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THE OSCILLATION OF CERTAIN HIGH ORDER PARTIAL DIFFERENCE EQUATIONS

BY

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Abstract

In this paper, some criteria for the oscillation of the high order partial difference equations of the form

$$T^i(x_{m,n}+ax_{m-k_1,n-l_1}-bx_{m+k_2,n+l_2}) = c(qx_{m-\sigma_1,n-\tau_1}+px_{m+\sigma_2,n+\tau_2})$$

are established, where $c=\pm 1, i\in N=\{1,2,3,\ldots\}$.

1. Introduction

Partial difference equation are difference equations that involve functions of two or more independent integer variables. Such equations are root in the random walk problems, molecular orbits problems, mathematical physics problems, and numerical difference approximation problems. In this paper, we consider the oscillation of certain high order partial difference equations of the form

$$T^i(x_{m,n}+ax_{m-k_1,n-l_1}-bx_{m+k_2,n+l_2}) = c(qx_{m-\sigma_1,n-\tau_1}+px_{m+\sigma_2,n+\tau_2}), \tag{E_i, c}$$

where $c=\pm 1, i\in N, a, b, p$ and q are nonnegative real numbers, k_j, l_j, σ_j,and τ_j are nonnegative integer numbers, $j=1,2$.

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Now we define an operator T as follows

$$Tx_{m,n}=(\Delta_1+\Delta_2+I)x_{m,n},\ \Delta_1 x_{m,n}=x_{m+1,n}-x_{m,n},\ \Delta_2 x_{m,n}=x_{m,n+1}-x_{m,n},$$

and $Ix_{m,n} = x_{m,n}$. That is to say $Tx_{m,n} = x_{m+1,n} + x_{m,n+1} - x_{m,n}$, and define $T^2 x_{m,n} = T(Tx_{m,n}), \ldots, T^i x_{m,n} = T(T^{i-1}x_{m,n})$.

When $i = 1$, and $a = b = p = 0$, $c = -1$, B. G. Zhang in [1-3] had studied the oscillation of the following equations

$$x_{m+1,n} + x_{m,n+1} - x_{m,n} + qx_{m-\sigma,n-\tau} = 0 \tag{1.1}$$

and

$$x_{m+1,n} + x_{m,n+1} - x_{m,n} + \sum_{i=1}^{u} q_i x_{m-\sigma_i,n-\tau_i} = 0. \tag{1.2}$$

R. P. Agrwal and S. R. Grace[6] had studied the oscillation of the following ordinary difference equations

$$\Delta^i(x_n + ax_{m-k} - bx_{n+l}) = c(qx_{m-\sigma} + px_{n+\tau}), \tag{1.3}$$

where a, b, p and q are nonnegative real numbers, and k, l, σ, τ are nonnegative integer numbers. These can be regarded as a type of special situation of Eq.(E_i, c), some others oscillation results about one order partial difference equation can be founded in [4,5] and the references therein.

But the oscillation of high order partial difference equations is of interest, and the analysis of high order partial difference equations is fewer in the past. The purpose of this paper is to establish some sufficient conditions, involving the coefficients and the arguments only, under which all solutions of Eq.(E_i, c) are oscillatory. The advantage of working with these condition is that they are explicit, and therefore easily verifiable.

It is evident that the method of monotone sequence which was constructed by R. P. Agarwal and S. R. Grace can't be used in handling partial difference equations (E_i, c). So we must use new technique to the case of partial difference equations. Furthermore, it will be noticed that our results improve all the corresponding theorems in [6] when the partial difference Eq.(E_i, c) are degenerated to the ordinary difference equations (1.3).

2. Main Results and Proofs

Let $\{f(m,n)\}$ be double-sequence, $m,n \in N_0^2$, N_0 is a set of nonnegative integer numbers. Define Z-transformation as follows

$$Z(f(m,n)) = F(z_1,z_2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f(m,n) z_1^{-m} z_2^{-n}, \tag{2.1}$$

suppose that (2.1) is convergence for $|z_i| > \gamma_i$, $i = 1,2$.

In domain $|z_i| > \gamma_1$ and $|z_i| > \gamma_2$, (2.1) define a complex analytic function about variables z_1 and z_2. By the property of Z-transformation and two dimensional analytic function, we have the following important lemma:

Lemma 1.([3]). *Consider the liner homogeneous partial difference equations*

$$x_{m+k,n+l} + \sum_{i=1}^{s_1}\sum_{j=1}^{s_2} q_{i,j} x_{m+k-i,n+l-j} = 0, \tag{2.2}$$

where k,l are nonnegative integer numbers, and $q_{i,j} \in R$, then the following propositions are equivalent

(1) *Every solution of Eq.*(2.2) *is oscillatory;*

(2) *The characteristic equation of* (2.2)

$$\lambda^k \mu^\lambda + \sum_{i=1}^{s_1}\sum_{j=1}^{s_2} q_{i,j} \lambda^{k-i} \mu^{l-j} = 0 \tag{2.3}$$

has no positive roots.

Now if i is odd, $c = \pm 1$, then we have

Theorem 1. *Suppose that $b > 0$, $p > 0$, $q > 0$, k_j, l_j, σ_j, τ_j, $j = 1,2$ are positive integer numbers, and assume that $i < \sigma_2 + \tau_{2,} k_2 > \sigma_2$, $l_2 > \tau_2$.*

If

$$p > \frac{(\sigma_2+\tau_2-i)^{\sigma_2+\tau_2-i}i^i}{\sigma_2^{\sigma_2}\tau_2^{\tau_2}} + a\frac{(k_1+\sigma_2+l_1+\tau_2-i)^{k_1+\sigma_2+l_1+\tau_2-i}i^i}{(k_1+\sigma_2)^{k_1+\sigma_2}(l_1+\tau_2)^{l_1+\tau_2}},$$

and either

$$q+p \ > \ b\frac{(k_2-\sigma_2)^{k_2-\sigma_2}\cdot(l_2-\tau_2)^{l_2-\tau_2}i^i}{(k_2-\sigma_2+l_2-\tau_2+i)^{k_2-\sigma_2+l_2-\tau_2+i}}$$

or

$$q \ > \ b\frac{(k_2+\sigma_1)^{k_2+\sigma_1}\cdot(l_2+\tau_1)^{l_2+\tau_1}i^i}{(k_2+\sigma_1+l_2+\tau_1+i)^{k_2+\sigma_1+l_2+\tau_1+i}},$$

then Eq.$(E_i,1)$ *is oscillatory.*

Proof. The oscillation of Eq.$(E_i,1)$ is equivalent to the equation

$$\begin{aligned} & T^i(x_{m-\sigma_2,n-\tau_2}+ax_{m-k_1-\sigma_2,n-l_1-\tau_2}-bx_{m+k_2-\sigma_2,n+l_2-\tau_2}) \\ = \ & qx_{m-\sigma_1-\sigma_2,n-\tau_1-\tau_2}+px_{m,n} \end{aligned} \tag{2.4}$$

or

$$\begin{aligned} & T^i(x_{m+\sigma_1,n+\tau_1}+ax_{m-k_1+\sigma_1,n-l_1+\tau_1}-bx_{m+k_2+\sigma_1,n+l_2+\tau_1}) \\ = \ & qx_{m,n}+px_{m+\sigma_2+\sigma_1,n+\tau_2+\tau_1}. \end{aligned} \tag{2.5}$$

Its characteristic equation polynomial is

$$\begin{aligned} F(\lambda,\mu) \ = \ & (\lambda+\mu-1)^i\lambda^{-\sigma_2}\mu^{-\tau_2}+a(\lambda+\mu-1)^i\lambda^{-k_1-\sigma_2}\mu^{-l_1-\tau_2} \\ & -b(\lambda+\mu-1)^i\lambda^{k_2-\sigma_2}\mu^{l_2-\tau_2}-q\lambda^{-\sigma_1-\sigma_2}\mu^{-\tau_1-\tau_2}-p \end{aligned} \tag{2.6}$$

or

$$\begin{aligned} \tilde{F}(\lambda,\mu) \ = \ & (\lambda+\mu-1)^i\lambda^{\sigma_1}\mu^{\tau_1}+a(\lambda+\mu-1)^i\lambda^{-k_1+\sigma_1}\mu^{-l_1+\tau_1} \\ & -b(\lambda+\mu-1)^i\lambda^{k_2+\sigma_1}\mu^{l_2+\tau_1}-q-p\lambda^{\sigma_2+\sigma_1}\mu^{\tau_2+\tau_1}. \end{aligned} \tag{2.7}$$

For $F(\lambda,\mu)$. (1) If $\lambda+\mu>1$, then

$$F(\lambda,\mu) < (\lambda+\mu-1)^i\lambda^{-\sigma_2}\mu^{-\tau_2}+a(\lambda+\mu-1)^i\lambda^{-k_1-\sigma_2}\mu^{-l_1-\tau_2}-p.$$

When $\lambda=\frac{\sigma_2}{\sigma_2+\tau_2-i}$, $\mu=\frac{\tau_2}{\sigma_2+\tau_2-i}$, $(\lambda+\mu-1)^i\lambda^{-\sigma_2}\mu^{-\tau_2}$ attains its maximum. When $\lambda=\frac{k_1+\sigma_2}{k_1+\sigma_2+l_1+\tau_2-i}$, $\mu=\frac{l_1+\tau_2}{k_1+\sigma_2+l_1+\tau_2-i}$, $(\lambda+\mu-1)^i\lambda^{-k_1-\sigma_2}\mu^{-l_1-\tau_2}$

attains its maximum. Therefor

$$F(\lambda,\mu) < \frac{(\sigma_2+\tau_2-i)^{\sigma_2+\tau_2-i}i^i}{\sigma_2^{\sigma_2}\tau_2^{\tau_2}} + a\frac{(k_1+\sigma_2+l_1+\tau_2-i)^{k_1+\sigma_2+l_1+\tau_2-i}i^i}{(k_1+\sigma_2)^{k_1+\sigma_2}(l_1+\tau_2)^{l_1+\tau_2}}.$$

Evidently, so long as

$$p > \frac{(\sigma_2+\tau_2-i)^{\sigma_2+\tau_2-i}i^i}{\sigma_2^{\sigma_2}\tau_2^{\tau_2}} + a\frac{(k_1+\sigma_2+l_1+\tau_2-i)^{k_1+\sigma_2+l_1+\tau_2-i}i^i}{(k_1+\sigma_2)^{k_1+\sigma_2}(l_1+\tau_2)^{l_1+\tau_2}},$$

we have $F(\lambda,\mu) < 0$.

(2) If $0 < \lambda+\mu < 1$, then

$$F(\lambda,\mu) < -b(\lambda+\mu-1)^i\lambda^{k_2-\sigma_2}\mu^{l_2-\tau_2} - q - p.$$

When $\lambda = \frac{k_2-\sigma_2}{k_2-\sigma_2+l_2-\tau_2+i}$, $\mu = \frac{l_2-\tau_2}{k_2-\sigma_2+l_2-\tau_2+i}$, $-(\lambda+\mu-1)^i\lambda^{k_2-\sigma_2}\mu^{l_2-\tau_2}$ attains its maximum. Therefor

$$F(\lambda,\mu) < -q-p+b\frac{(k_2-\sigma_2)^{k_2-\sigma_2}(l_2-\tau_2)^{l_2-\tau_2}i^i}{(k_2-\sigma_2+l_2-\tau_2+i)^{k_2-\sigma_2+l_2-\tau_2+i}}.$$

Provided that

$$q+p > b\frac{(k_2-\sigma_2)^{k_2-\sigma_2}(l_2-\tau_2)^{l_2-\tau_2}\cdot i^i}{(k_2-\sigma_2+l_2-\tau_2+i)^{k_2-\sigma_2+l_2-\tau_2+i}},$$

we have $F(\lambda,\mu) < 0$.

For $\widetilde{F}(\lambda,\mu)$, if $0 < \lambda+\mu < 1$, then

$$\widetilde{F}(\lambda,\mu) < -b(\lambda+\mu-1)^i\lambda^{k_2+\sigma_1}\mu^{l_2+\tau_1} - q.$$

When $\lambda = \frac{k_2+\sigma_1}{k_2+\sigma_1+l_2+\tau_1+i}$, $\mu = \frac{l_2+\tau_1}{k_2+\sigma_1+l_2+\tau_1+i}$, $-(\lambda+\mu-1)^i\lambda^{k_2+\sigma_1}\mu^{l_2+\tau_1}$ attains its maximum. Therefor, provided that

$$q > b\frac{(k_2+\sigma_1)^{k_2+\sigma_1}(l_2+\tau_1)^{l_2+\tau_1}i^i}{(k_2+\sigma_1+l_2+\tau_1+i)^{k_2+\sigma_1+l_2+\tau_1+i}},$$

we have $\widetilde{F}(\lambda,\mu) < 0$.

That is to say, characteristic equation $F(\lambda,\mu) = 0$ (or $\widetilde{F}(\lambda,\mu) = 0$) have no positive real roots, hence $(E_i, 1)$ is oscillatory. □

Theorem 2. *Assume that* $b > 0$, $\sigma_1 - k_1 \geq 1$, $\tau_1 - l_1 \geq 1$ *and* $k_2 - \sigma_2 + l_2 - \tau_2 > i$, *if*

$$q > \frac{\sigma_1^{\sigma_1}\tau_1^{\tau_1} i^i}{(\sigma_1 + \tau_1 + i)^{\sigma_1+\tau_1+i}} + a\frac{(\sigma_1 - k_1)^{\sigma_1-k_1}(\tau_1 - l_1)^{\tau_1-l_1} \cdot i^i}{(\sigma_1 - k_1 + \tau_1 - l_1 + i)^{\sigma_1-k_1+\tau_1-l_1+i}}$$

and

$$p > b\frac{(k_2 + \sigma_2 + l_2 - \tau_2 - i)^{k_2+\sigma_2+l_2-\tau_2-i}}{(k_2 - \sigma_2)^{k_2-\sigma_2}(l_2 - \tau_2)^{l_2-\tau_2}},$$

then $(E_i, -1)$ *is oscillatory.*

Proof. The oscillation of Eq.$(E_i, -1)$ is equivalent to the equation

$$\begin{aligned} T^i(x_{m-\sigma_2,n-\tau_2} + ax_{m-k_1-\sigma_2,n-l_1-\tau_2} - bx_{m+k_2-\sigma_2,n+l_2-\tau_2}) \\ +qx_{m-\sigma_1-\sigma_2,n-\tau_1-\tau_2} + px_{m,n} = 0 \end{aligned} \tag{2.8}$$

or

$$\begin{aligned} T^i(x_{m+\sigma_1,n+\tau_1} + ax_{m-k_1+\sigma_1,n-l_1+\tau_1} - bx_{m+k_2+\sigma_1,n+l_2+\tau_1}) \\ +qx_{m,n} + px_{m+\sigma_2+\sigma_1,n+\tau_2+\tau_1} = 0. \end{aligned} \tag{2.9}$$

Its characteristic equation polynomial is

$$\begin{aligned} F(\lambda, \mu) &= (\lambda + \mu - 1)^i \lambda^{-\sigma_2}\mu^{-\tau_2} + a(\lambda + \mu - 1)^i \lambda^{-k_1-\sigma_2}\mu^{-l_1-\tau_2} \\ &\quad -b(\lambda + \mu - 1)^i \lambda^{k_2-\sigma_2}\mu^{l_2-\tau_2} + q\lambda^{-\sigma_1-\sigma_2}\mu^{-\tau_1-\tau_2} + p \end{aligned} \tag{2.10}$$

or

$$\begin{aligned} \tilde{F}(\lambda, \mu) &= (\lambda + \mu - 1)^i \lambda^{\sigma_1}\mu^{\tau_1} + a(\lambda + \mu - 1)^i \lambda^{-k_1+\sigma_1}\mu^{-l_1+\tau_1} \\ &\quad -b(\lambda + \mu - 1)^i \lambda^{k_2+\sigma_1}\mu^{l_2+\tau_1} + q + p\lambda^{\sigma_2+\sigma_1}\mu^{\tau_2+\tau_1}. \end{aligned} \tag{2.11}$$

For $F(\lambda, \mu)$. If $\lambda + \mu > 1$, then

$$F(\lambda, \mu) > -b(\lambda + \mu - 1)^i \lambda^{k_2-\sigma_2}\mu^{l_2-\tau_2} + p.$$

When $\lambda = \frac{k_2-\sigma_2}{k_2-\sigma_2+l_2-\tau_2-i}$, $\mu = \frac{l_2-\tau_2}{k_2-\sigma_2+l_2-\tau_2-i}$, $(\lambda+\mu-1)^i\lambda^{k_2-\sigma_2}\mu^{l_2-\tau_2}$ attains its maximum. Therefor

$$F(\lambda, \mu) > -b\frac{(k_2 - \sigma_2 + l_2 - \tau_2 - i)^{k_2-\sigma_2+l_2-\tau_2-i} i^i}{(k_2 - \sigma_2)^{k_2-\sigma_2} \cdot (l_2 - \tau_2)^{l_2-\tau_2}} + p.$$

So long as

$$p > b\frac{(k_2-\sigma_2+l_2-\tau_2-i)^{k_2-\sigma_2+l_2-\tau_2-i}\cdot i^i}{(k_2-\sigma_2)^{k_2-\sigma_2}\cdot(l_2-\tau_2)^{l_2-\tau_2}},$$

we have $F(\lambda,\mu)>0$.

For $\tilde{F}(\lambda,\mu)$. If $0<\lambda+\mu<1$, then

$$\tilde{F}(\lambda,\mu) > (\lambda+\mu-1)^i\lambda^{\sigma_1}\mu^{\tau_1}+a(\lambda+\mu-1)^i\lambda^{-k_1+\sigma_1}\mu^{-l_1+\tau_1}+q.$$

When $\lambda=\frac{\sigma_1}{\sigma_1+\tau_1+i}$, $\mu=\frac{\tau_1}{\sigma_1+\tau_1+i}$, $(\lambda+\mu-1)^i\lambda^{\sigma_1}\mu^{\tau_1}$ attains its minimum. When $\lambda=\frac{-k_1+\sigma_1}{-k_1+\sigma_1-l_1+\tau_1-i}$, $\mu=\frac{-l_1+\tau_1}{-k_1+\sigma_1-l_1+\tau_1-i}$, $(\lambda+\mu-1)^i\lambda^{-k_1+\sigma_1}\mu^{-l_1+\tau_1}$ attains its maximum. Therefor

$$\tilde{F}(\lambda,\mu) > -a\frac{(-k_1+\sigma_1)^{-k_1+\sigma_1}(-l_1+\tau_1)^{-l_1+\tau_1}i^i}{(-k_1+\sigma_1-l_1+\tau_1+i)^{-k_1+\sigma_1-l_1+\tau_1+i}}-\frac{\sigma_1^{\sigma_1}\tau_1^{\tau_1}i^i}{(\sigma_1+\tau_1+i)^{\sigma_1+\tau_1+i}}+q.$$

Provided that

$$q > a\frac{(-k_1+\sigma_1)^{-k_1+\sigma_1}(-l_1+\tau_1)^{-l_1+\tau_1}i^i}{(-k_1+\sigma_1-l_1+\tau_1+i)^{-k_1+\sigma_1-l_1+\tau_1+i}}+\frac{\sigma_1^{\sigma_1}\tau_1^{\tau_1}i^i}{(\sigma_1+\tau_1+i)^{\sigma_1+\tau_1+i}},$$

we have $\widetilde{F}(\lambda,\mu)>0$.

In a word, characteristic equation $F(\lambda,\mu)=0$ (or $\widetilde{F}(\lambda,\mu)=0$) have no positive real roots, then $(E_i,-1)$ is oscillatory. □

Next if i is even, $c=\pm1$, then we have

Theorem 3. *Suppose that $b>0$, $\sigma_1-k_1\ge 1$, $\tau_1-l_1\ge 1$, and $\sigma_2+\tau_2>i$, if*

$$q > \frac{\sigma_1^{\sigma_1}\tau_1^{\tau_1}\cdot i^i}{(\sigma_1+\tau_1+i)^{\sigma_1+\tau_1+i}}+a\cdot\frac{(\sigma_1-k_1)^{\sigma_1-k_1}(\tau_1-l_1)^{\tau_1-l_1}}{(\sigma_1-k_1+\tau_1-l_1+i)^{\sigma_1-k_1+\tau_1-l_1+i}}$$

and

$$p > \frac{(\sigma_2+\tau_2-i)^{\sigma_2+\tau_2-i}i^i}{\sigma_2^{\sigma_2}\tau_2^{\tau_2}}+a\frac{(k_1+\sigma_2+l_1+\tau_2-i)^{k_1+\sigma_2+l_1+\tau_2-i}\cdot i^i}{(k_1+\sigma_2)^{k_1+\sigma_2}(l_1+\tau_2)^{l_1+\tau_2}},$$

then $(E_i,1)$ is oscillatory.

Proof. Just the same as in the proof of Theorem 1. The oscillation of Eq.$(E_i, 1)$ is equivalent to Eq.(2.4) and (2.5). Its characteristic equation polynomial is (2.6) or (2.7).

For $F(\lambda,\mu)$, if $\lambda+\mu>1$, then

$$F(\lambda,\mu)<(\lambda+\mu-1)^i\lambda^{-\sigma_2}\mu^{-\tau_2}+a(\lambda+\mu-1)^i\lambda^{-k_1-\sigma_2}\mu^{-l_1-\tau_2}-p.$$

When $\lambda=\frac{\sigma_2}{\sigma_2+\tau_2-i},\mu=\frac{\tau_2}{\sigma_2+\tau_2-i}$, $(\lambda+\mu-1)\lambda^{-\sigma_2}\mu^{-\tau_2}$ attains its maximum. When $\lambda=\frac{k_1+\sigma_2}{k_1+\sigma_2+l_1+\tau_2-i},\mu=\frac{l_1+\tau_2}{k_1+\sigma_2+l_1+\tau_2-i}$, $(\lambda+\mu-1)\lambda^{-k_1-\sigma_2}\mu^{-l_1-\tau_2}$ attains its maximum. Hence

$$F(\lambda,\mu)<\frac{(\sigma_2+\tau_2-i)^{\sigma_2+\tau_2-i}i^i}{\sigma_2^{\sigma_2}\tau_2^{\tau_2}}+a\frac{(k_1+\sigma_2+l_1+\tau_2-i)^{k_1+\sigma_2+l_1+\tau_2-i}.i^i}{(k_1+\sigma_2)^{k_1+\sigma_2}(l_1+\tau_2)^{l_1+\tau_2}}-p.$$

So long as

$$p>\frac{(\sigma_2+\tau_2-i)^{\sigma_2+\tau_2-i}i^i}{\sigma_2^{\sigma_2}\tau_2^{\tau_2}}+a\frac{(k_1+\sigma_2+l_1+\tau_2-i)^{k_1+\sigma_2+l_1+\tau_2-i}.i^i}{(k_1+\sigma_2)^{k_1+\sigma_2}(l_1+\tau_2)^{l_1+\tau_2}},$$

we have $F(\lambda,\mu)<0$.

For $\widetilde{F}(\lambda,\mu)$. If $0<\lambda+\mu<1$, then

$$\tilde{F}(\lambda,\mu)<(\lambda+\mu-1)^i\lambda^{\sigma_1}\mu^{\tau_1}+a(\lambda+\mu-1)^i\lambda^{\sigma_1-k_1}\mu^{\tau_1-l_1}-q.$$

When $\lambda=\frac{\sigma_1}{\sigma_1+\tau_1+i},\mu=\frac{\tau_1}{\sigma_1+\tau_1+i}$, $(\lambda+\mu-1)^i\lambda^{\sigma_1}\mu^{l_1}$ attains its maximum. When $\lambda=\frac{\sigma_1-k_2}{\sigma_1-k_1+\tau_1-l_1+i},\mu=\frac{\tau_1-l_1}{\sigma_1-k_1+\tau_1-l_1+i}$, $(\lambda+\mu-1)^i\lambda^{\sigma_1-k_1}\mu^{\tau_1-l_1}$ attains its maximum. Therefor, provided that

$$q>\frac{\sigma_1^{\sigma_1}\tau_1^{\tau_1}i^i}{(\sigma_1+\tau_1+i)^{\sigma_1+\tau_1+i}}+a\frac{(\sigma_1-k_1)^{\sigma_1-k_1}(\tau_1-l_1)^{\tau_1-l_1}i^i}{(\sigma_1-k_1+\tau_1-l_1+i)^{\sigma_1-k_1+\tau_1-l_1+i}},$$

We have $\widetilde{F}(\lambda,\mu)<0$.

That is to say, characteristic equation $F(\lambda,\mu)=0$ (or $\widetilde{F}(\lambda,\mu)=0$) have no positive real roots, hence $(E_i,1)$ is oscillatory. □

Theorem 4. *Assume that* $b>0$, $\sigma_2-k_2+\tau_2-l_2>i$, *if*

$$q>b\frac{(k_2+\sigma_1)^{k_2+\sigma_1}(l_2+\tau_1)^{l_2+\tau_1}i^i}{(k_2+\sigma_1+l_2+\tau_1+i)^{k_2+\sigma_1+l_1+\tau_1+i}}$$

and

$$p > b\frac{(\sigma_2 - k_2 + \tau_2 - l_2 - i)^{\sigma_2 - k_2 + \tau_2 - l_2 - i} i^i}{(\sigma_2 - k_2)^{\sigma_2 - k_2}(\tau_2 - l_2)^{\tau_2 - l_2}},$$

then $(E_i, -1)$ *is oscillatory.*

Proof. Just the same as Theorem 2. The oscillation of Eq.$(E_i, -1)$ is equivalent to Eq.(2.8) or (2.9). Its characteristic equation polynomial is (2.10) or (2.11).

For $F(\lambda,\mu)$, if $\lambda+\mu > 1$, then

$$F(\lambda,\mu) > -b(\lambda+\mu-1)^i \lambda^{-(\sigma_2-k_2)}\mu^{-(\tau_2-l_2)} + p.$$

When $\lambda = \frac{\sigma_2-k_2}{\sigma_2-k_2+\tau_2-l_2-i}, \mu = \frac{\tau_2-l_2}{\sigma_2-k_2+\tau_2-l_2-i}, (\lambda+\mu-1)^i \lambda^{-(\sigma_{2-}-k_2)}\mu^{-(\tau_2-l_2)}$ attains its maximum. Hence

$$F(\lambda,\mu) > -b\frac{(\sigma_2 - k_2 + \tau_2 - l_2 - i)^{\sigma_2 - k_2 + \tau_2 - l_2 - i} i^i}{(\sigma_2 - k_2)^{\sigma_2 - k_2}(\tau_2 - l_2)^{\tau_2 - l_2}} + p,$$

provided that

$$p > b\frac{(\sigma_2 - k_2 + \tau_2 - l_2 - i)^{\sigma_2 - k_2 + \tau_2 - l_2 - i} i^i}{(\sigma_2 - k_2)^{\sigma_2 - k_2}(\tau_2 - l_2)^{\tau_2 - l_2}},$$

we have $F(\lambda,\mu) > 0$.

For $\tilde{F}(\lambda,\mu)$. If $0 < \lambda+\mu < 1$, then

$$\tilde{F}(\lambda,\mu) > -b(\lambda+\mu-1)^i \lambda^{k_2+\sigma_1}\mu^{l_2+\tau_1} + q.$$

When $\lambda = \frac{k_2+\sigma_1}{k_2+\sigma_1+l_2+\tau_1+i}, \mu = \frac{l_2+\tau_1}{k_2+\sigma_1+l_2+\tau_1+i}, (\lambda+\mu-1)^i \lambda^{k_2+\sigma_1}\mu^{l_2+\tau_1}$ attains its maximum. Hence

$$\tilde{F}(\lambda,\mu) > -b\frac{(k_2+\sigma_1)^{k_2+\sigma_1}(l_2+\tau_1)^{l_2+\tau_1} i^i}{(k_2+\sigma_1+l_2+\tau_1+i)^{k_2+\sigma_1+l_2+\tau_1+i}} + q.$$

So long as

$$q > b\frac{(k_2+\sigma_1)^{k_2+\sigma_1}(l_2+\tau_1)^{l_2+\tau_1} i^i}{(k_2+\sigma_1+l_2+\tau_1+i)^{k_2+\sigma_1+l_2+\tau_1+i}},$$

we have $\tilde{F}(\lambda,\mu) > 0$.

In a word, characteristic equation $F(\lambda,\mu)=0$ (or $\widetilde{F}(\lambda,\mu)=0$) have no positive real roots, hence $(E_i,-1)$ is oscillatory. □

3. Some Remarks

1. It is remarkable that our results are also valid if we define operation

$$Tx_{m,n}=(r\Delta_1+s\Delta_2+tI),$$

where $\Delta_1 x_{m,n}=x_{m+1,n}-x_{m,n}$, $\Delta_2 x_{m,n}=x_{m,n+1}-x_{m,n}$, $Ix_{m,n}=x_{m,n}$, $s,t>0$, and $r+s-t>0$.

As an example, we verify the equations $(E_i,1)$. Its characteristic equation polynomial is

$$\begin{aligned}F(\lambda,\mu) &= [r\lambda+s\mu-(r+s-t)]^i\lambda^{-\sigma_2}\mu^{-\tau_2}\\ &\quad +a[r\lambda+s\mu-(r+s-t)]^i\lambda^{-k_1-\sigma_2}\mu^{-l_1-\tau_2}\\ &\quad -b[r\lambda+s\mu-(r+s-t)]^i\lambda^{k_2-\sigma_2}\mu^{l_2-\tau_2}-q\lambda^{-\sigma_1-\sigma_2}\mu^{-\tau_1-\tau_2}-p\end{aligned}$$

or

$$\begin{aligned}\tilde{F}(\lambda,\mu) &= [r\lambda+s\mu-(r+s-t)]^i\lambda^{\sigma_1}\mu^{\tau_1}\\ &\quad +a[r\lambda+s\mu-(r+s-t)]^i\lambda^{-k_1+\sigma_1}\mu^{-l_1+\tau_1}\\ &\quad -b[r\lambda+s\mu-(r+s-t)]^i\lambda^{k_2+\sigma_1}\mu^{l_2+\tau_1}-q-p\lambda^{\sigma_2+\sigma_1}\mu^{\tau_2+\tau_1}.\end{aligned}$$

(1) If $r\lambda+s\mu>r+s-t$, then

$$\begin{aligned}F(\lambda,\mu) &< [r\lambda+s\mu-(r+s-t)]\lambda^{-\sigma_2}\mu^{-\tau_2}\\ &\quad +a[r\lambda+s\mu-(r+s-t)]^i\lambda^{-k_1-\sigma_2}\mu^{-l_1-\tau_2}-p.\end{aligned}$$

When $\lambda=\frac{r+s-t}{r}\frac{\sigma_2}{\sigma_2+\tau_2-i}$, $\mu=\frac{r+s-t}{s}\frac{\tau_2}{\sigma_2+\tau_2-i}$, $[r\lambda+s\mu-(r+s-t)]^i\lambda^{-\sigma_2}\mu^{-\tau_2}$ attains its maximum $\frac{r^{\sigma_2}s^{\tau_2}}{(r+s-t)^{\sigma_2+\tau_2-i}}\frac{(\sigma_2+\tau_2-i)^{\sigma_2+\tau_2-i}i^i}{\sigma_2^{\sigma_2}\tau_2^{\tau_2}}$.

When $\lambda=\frac{r+s-t}{r}\frac{k_1+\sigma_2}{k_1+\sigma_2+l_1+\tau_2-i}$, $\mu=\frac{r+s-t}{r}\frac{l_1+\tau_2}{k_1+\sigma_2+l_1+\tau_2-i}$, $[r\lambda+s\mu-(r+s-t)]^i\lambda^{-k_1-\sigma_2}\mu^{-l_1-\tau_2}$ attains its maximum

$$\frac{r^{k_1+\sigma_2}s^{l_1+\tau_2}}{(r+s-t)^{k_1+\sigma_2+l_1+\tau_2-i}}\frac{(k_1+\sigma_2+l_1+\tau_2-i)^{k_1+\sigma_2+l_1+\tau_2-i}i^i}{(k_1+\sigma_2)^{k_1+\sigma_2}(l_1+\tau_2)^{l_1+\tau_2}}.$$

Therefor

$$F(\lambda,\mu) < \frac{r^{\sigma_2}s^{\tau_2}}{(r+s-t)^{\sigma_2+\tau_2-i}}\frac{(\sigma_2+\tau_2-i)^{\sigma_2+\tau_2-i}i^i}{\sigma_2^{\sigma_2}\tau_2^{\tau_2}}$$
$$+a\frac{r^{k_1+\sigma_2}s^{l_1+\tau_2}}{(r+s-t)^{k_1+\sigma_2+l_1+\tau_2-i}}\frac{(k_1+\sigma_2+l_1+\tau_2-i)^{k_1+\sigma_2+l_1+\tau_2-i}i^i}{(k_1+\sigma_2)^{k_1+\sigma_2}(l_1+\tau_2)^{l_1+\tau_2}}-p.$$

Evidently, so long as

$$p > \frac{r^{\sigma_2}s^{\tau_2}}{(r+s-t)^{\sigma_2+\tau_2-i}}\frac{(\sigma_2+\tau_2-i)^{\sigma_2+\tau_2-i}i^i}{\sigma_2^{\sigma_2}\tau_2^{\tau_2}}$$
$$+a\frac{r^{k_1+\sigma_2}s^{l_1+\tau_2}}{(r+s-t)^{k_1+\sigma_2+l_1+\tau_2-i}}\frac{(k_1+\sigma_2+l_1+\tau_2-i)^{k_1+\sigma_2+l_1+\tau_2-i}i^i}{(k_1+\sigma_2)^{k_1+\sigma_2}(l_1+\tau_2)^{l_1+\tau_2}},$$

we have $F(\lambda,\mu) < 0$.

(2) If $0 < r\lambda + s\mu < r+s-t$.

(I) If i is odd, then

$$F(\lambda,\mu) < -b[r\lambda+s\mu-(r+s-t)]^i\lambda^{k_2-\sigma_2}\mu^{l_2-\tau_2}-q-p.$$

When $\lambda = \frac{r+s-t}{r}\frac{k_2-\sigma_2}{k_2-\sigma_2+l_2-\tau_2+i}$, $\mu = \frac{r+s-t}{s}\frac{l_2-\tau_2}{k_2-\sigma_2+l_2-\tau_2+i}$, $-[r\lambda+s\mu-(r+s-t)]^i\lambda^{k_2-\sigma_2}\mu^{l_2-\tau_2}$ attains its maximum

$$b\frac{(r+s-t)^{k_2-\sigma_2+l_2-\tau_2+i}}{r^{k_2-\sigma_2}s^{l_2-\tau_2}}\frac{(k_2-\sigma_2)^{k_2-\sigma_2}(l_2-\tau_2)^{l_2-\tau_2}i^i}{(k_2-\sigma_2+l_2-\tau_2+i)^{k_2-\sigma_2+l_2-\tau_2+i}}.$$

Hence

$$F(\lambda,\mu) < -q-p+b\frac{(r+s-t)^{k_2-\sigma_2+l_2-\tau_2+i}}{r^{k_2-\sigma_2}s^{l_2-\tau_2}}\frac{(k_2-\sigma_2)^{k_2-\sigma_2}(l_2-\tau_2)^{l_2-\tau_2}i^i}{(k_2-\sigma_2+l_2-\tau_2+i)^{k_2-\sigma_2+l_2-\tau_2+i}}.$$

Provided that

$$q+p > b\frac{(r+s-t)^{k_2-\sigma_2+l_2-\tau_2+i}}{r^{k_2-\sigma_2}s^{l_2-\tau_2}}\frac{(k_2-\sigma_2)^{k_2-\sigma_2}(l_2-\tau_2)^{l_2-\tau_2}\cdot i^i}{(k_2-\sigma_2+l_2-\tau_2+i)^{k_2-\sigma_2+l_2-\tau_2+i}},$$

we have $F(\lambda,\mu) < 0$.

For $\widetilde{F}(\lambda,\mu)$. We have

$$\widetilde{F}(\lambda,\mu) < -b[r\lambda+s\mu-(r+s-t)]^i\lambda^{k_2+\sigma_1}\mu^{l_2+\tau_1}-q.$$

When $\lambda=\frac{r+s-t}{r}\frac{k_2+\sigma_1}{k_2+\sigma_1+l_2+\tau_1+i}$, $\mu=\frac{r+s-t}{s}\frac{l_2+\tau_1}{k_2+\sigma_1+l_2+\tau_1+i}$, $-[r\lambda+s\mu-(r+s-t)]^i\lambda^{k_2+\sigma_1}\mu^{l_2+\tau_1}$ attains its maximum

$$\frac{(r+s-t)^{k_2+\sigma_1+l_2+\tau_1+i}}{r^{k_2+\sigma_1}s^{l_2+\tau_1}}\frac{(k_2+\sigma_1)^{k_2+\sigma_1}(l_2+\tau_1)^{l_2+\tau_1}i^i}{(k_2+\sigma_1+l_2+\tau_1+i)^{k_2+\sigma_1+l_2+\tau_1+i}}.$$

Therefor provided that

$$q>b\frac{(r+s-t)^{k_2+\sigma_1+l_2+\tau_1+i}}{r^{k_2+\sigma_1}s^{l_2+\tau_1}}\frac{(k_2+\sigma_1)^{k_2+\sigma_1}(l_2+\tau_1)^{l_2+\tau_1}i^i}{(k_2+\sigma_1+l_2+\tau_1+i)^{k_2+\sigma_1+l_2+\tau_1+i}},$$

we have $\widetilde{F}(\lambda,\mu)<0$.

(II) If i is even, then

$$\tilde{F}(\lambda,\mu)<[r\lambda+s\mu-t]^i\lambda^{\sigma_1}\mu^{\tau_1}+a[r\lambda+s\mu-t]^i\lambda^{\sigma_1-k_1}\mu^{\tau_1-l_1}-q.$$

When $\lambda=\frac{r+s-t}{r}\frac{\sigma_1}{\sigma_1+\tau_1+i}$, $\mu=\frac{r+s-t}{s}\frac{\tau_1}{\sigma_1+\tau_1+i}$, $[r\lambda+s\mu-t]^i\lambda^{\sigma_1}\mu^{l_1}$ attains its maximum $\frac{(r+s-t)^{\sigma_1+\tau_1+i}}{r^{\sigma_1}s^{\tau_1}}\frac{\sigma_1^{\sigma_1}\tau_1^{\tau_1}i^i}{(\sigma_1+\tau_1+i)^{\sigma_1+\tau_1+i}}$. When $\lambda=\frac{r+s-t}{r}\frac{\sigma_1-k_2}{\sigma_1-k_1+\tau_1-l_1+i}$, $\mu=\frac{r+s-t}{s}\frac{\tau_1-l_1}{\sigma_1-k_1+\tau_1-l_1+i}$, $[r\lambda+s\mu-t]^i\lambda^{\sigma_1-k_1}\mu^{\tau_1-l_1}$ attains its maximum

$$\frac{(r+s-t)^{\sigma_1-k_1+\tau_1-l_1+i}}{r^{\sigma_1-k_1}s^{\tau_1-l_1}}\frac{(\sigma_1-k_1)^{\sigma_1-k_1}(\tau_1-l_1)^{\tau_1-l_1}i^i}{(\sigma_1-k_1+\tau_1-l_1+i)^{\sigma_1-k_1+\tau_1-l_1+i}}.$$

Therefor, provided that

$$\begin{aligned} q \;>\; & \frac{(r+s-t)^{\sigma_1+\tau_1+i}}{r^{\sigma_1}s^{\tau_1}}\frac{\sigma_1^{\sigma_1}\tau_1^{\tau_1}i^i}{(\sigma_1+\tau_1+i)^{\sigma_1+\tau_1+i}} \\ & +a\frac{(r+s-t)^{\sigma_1-k_1+\tau_1-l_1+i}}{r^{\sigma_1-k_1}s^{\tau_1-l_1}}\frac{(\sigma_1-k_1)^{\sigma_1-k_1}(\tau_1-l_1)^{\tau_1-l_1}i^i}{(\sigma_1-k_1+\tau_1-l_1+i)^{\sigma_1-k_1+\tau_1-l_1+i}}, \end{aligned}$$

We have $\widetilde{F}(\lambda,\mu)<0$.

From the above discussion we can obtain

Theorem 5. *Suppose that $b>0$, $p>0$, $q>0$, k_j, l_j, σ_j, τ_j, $(j=1,2)$ are positive integer numbers, and assume that $i<\sigma_2+\tau_2$, $k_2>\sigma_2$, $l_2>\tau_2$,*

and

$$p > \frac{r^{\sigma_2}s^{\tau_2}}{(r+s-t)^{\sigma_2+\tau_2-i}}\frac{(\sigma_2+\tau_2-i)^{\sigma_2+\tau_2-i}i^i}{\sigma_2^{\sigma_2}\tau_2^{\tau_2}} + a\frac{r^{k_1+\sigma_2}s^{l_1+\tau_2}}{(r+s-t)^{k_1+\sigma_2+l_1+\tau_2-i}}\frac{(k_1+\sigma_2+l_1+\tau_2-i)^{k_1+\sigma_2+l_1+\tau_2-i}i^i}{(k_1+\sigma_2)^{k_1+\sigma_2}(l_1+\tau_2)^{l_1+\tau_2}}.$$

If

$$q+p > b\frac{(r+s-t)^{k_2-\sigma_2+l_2-\tau_2+i}}{r^{k_2-\sigma_2}s^{l_2-\tau_2}}\frac{(k_2-\sigma_2)^{k_2-\sigma_2}(l_2-\tau_2)^{l_2-\tau_2}\cdot i^i}{(k_2-\sigma_2+l_2-\tau_2+i)^{k_2-\sigma_2+l_2-\tau_2+i}}$$

for i is odd and

$$q > \frac{(r+s-t)^{\sigma_1+\tau_1+i}}{r^{\sigma_1}s^{\tau_1}}\frac{\sigma_1^{\sigma_1}\tau_1^{\tau_1}i^i}{(\sigma_1+\tau_1+i)^{\sigma_1+\tau_1+i}} + a\frac{(r+s-t)^{\sigma_1-k_1+\tau_1-l_1+i}}{r^{\sigma_1-k_1}s^{\tau_1-l_1}}\frac{(\sigma_1-k_1)^{\sigma_1-k_1}(\tau_1-l_1)^{\tau_1-l_1}i^i}{(\sigma_1-k_1+\tau_1-l_1+i)^{\sigma_1-k_1+\tau_1-l_1+i}}$$

for i is even, then Eq.$(E_i,1)$ is oscillatory.

If double-sequence $x_{m,n}$ is degenerated to single-sequence x_m, that is to say, when the partial difference equations $(E_i,1)$ is degenerated to the ordinary difference equations

$$\Delta^i(x_n + ax_{m-k} - bx_{n+l}) = c(qx_{m-\sigma} + px_{n+\tau}),$$

(In fact, at this moment, $l_1 = l_2 = \tau_1 = \tau_2 = 0$, and $r = 1, s = t = 0$ in Eq.$(E_i,1)$), then from Theorem 5 we can easily obtain

Corollary 1. *Suppose that $\sigma_2 > i, k_2 > \sigma_2, b > 0$, and*

$$p > \frac{(\sigma_2-i)^{\sigma_2-i}i^i}{\sigma_2^{\sigma_2}} + a\frac{(k_1+\sigma_2-i)^{k_1+\sigma_2-i}i^i}{(k_1+\sigma_2)^{k_1+\sigma_2}}.$$

If

$$q+p > b\frac{(k_2-\sigma_2)^{k_2-\sigma_2}\cdot i^i}{(k_2-\sigma_2+i)^{k_2-\sigma_2+i}} \text{ for } m \text{ is odd}$$

and

$$q > \frac{\sigma_1^{\sigma_1}\tau_1^{\tau_1}i^i}{(\sigma_1+\tau_1+i)^{\sigma_1+\tau_1+i}} + a\frac{(\sigma_1-k_1)^{\sigma_1-k_1}i^i}{(\sigma_1-k_1+i)^{\sigma_1+i}} \quad \textit{for } m \textit{ is even,}$$

then $Eq.(E_i,-1)$ *is oscillatory.*

Since

$$\frac{(\sigma_2-i)^{\sigma_2-i}i^i}{\sigma_2^{\sigma_2}} + a\frac{(k_1+\sigma_2-i)^{k_1+\sigma_2-i}i^i}{(k_1+\sigma_2)^{k_1+\sigma_2}} < (1+a)\frac{(\sigma_2-i)^{\sigma_2-i}i^i}{\sigma_2^{\sigma_2}}$$

and

$$\frac{\sigma_1^{\sigma_1}\tau_1^{\tau_1}i^i}{(\sigma_1+\tau_1+i)^{\sigma_1+\tau_1+i}} + a\frac{(\sigma_1-k_1)^{\sigma_1-k_1}i^i}{(\sigma_1-k_1+i)^{\sigma_1+i}} < (1+a)\frac{(\sigma_1-k_1)^{\sigma_1-k_1}i^i}{(\sigma_1-k_1+i)^{\sigma_1+i}},$$

hence Corollary 1 is essentially new, which improve Theorem 1 in [6].

Making use of the same discussion as above we can obtain

Corollary 2. *Let* $b>0$, $\sigma_2 > k_2+i$, $\sigma_1 > k_1$, *and*

$$p > b\frac{i^i(\sigma_2-k_2-i)^{\sigma_2-k_2-i}}{(\sigma_2-k_2)^{\sigma_2-k_2}}.$$

If

$$q > \frac{\sigma_1^{\sigma_1}i^i}{(\sigma_1+i)^{\sigma_1+i}} + a\frac{(\sigma_1-k_1)^{\sigma_1-k_1}\cdot i^i}{(\sigma_1-k_1+i)^{\sigma_1-k_1+i}} \quad \textit{for } i \textit{ is odd}$$

and

$$q > b\frac{(k_2+\sigma_1)^{k_2+\sigma_1}i^i}{(k_2+\sigma_1+i)^{k_2+\sigma_1+i}} \quad \textit{for } i \textit{ is even,}$$

then $Eq.(E_i,-1)$ *is oscillatory.*

Since

$$\frac{\sigma_1^{\sigma_1}i^i}{(\sigma_1+i)^{\sigma_1+i}} + a\frac{(\sigma_1-k_1)^{\sigma_1-k_1}\cdot i^i}{(\sigma_1-k_1+i)^{\sigma_1-k_1+i}} < (1+a)\frac{(\sigma_1-k_1)^{\sigma_1-k_1}\cdot i^i}{(\sigma_1-k_1+i)^{\sigma_1-k_1+i}},$$

therefor our Corollary 2 is essentially new, which improve Theorem 2 in [6]. Making use of the same discussion as above we can improve some other theorems in ordinary difference equation theory, such as:

$$\begin{aligned}\Delta^i(x_n + ax_{m-k} - bx_{n+l}) &= c(qx_{m-\sigma} + px_{n+\tau}),\\ \Delta^i(x_n - ax_{m-k} - bx_{n+l}) &= c(qx_{m-\sigma} + px_{n+\tau})\end{aligned}$$

and

$$\Delta^i(x_n + ax_{m-k} + bx_{n+l}) = c(qx_{m-\sigma} + px_{n+\tau}).$$

We omit the details here.

2. It can be extended to operation

$$Tx_{m,n} = (r\Delta_1^g + s\Delta_2^h + tI),$$

where $\Delta_1^g x_{m,n} = x_{m+g,n} - x_{m,n}$, $\Delta_2^h x_{m.n} = x_{m,n+h} - x_{m,n}$, $Ix_{m,n} = x_{m,n}$. $s, t > 0$, and $r + s - t > 0$. Here we omit the details.

3. Using the same technique and method, it is remarkable that our results are still valid for Eq.$(E_{i,}c)$ if $p = 0$ or $q = 0$ (but not $p = q = 0$) and provided that $ab = 0$. The details are omitted.

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