \text{convolution} \\
&= (-1)^{|\alpha|}\int g(y)(h * \partial^\alpha \varphi)(y)dy \\
&= (-1)^{|\alpha|}\iint g(y)h(x)\partial_y^\alpha \varphi(y-x)dxdy \\
&= (-1)^{|\alpha|}\iint g(y)\ \frac{1}{\lambda^\nu}\ \rho\left(\frac{x}{\lambda}\right)\partial_y^\alpha \varphi(y-x)dxdy \\
&= (-1)^{|\alpha|}\iint g(y)\ \rho(z)\partial_y^\alpha \varphi(y-\lambda z)dydz \qquad \text{where } x = \lambda z.
\end{aligned}$$

Replacing y, z by x, y respectively, the result follows. □

3.6. Lemma. *Let $\Omega = \cup_{i\in I}\Omega_i$ be covered by open sets Ω_i. Then a detection T on Ω is moderate iff all restrictions $T|\mathcal{D}(\Omega_i)$ are moderate.*

Proof. Suppose that all restrictions $T|\mathcal{D}(\Omega_i)$ are moderate. To show that T is moderate on Ω, let K be a compact subset of Ω and let α be a multi-index. There is a finite subset J of I such that $K \subset \cup_{j\in J}\Omega_j$. Since Ω is locally compact, for each $j \in J$ there is a compact subset K_j of Ω_j such that $K = \cup_{j\in J}K_j$. Choose integers n_j for $T|\mathcal{D}(\Omega_j)$ according to §3.2. Let $n = \max_{j\in J} n_j$. Pick any $\rho \in A_n$. Select M_j and $r_j \in (0,1)$ according to §3.2. Let $M = \max_{j\in J} M_j$ and $r = \min_{j\in J} r_j$. Finally take any $x \in K$ and any $\lambda \in (0,r)$. Then $x \in K_j$ for some $j \in J$. Since $0 < \lambda < r_j < 1$, we have

$$|(\ \partial^\alpha T\ )\,(\rho_{\lambda x})| \le \frac{M_j}{\lambda^{n_j}} \le \frac{M}{\lambda^n}.$$

Therefore T is moderate on Ω. The converse is obvious. $\square$

3.7. Theorem. *Every distribution T is a moderate detection.*

Proof. Consider the special case when T is a continuous function g on Ω. Let K be a compact subset of Ω and α be a multi-index. Fix $n = |\alpha|$. Select any $\rho \in A_n$. Choose $r > 0$ such that $Q = K + [0,r]\operatorname{supp}\rho$ is a compact subset of Ω. Let $M = \left(\sup_{x\in Q}|g(x)|\right)\int|\partial^\alpha\rho(y)|dy$. Pick any x in K and any λ in $(0,r)$. By §3.4 d, we have

$$|(\partial^\alpha T)(\rho_{\lambda x})| \le \frac{1}{\lambda^{|\alpha|}}\int|g(x+\lambda y)|\;|\partial^\alpha\rho(y)|dy \le \frac{M}{\lambda^n}.$$

Hence g is a moderate detection. In general, let T be a distribution. Then each $a \in \Omega$ has a neighborhood V and a continuous function g on V such that $T = \partial^\beta g$ on V for some multi-index β. Then g and hence $T = \partial^\beta g$ are moderate detections on V. The result follows from the last lemma. $\square$

4. Null Detections

4.1. A detection S is *null* if $\forall K \le \Omega$, $\forall \alpha \in \mathbb{N}^\nu$, $\exists n \in \mathbb{N}$, $\forall q \ge n$, $\forall \rho \in A_q$, $\exists M > 0$, $\exists r > 0$, $\forall x \in K$, $\forall \lambda \in (0,r)$, we have

$$|(\partial^\alpha S)(\rho_{\lambda x})| \le M\lambda^{q-n}.$$

Since $A_{q+1} \subset A_q$, it is easy to verify the following theorem.

4.2. Theorem. *The set $null(\Omega)$ of all null detections on Ω is an ideal of the algebra $md(\Omega)$ of moderate detections. Furthermore, it is invariant under partial differentiation.*

4.3. Lemma. *For every $T \in \mathcal{D}'(\Omega)$, $\varphi \in \mathcal{D}(\Omega)$ and $\rho \in A_0$, we have*

$$< T,\varphi > = \lim_{\lambda\to 0}\int < T,\rho_{\lambda x} > \varphi(x)dx\,.$$

Proof. By linearity in φ and smooth partition of unity, we may assume that $\operatorname{supp}\varphi$ is small enough so that $T = \partial^\alpha g$ where g is a continuous function on an open neighborhood V of $\operatorname{supp}\varphi$. Choose $r_1 > 0$ by §3.5. We may assume that $K = \operatorname{supp}\varphi + [0, r_1]\operatorname{supp}\rho$ is a compact subset of V. By uniform continuity of $\partial^\alpha\varphi$, for every $\varepsilon > 0$ there is $r_2 > 0$ such that for all $x, y \in K$ with $\|x - y\| \leq r_2$ we have $|\partial^\alpha\varphi(x) - \partial^\alpha\varphi(y)| \leq \varepsilon$. Let $r = \min\{r_1, r_2\}$. Then for each $\lambda \in (0, r)$ and $\rho \in A_0$, we obtain

$$\begin{aligned}
&\left|\int < T, \rho_{\lambda x} > \varphi(x)dx - < T, \varphi >\right| \\
\leq\ &\left|(-1)^{|\alpha|}\iint g(x)\rho(y)\partial_x^\alpha\varphi(x - \lambda y)dxdy - (-1)^{|\alpha|}\int g(x)\partial_x^\alpha\varphi(x)dx\right| \\
\leq\ &\left|(-1)^{|\alpha|}\iint g(x)\rho(y)\{\partial_x^\alpha\varphi(x - \lambda y) - \partial_x^\alpha\varphi(x)\}dxdy\right| \\
\leq\ &\varepsilon\iint |g(x)\rho(y)|dxdy
\end{aligned}$$

which is independent of the choice of λ. This completes the proof. □

4.4. Theorem. *If a distribution S is a null detection, then $S = 0$.*

Proof. Let $\varphi \in \mathcal{D}(\Omega)$ be given. For $K = \operatorname{supp}\varphi$ and $\alpha = 0$, choose n according to §4.1. For $q = n + 1$ and $\rho \in A_q$, fix $M, r > 0$ by §4.1. Then for all $\lambda \in (0, r)$, we have

$$\left|\int < S, \rho_{\lambda x} > \ \varphi(x)dx\right| \leq M\lambda\int |\varphi(x)|dx.$$

Hence $< S, \varphi >= 0$ by the last lemma. Since φ is arbitrary, we get $S = 0$. □

5. Detectors

5.1. For the ideal $null(\Omega)$ of the algebra $md(\Omega)$, the equivalence classes of the quotient algebra $dtr(\Omega) = md(\Omega)/null(\Omega)$ are called *detectors* on Ω. For each moderate detection T, let $[T]$ denote the equivalence class containing T. Since $null(\Omega)$ is invariant under all partial differential operators, for each multi-index α the formula $\partial^\alpha[T] = [\partial^\alpha T]$ is independent of the choice of T in $[T]$. Hence the partial derivatives of detectors are well-defined.

5.2. Since every distribution T is a moderate detection, the linear map $T \to [T] : \mathcal{D}'(\Omega) \to dtr(\Omega)$ is injective by §4.4. Therefore the distribution space $\mathcal{D}'(\Omega)$ is identified as a subset of $dtr(\Omega)$. Since every continuous function g is a distribution, the equivalence class $[g]$ is a detector.

5.3. A detector H *admits* a distribution S if $\exists T \in H$, $\forall \varphi \in \mathcal{D}(\Omega)$, $\exists n \geq 0$, $\forall q \geq n$, $\forall \rho \in A_q$, we have

$$< S, \varphi > = \lim_{\lambda \to 0} \int T(\rho_{\lambda x})\varphi(x)dx.$$

Because every null detection admits the zero distribution, S is independent of the choice of $T \in H$. Clearly a detector admits at most one distribution. Since the above formula is linear in T, the set $ad(\Omega)$ of all admissible detectors forms a vector subspace of $dtr(\Omega)$. Because every distribution admits itself, we have $\mathcal{D}'(\Omega) \subset ad(\Omega) \subset dtr(\Omega)$. As the linear map $H \to S : ad(\Omega) \to \mathcal{D}'(\Omega)$ is an idempotent, it behaves like a projection.

5.4. Theorem. *For all continuous functions on Ω, the product detector $[f][g]$ admits the distribution fg which is the usual pointwise product of continuous functions.*

Proof. Let T_f, T_g, T_{fg} be the distributions identified with f, g, fg respectively. Then for all $\varphi, \rho \in \mathcal{D}(\Omega)$ with $\int \rho(x)dx = 1$, we have

$$\begin{aligned}
&\int (T_f T_g)(\rho_{\lambda x})\varphi(x)dx - < T_{fg}, \varphi > \\
= &\int T_f(\rho_{\lambda x})T_g(\rho_{\lambda x})\varphi(x)dx - \int f(x)g(x)\varphi(x)dx \\
= &\int \left\{ \int f(x+\lambda y)\rho(y)dy \right\} \left\{ \int g(x+\lambda z)\rho(z)dz \right\} \varphi(x)dx \\
&\qquad\qquad - \int f(x)g(x)\varphi(x)dx \int \rho(y)dy \int \rho(z)dz \\
= &\iiint \{f(x+\lambda y)g(x+\lambda z) - f(x)g(x)\}\varphi(x)\rho(y)\rho(z)dxdydz\ .
\end{aligned}$$

The first term $f(x+\lambda y)g(x+\lambda z) - f(x)g(x)$ is small by uniform continuity of f, g on some compact neighborhood of $\operatorname{supp}\varphi$. The other terms are independent of λ. Therefore $[T_f][T_g]$ admits T_{fg}. □

5.5. Theorem. *Let f be a smooth function and S a distribution on Ω. Then the product detector $[f][S]$ admits the distribution fS which is the usual product in the classical distribution theory.*

Proof. We want to show that $\forall\, \varphi \in \mathcal{D}(\Omega)$, $\exists n \geq 0$, $\forall\, q \geq n$, $\forall\, \rho \in A_q$, we have

$$< fS, \varphi > = \lim_{\lambda \to 0} \int (T_f S)(\rho_{\lambda x}) \varphi(x) dx.$$

In view of §4.3, it suffices to prove that

$$\lim_{\lambda \to 0} \int S(\rho_{\lambda x}) \{f(x)\varphi(x)\} dx = \lim_{\lambda \to 0} \int (T_f S)(\rho_{\lambda x}) \varphi(x) dx.$$

By linearity in φ, we may assume that $\operatorname{supp} \varphi$ is small enough such that $S = \partial^\alpha g$ for some continuous function g on a neighborhood of $\operatorname{supp} \varphi$. It follows from §3.5 that

$$\begin{aligned} &\int S(\rho_{\lambda x}) \{f(x)\varphi(x)\} dx \\ = \;& (-1)^{|\alpha|} \iint g(x)\rho(y) \partial_x^\alpha [f(x - \lambda y)\varphi(x - \lambda y)] dx dy. \end{aligned} \tag{1}$$

On the other hand, observe that

$$\begin{aligned} &\int (T_f S)(\rho_{\lambda x}) \varphi(x) dx \\ = \;& \int < T_f, \rho_{\lambda x} > < S, \rho_{\lambda x} > \;\; \varphi(x) dx \\ = \;& \int (\partial^\alpha g)(\rho_{\lambda x}) \left\{ \varphi(x) \int f(x + \lambda z)\rho(z) dz \right\} dx \qquad \text{by §3.4d} \\ = \;& (-1)^{|\alpha|} \iint g(x)\rho(y) \partial_x^\alpha \left\{ \varphi(x - \lambda y) \int f(x - \lambda y + \lambda z)\rho(z) dz \right\} dx dy \\ & \qquad \text{by §3.5} \\ = \;& (-1)^{|\alpha|} \iiint g(x)\rho(y)\rho(z) \partial_x^\alpha [\, f(x - \lambda y + \lambda z)\varphi(x - \lambda y)\,] dx dy dz. \end{aligned} \tag{2}$$

Since $\int \rho(z) dz = 1$, the difference (1)–(2) is

$$(-1)^{\alpha|} \iiint g(x)\rho(y)\rho(z) \partial_x^\alpha \{[f(x - \lambda y) - f(x - \lambda y + \lambda z)]\varphi(x - \lambda y)\} dx dy dz.$$

The first three terms $g(x)\rho(y)\rho(z)$ are independent of λ. The last term is

$$\sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \left\{ \partial_x^\beta f(x - \lambda y) - \partial_x^\beta f(x - \lambda y + \lambda z) \right\} \partial_x^{\alpha - \beta} \varphi(x - \lambda y).$$

The term enclosed by braces is small by uniform continuity of the function $x \to \partial_x^\beta f(x)$ on any compact neighborhood of $\operatorname{supp}\varphi$ while the second term $\partial_x^{\alpha-\beta}\varphi(x-\lambda y)$ is bounded. Therefore the difference (1)–(2) is small. This completes the proof. $\square$

6. Compact Detections

6.1. The strong dual $\mathcal{E}'(\Omega)$ of the space $\mathcal{E}(\Omega)$ of smooth functions on Ω is the space of distributions with compact support. A holomorphic function T on $\mathcal{E}'(\Omega)$ is called a *compact detection*. Clearly the set $kd(\Omega)$ of all compact detections on Ω is an algebra.

6.2. Let T be a compact detection on Ω. Because the embedding $\mathcal{D}(\Omega) \to \mathcal{E}'(\Omega)$ is a continuous linear map, the restriction $T|\mathcal{D}(\Omega)$ is also holomorphic. Since T is continuous on $\mathcal{E}'(\Omega)$ and $\mathcal{D}(\Omega)$ is dense in $\mathcal{E}'(\Omega)$, the linear map $T \to T|\mathcal{D}(\Omega) : kd(\Omega) \to dt(\Omega)$ is injective. This allows us to identify $kd(\Omega)$ as a subspace of $dt(\Omega)$. On the other hand, a detection $S : \mathcal{D}(\Omega) \to \mathbb{C}$ is compact iff it has an extension T over $\mathcal{E}'(\Omega)$ which is holomorphic with respect to the strong topology.

6.3. Lemma. *The delta map $\delta : \Omega \to \mathcal{E}'(\Omega)$ given by $\delta(x) = \delta_x$ is a smooth map. Furthermore for every multi-index α, we have*

$$(\partial^\alpha \delta)(x) = (-1)^{|\alpha|}\partial^\alpha \delta_x.$$

Proof. By induction on $|\alpha|$, for each $g \in \mathcal{E}(\Omega)$ observe that

$$\begin{aligned}
&\lim_{\lambda\to 0}\left\langle \frac{\partial^\alpha\delta(x+\lambda e_j)-\partial^\alpha\delta(x)}{\lambda}, g\right\rangle \\
=\ &\lim_{\lambda\to 0}\,(-1)^{|\alpha|}\left\langle \frac{\partial^\alpha\delta_{x+\lambda e_j}-\partial^\alpha\delta_x}{\lambda}, g\right\rangle \\
=\ &\lim_{\lambda\to 0}\,(-1)^{|\alpha|}(-1)^{|\alpha|}\left\langle \frac{\delta_{x+\lambda e_j}-\delta_x}{\lambda}, \partial^\alpha g\right\rangle \\
=\ &\lim_{\lambda\to 0}\,\frac{\partial^\alpha g(x+\lambda e_j)-\partial^\alpha g(x)}{\lambda}
\end{aligned}$$

$$\begin{aligned} &= \partial_j \partial^\alpha g(x) \\ &= < \delta_x, \partial_j \partial^\alpha g > \\ &= (-1)^{|\alpha|+1} < \partial_j \partial^\alpha \delta_x, g > \quad . \end{aligned}$$

Hence the convergence

$$\frac{\partial^\alpha \delta(x+\lambda e_j) - \partial^\alpha \delta(x)}{\lambda} \to (-1)^{|\alpha|+1} \partial_j \partial^\alpha \delta_x$$

is weakly in the Montel space $\mathcal{E}'(\Omega)$; so it converges strongly in $\mathcal{E}'(\Omega)$. Therefore we have $(\partial_j \partial^\alpha \delta)(x) = (-1)^{|\alpha|+1} \partial_j \partial^\alpha \delta_x$. This completes the proof. □

6.4. For each compact detection T, define $\xi T(x) = T(\delta_x)$ for every x in Ω. Then ξT is a smooth function on Ω. Since $\mathcal{E}(\Omega)$ is reflexive, the map $\xi : kd(\Omega) \to \mathcal{E}(\Omega)$ is a linear surjection. Hence we get an identification induced by the natural isomorphism $kd(\Omega)/ker(\xi) \simeq \mathcal{E}(\Omega)$.

6.5. The *partial derivative* $\partial_j T$ of a compact detection T on Ω is defined by

$$(\partial_j T)(S) = -[(DT)S](\partial_j S)$$

for all $S \in \mathcal{E}'(\Omega)$. Then $\partial_j T$ is a compact detection satisfying

$$(\partial_j T)|\mathcal{D}(\Omega) = \partial_j[T|\mathcal{D}(\Omega)]$$

where the second ∂_j is defined in §2.3. Since both $\partial_i \partial_j T$ and $\partial_j \partial_i T$ are continuous on $\mathcal{E}'(\Omega)$ and they agree on the dense set $\mathcal{D}(\Omega)$, we have $\partial_i \partial_j T = \partial_j \partial_i T$ on $\mathcal{E}'(\Omega)$. Consequently for every multi-index α, the partial derivative $\partial^\alpha T$ is well-defined. Furthermore we have $\xi(\partial^\alpha T) = \partial^\alpha(\xi T)$ where the second ∂^α is an ordinary partial differential operator on the smooth function ξT. In fact for every $x \in \Omega$, it follows by §6.3 and the Chain Rule that

$$\xi(\partial_j T)(x) = (\partial_j T)(\delta_x) = -[(DT)(\delta_x)](\partial_j \delta_x)$$

and

$$\partial_j(\xi T)(x) = \partial_j(T \circ \delta)(x) = DT(\delta_x)(\partial_j \delta)(x) = DT(\delta_x)(-\partial_j \delta_x).$$

The general case follows by induction on $|\alpha|$.

6.6. Lemma. *For each $\rho \in A_q$ and each $x \in \Omega$, there is $r > 0$ such that the set*

$$B = \left\{ \frac{\rho_{\lambda x} - \delta_x}{\lambda^{q+1}} : \lambda \in (0, r) \right\}$$

is bounded in $\mathcal{E}'(\Omega)$.

Proof. Let $r > 0$ be any small number such that $x + [0, r] \operatorname{supp} \rho \subset \Omega$. It suffices to show that B is weakly bounded, i.e. for each $f \in \mathcal{E}(\Omega)$ the set $< f, B >$ is bounded in $\mathbb{C}$. Now by Taylor's formula, for each small $y \in \mathbb{R}^\nu$ we have

$$f(x+y) = \sum_{|\alpha| \le q} \frac{1}{\alpha!} \partial^\alpha f(x) y^\alpha + \sum_{|\beta| = q+1} g_\beta(x+y) y^\beta .$$

where g_β are some smooth functions on Ω. Then by §3.4, we obtain

$$< f, \rho_{\lambda x} - \delta_x > = \int f(x + \lambda z)\rho(z)dz - \int f(x)\rho(z)dz$$

$$= \sum_{0 < |\alpha| \le q} \int \frac{1}{\alpha!} \partial^\alpha f(x)(\lambda z)^\alpha \rho(z)dz + \sum_{|\beta| = q+1} \int g_\beta(x + \lambda z)(\lambda z)^\beta \rho(z)dz$$

Since $\rho \in A_q$, we have $\int z^\alpha \rho(z)dz = 0$ for all α with $0 < |\alpha| \le q$. Hence the first term vanishes. Therefore we have

$$< f, \rho_{\lambda x} - \delta_x > = \lambda^{q+1} \sum_{|\beta| = q+1} \int g_\beta(x + \lambda z) z^\beta \rho(z)dz \,.$$

Since all g_β are bounded on the compact set $x + [0, r] \operatorname{supp} \rho$, the set B is weakly bounded in $\mathcal{E}'(\Omega)$ and hence strongly bounded in $\mathcal{E}'(\Omega)$. □

6.7. Theorem. *Let S, T be compact detections on Ω. The following statements are equivalent.*

(a) $[S] = [T]$, *i.e. they represent the same detector.*

(b) $S(\delta_x) = T(\delta_x)\,,\ \forall\, x \in \Omega$, *i.e. they have the same value at every point.*

Proof. By linearity, we may assume $S = 0$.

($a \Rightarrow b$) Let T be a null compact detection. Take any $x \in \Omega$. For $K = \{x\}$ and $\alpha = 0$, choose n by §4.1. Take $q = n + 1$ and $\rho \in A_q$. There are $r, M > 0$ such that for all $\lambda \in (0, r)$, we have $\operatorname{supp} \rho_{\lambda x} \subset \Omega$

and $|T(\rho_{\lambda x})| \le M\lambda$. It follows from §6.6 that $\rho_{\lambda x} - \delta_x \in \lambda B$ where B is a bounded set in $\mathcal{E}'(\Omega)$ and hence $\rho_{\lambda x} \to \delta_x$ in $\mathcal{E}'(\Omega)$ as $\lambda \to 0$. As a holomorphic function, T is continuous on $\mathcal{E}'(\Omega)$. Therefore we obtain $T(\delta_x) = \lim_{\lambda\to 0} T(\rho_{\lambda x}) = 0$.

($b \Rightarrow a$) Suppose that T is a compact detection with $T(\delta_x) = 0$ for all $x \in \Omega$. To show that T is null, let K be a compact subset of Ω and let α be a multi-index. Fix $n \ge 0$. Take any $q \ge n$ and $\rho \in A_q$. There is $r \in (0,1)$ such that $K + [0,r]\operatorname{supp}\rho \subset \Omega$ and that §6.6 holds. Since $(T \circ \delta)(x) = 0$, by §6.3 we have

$$(\partial_j T)(\delta_x) = -DT(\delta_x)(\partial_j \delta_x) = DT(\delta_x)(\partial_j \delta)(x) = \partial_j (T \circ \delta)(x) = 0.$$

By induction, we get $S(\delta_x) = 0$ for all $x \in \Omega$ where $S = \partial^\alpha T$. Now pick any $x \in K$ and any $\lambda \in (0,r)$. For $h = \rho_{\lambda x} - \delta_x \in \lambda^{q+1} B$, we have

$$S(\rho_{\lambda x}) = S(\rho_{\lambda x}) - S(\delta_x) \in \overline{\text{co}}\,\{DS(\delta_x + th)h : t \in [0,1]\}$$

the closed convex hull. By [15, 2.3], the family

$$\{DS(\delta_x + th) : t \in [0,1]\}$$

is equicontinuous. From §6.6, the set

$$\{DS(\delta_x + th)b : t \in [0,1]\,, b \in B\}$$

is bounded. There is $M_x > 0$ such that $|DS(\delta_x + th)b| < M_x$, that is $|S(\rho_{\lambda x})| < \lambda^{q+1} M_x$. By §3.4 c, the function $x \to S(\rho_{\lambda x})$ is continuous. Thus there is an open neighborhood U_x of x such that $\operatorname{supp}\rho_{\lambda y} \subset \Omega$ and $|S(\rho_{\lambda y})| < \lambda^{q+1} M_x$ for all $y \in U_x$. By compactness of K, we may choose a finite open cover from U_x and define M as the maximum of the corresponding M_x. Hence for any $y \in K$ and any $\lambda \in (0,r)$, we have $|S(\rho_{\lambda y})| \le \lambda^{q+1} M$, that is $|(\partial^\alpha T)(\rho_{\lambda y})| \le M\lambda^{q-n}$. Therefore T is a null detection, that is $[T] = 0$. □

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School of Mathematics and Statistics, University of Western Australia, Nedlands, W.A., 6907, Australia.

E-mail: twma@maths.uwa.edu.au