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OSCILLATION THEOREMS FOR NEUTRAL DELAY DIFFERENTIAL EQUATIONS OF ODD ORDER

BY

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Abstract

Sufficient conditions are established ensuring oscillation of all solutions of the *odd*-order neutral delay differential equation

$$(x(t)-px(t-\tau))^{(n)}+\sum_{i=1}^{m}p_ix(t-\sigma_i)=0,$$

where $p\in(-\infty,\infty)$, σ_i, p_i, $\tau\in(0,\infty)$. The present result improves some recent results.

1. Introduction

Establishing sufficient conditions in terms of coefficients and deviating arguments ensuring oscillation of all solutions of the *odd*-order neutral delay differential equation

$$(x(t)-p(t)x(t-\tau))^{(n)}+\sum_{i=1}^{m}p_i(t)x(t-\sigma_i)=0,\tag{1.1}$$

where τ, $\sigma_i\in(0,\infty)$, p, $p_i\in C(R,(0,\infty))$is a subject of many investigations.

By a solution of (1.1) on $[T_y,\infty)$, $T_y\geq 0$, we mean a function $y\in C\left([T_y-r,\infty),R\right)$ such that $y(t)-p(t)y(t-\tau)$ is n times continuously differentiable and (1.1) is satisfied identically for $t\geq T_y$ where $r=\max_{1\leq i\leq m}$

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$\{\tau, \sigma_i\}$. Such a solution of (1.1) is said to be *oscillatory* if it has arbitrarily large zeros; otherwise, it is called *nonoscillatory*.

From the review of the literature it appears that the case when $p(t) \equiv 0$, $n = 1$, all solutions of (1.1) are oscillatory if

$$\liminf_{t\to\infty} \sum_{i=1}^{m} p_i(t)\sigma_i > \frac{1}{e}. \tag{1.2}$$

Similarly, if $p(t) \equiv 0$, $n = 1$ and $m = 1$, all solutions of (1.1) are oscillatory if

$$\liminf_{t\to\infty} \int_{t-\sigma_1}^{t} p_1(s)ds > \frac{1}{e}. \tag{1.3}$$

Assuming $p_i(t) = p_i \in (O, \infty)$, (1.2) and (1.3) reduces respectively to

$$\sum_{i=1}^{m} p_i\sigma_i > \frac{1}{e}$$

and

$$p_1\sigma_1 > \frac{1}{e} \tag{1.4}$$

where the later is a sharp condition for oscillation. Indeed, it is established by an counter example that the former is not a necessary condition for oscillation.

In [13], authors have further extended the results for *odd*-order equations of the form

$$x^{(n)}(t) + p_1 x(t - \sigma_1) = 0$$

replacing (1.4) by

$$p_1^{1/n}\sigma_1 > \frac{n}{e}.$$

But a similar extension of it, for equations with several deviating arguments and general *odd*-order equations is yet to be found out. One of the results of this paper gives the required extension.

In a recent paper [19], Zang established that every solution of (1.1) oscillates if

$$n = 1 \tag{1.5}$$

$$0 < p < 1 \tag{1.6}$$

$$\tau,\ \sigma_i,\ p_i \in (0, \infty), \quad i = 1, 2, \ldots, m \tag{1.7}$$

$$\tau > \sigma_i, \quad i = 1, 2, \ldots, m \tag{1.8}$$

and

$$\tau\Big(\sum_{i=1}^{m} p_i\Big) > F(\lambda) = \frac{(1 - p\lambda)^2}{\lambda} \tag{1.9}$$

where λ is the unique real root of the equation

$$1 - py = \ln(y), \quad 1 < y < \frac{1}{p}. \tag{1.10}$$

The method adopted by him is not only complicated but also prevents n, p and τ to take values in other ranges.

Obviously, we prove that, if n is an odd natural number and (1.6) – (1.7) hold then every solution of (1.1) oscillates, if

$$\Big(\frac{\tau}{n}\Big)^{(n)} \sum_{i=1}^{m} p_i \exp(\tau_i) > \frac{(1 - p\lambda)^{n+1}}{\lambda} \tag{1.11}$$

where λ is the unique real root of

$$n(1 - p\lambda) = \ln(\lambda), \quad 1 < \lambda < \frac{1}{p} \tag{1.12}$$

and

$$\tau_i = \begin{cases} \frac{1}{\tau} \ln\Big[\frac{1}{(1-p)} \left(\frac{e}{n}\right)^n \sum_{i=1}^{m} p_i \sigma_i^n\Big](\sigma_i - \tau), & \sigma_i \geq \tau \\ \frac{1}{\tau} \ln\left(\frac{1}{p}\right) (\sigma_i - \tau), & \sigma_i < \tau \end{cases} \tag{1.13}$$

Indeed, if $n = 1$ and (1.8) holds then $\tau_i > 0$. Consequently, (1.11) reduces to

$$\tau\Big(\sum_{i=1}^{m} p_i \exp(\tau_i)\Big) > \frac{(1 - p\lambda)^2}{\lambda}, \tag{1.14}$$

which is a weaker condition than that of (1.9). Similar results are obtained when p does not satisfy (1.6). Examples are cited to demonstrate the generalization.

2. Odd Order Equations

Theorem 1. *If n is odd,* (1.6), (1.7) *and* (1.11) - (1.13) *are satisfied, then every solution of* (1.1) *oscillates.*

Proof. To the contrary, suppose that (1.1) admits a nonoscillatory solution. By Wang [17], the associated characteristic equation

$$G(\mu) = -\mu^n(1 - pe^{\mu\tau}) + \sum_{i=1}^{m} p_i e^{\mu\sigma_i} = 0 \tag{2.1}$$

admits a real root, say, μ_0. Since $G(\mu) > 0$ for $\mu \le 0$, it follows that $\mu_0 > 0$ and consequently,

$$\sum_{i=1}^{m} p_i \frac{e^{\mu_0\sigma_i}}{\mu_0^n} \;=\; (1 - pe^{\mu_0\tau})\,. \tag{2.2}$$

Since

$$\min_{x>0}\,(e^x/x^n) \;=\; (e/n)^n,$$

from (2.2) it follows that

$$\left(\frac{e}{n}\right)^n \sum_{i=1}^{m} p_i\sigma_i^n \le \sum_{i=1}^{m} p_i \frac{e^{\mu_0\sigma_i}}{\mu_0^n} = (1 - pe^{\mu_0\tau}) \le (1-p)e^{\mu_0\tau}.$$

Consequently,

$$\mu_0 > \frac{1}{\tau}\ln\Big[\frac{1}{(1-p)}\left(\frac{e}{n}\right)^n \sum_{i=1}^{m} p_i\sigma_i^n\Big]. \tag{2.3}$$

Again, the positiveness of the left hand side of (2.2) shows that

$$\mu_0 < \frac{1}{\tau}\ln\left(\frac{1}{p}\right). \tag{2.4}$$

From (2.3), (2.4) and the definition of τ_i in (1.13) gives that

$$\mu_0(\sigma_i - \tau) \geq \tau_i \quad i = 1, 2, \ldots, m. \tag{2.5}$$

Multiplying (2.2) throughout by $\exp(-\mu_0\tau)\left(\frac{\tau}{n}\right)^n$ then using (2.5) in the resulting equation it yields

$$\left(\frac{\tau}{n}\right)^n \sum_{i=1}^{m} p_i \exp(\tau_i) \leq \left(\frac{\tau}{n}\right)^n \frac{\mu_0^n(1 - pe^{\mu_0\tau})}{e^{\mu_0\tau}} \tag{2.6}$$

Putting $\xi = \exp(\mu_0\tau)$ in the right hand side of (2.6) and denoting it by $K(\xi)$ we see that

$$K(X) = \frac{1}{n^n}\frac{(\ln X)^n(1 - pX)}{X}. \tag{2.7}$$

It is easy to verify that $K'(X) = 0$ if and only if X satisfies (1.12). That is $X = \lambda$. Furthermore, $K''(\lambda) < 0$ shows that $X = \lambda$ is a point of local maximum of $K(X)$. Consequently,

$$K(\xi) \leq K(\lambda) \ = \ \frac{1}{n^n}\frac{(\ln \lambda)^n(1 - p\lambda)}{\lambda}. \tag{2.8}$$

Using (1.12) in (2.8) we obtain

$$K(\xi) \ \leq \ \frac{(1 - p\lambda)^{n+1}}{\lambda}. \tag{2.9}$$

Combining (2.6) and (2.9) the resultant inequality contradicts (1.11).

This completes the proof. □

Corollary 1. *If the hypotheses of Theorem 1 are satisfied rplacing* (1.11) *to* (1.13) *by*

$$\left(\frac{\tau}{n}\right)^n \sum_{i=1}^{m} p_i \exp(\tau_i) > \left(\frac{n+1}{ne}\right)^{n+1}$$

then every solution of (1.1) *oscillates.*

Proof. The proof follows directly from Theorem 1 if

$$F(\lambda) = \frac{(1 - p\lambda)^{n+1}}{\lambda} \leq \left(\frac{n+1}{ne}\right)^{n+1} \tag{2.10}$$

where λ satisfies (1.12). Indeed, from (1.12)

$$1 - p\lambda = \frac{1}{n} \ln \lambda$$

and subsequently, from (2.10) we get

$$F(\lambda) = \frac{(1-p\lambda)^{n+1}}{\lambda} = \frac{1}{\lambda}\left(\frac{\ln \lambda}{n}\right)^{n+1}. \tag{2.11}$$

Sinece

$$\max_{x>1} \frac{(\ln x)^{n+1}}{x} = \left(\frac{n+1}{e}\right)^{n+1}, \tag{2.12}$$

from (2.11) and (2.12) we get

$$F(\lambda) \leq \left(\frac{n+1}{ne}\right)^{n+1}.$$

This completes the proof. □

Corollary 2. *If the hypotheses of Theorem 1 are satisfied replacing* (1.11) *to* (1.13) *by*

$$\left(\frac{\tau}{n}\right)^n \sum_{i=1}^{m} p_i \exp(\tau_i) > (1-p)^{n+1}, \tag{2.13}$$

then every solution of (1.1) *oscillates.*

Proof. In Theorem 1 we observe that F is a decreasing function of λ $(1 < \lambda < 1/p)$. Hence

$$F(\lambda) < F(1) = (1-p)^{n+1} \tag{2.14}$$

Now the proof follows from (2.13), (2.14) and Theorem 1. □

Corollary 3. *If the hypotheses of Theorem 1 are satisfied replacing* (1.11)–(1.13) *by*

$$\left(\frac{\tau}{n}\right)^n \sum_{i=1}^{m} p_i \exp(\tau_i) > \frac{(1-p)^{n+1}}{(1+np)^n(1+n)} \tag{2.15}$$

then every solution of (1.1) *oscillates.*

Proof. In theorem 1,

$$F(\lambda)=\frac{(1-p\lambda)^{n+1}}{\lambda}$$

where λ satisfies

$$H(\lambda)=n(1-p\lambda)-\ln\lambda=0,\quad 1<\lambda<1/p.$$

Clearly, $F(x)$ is decreasing for $1<x<1/p$, $H'(y)<0$ and $H''(y)>0$ for $1<y<1/p$. Suppose that $\lambda=1+c$ for some $c>0$. Expanding H by Taylors theorem (for some $1\le\alpha\le 1+c$),

$$\begin{aligned}0=H(\lambda) &= H(1)+cH'(1)+\frac{c^2}{2!}H''(\alpha)\\ &> H(1)+cH'(1)\\ &= n(1-p)+c(-np-1). \qquad (2.16)\end{aligned}$$

Consequently, from (2.16) it yields

$$c>\frac{n(-p)}{1+np}$$

and

$$\lambda=1+c>\frac{(1+n)}{1+np}. \qquad (2.17)$$

Since F is decreasing

$$F(\lambda)>F\left(\frac{1+n}{1+np}\right)=\frac{(1-p)^{n+1}}{(1+np)^n(1+n)}. \qquad (2.18)$$

Now the proof follows from (2.15), (2.18) and Theorem 1. □

Theorem 2. *Suppose that* (1.7) *is satisfied, n is odd and $p=0$. If*

$$\left(\frac{e}{n}\right)^n\sum_{i=1}^m p_i\sigma_i^n>1 \qquad (2.19)$$

then every solution of (1.1) *oscillates.*

Proof. To the contrary, by Wang [17], the associated characteristic equation (2.1) admits a real root μ_0. Proceeding in the lines of Theorem 1 we obtain (2.2). Since

$$\min_{x>0} \frac{e^x}{x^n} = \left(\frac{e}{n}\right)^n \tag{2.20}$$

from (2.2), (2.20) and (2.19) the contradiction follows.

Note. When $n = 1$, (2.19) reduces to (1.3). □

Theorem 3. *Suppose that* (1.7) *and* (1.8) *hold,* $p > 1$ *and* n *is odd. If*

$$\left(\frac{\tau}{n}\right)^n \sum_{i=1}^{m} p_i \exp(\tau_i) > F(\lambda) = \frac{(1-p\lambda)^{n+1}}{\lambda}, \tag{2.21}$$

where λ *is the unique real root of*

$$n(1-py) = \ln(y), \quad \frac{1}{p} < y < 1$$

and

$$\tau_i = \Big(\frac{1}{(p-1)} \sum_{i=1}^{m} p_i\Big)(\tau - \sigma_i),$$

then every solution of (1.1) *oscillates.*

Proof. On the contrary and by Wang [17], the associated characteristic equation admits a real root $\mu_0 < 0$. That is,

$$\sum_{i=1}^{m} p_i e^{\mu_0 \sigma_i} = \mu_0^n \left(1 - p e^{\mu_0 \tau}\right) < (-\mu_0)^n (p-1) e^{\mu_0 \tau}. \tag{2.22}$$

Dividing (2.22) throughout by $e^{\mu_0 \tau}$ and using (1.8) we get

$$\mu_0 < -\Big[\frac{1}{(p-1)} \sum_{i=1}^{m} p_i\Big]^{1/n}. \tag{2.23}$$

From (2.23) it follows that

$$\mu_0(\sigma_i - \tau) > \tau_i. \tag{2.24}$$

Dividing the characteristic equation $G(\mu_0) = 0$ throughout by $\left(\frac{\tau}{n}\right)^n e^{\mu_0\tau}$, then using (2.24) and proceeding exactly in the lines of Theorem 1 we obtain a contradiction.

This completes the proof. □

Theorem 4. *Suppose that* (1.7) *is satisfied. If n is odd and $p = 1$, then every solution of* (1.1) *oscillates.*

Proof. If possible, let us suppose that (1.1) admits a nonoscillatory solution. Consequently, the characteristic equation

$$G(\mu) = -\mu^n (1 - e^{\mu\tau}) + \sum_{i=1}^{m} p_i e^{\mu\sigma_i} = 0$$

admits a real root, say, μ_0. Since $G(\mu) > 0$ for all $\mu \geq 0$ it follows that $\mu_0 < 0$. Hence

$$\mu_0^n (1 - e^{\mu_0\tau}) = \sum_{i=1}^{m} p_i e^{\mu_0\sigma_i}. \tag{2.25}$$

The right hand side of (2.25) is positive implies that

$$1 - e^{\mu_0\tau} < 0.$$

That is

$$e^{\mu_0\tau} > 1,$$

which is impossible. This completes the proof. □

Theorem 5. *Suppose that* (2.7) *is satisfied, n is odd, $p < 0$ and $\sigma_i \leq \tau$ $(i = 1, 2, \ldots, m)$. If*

$$\left(\frac{e}{n}\right)^n \sum_{i=1}^{m} p_i \sigma_i^n \geq (1-p) \exp\left\{\left(\frac{1}{(-p)} \sum_{i=1}^{m} p_i\right)^{1/n} \tau\right\}. \tag{2.26}$$

then every solution of (1.1) *oscillates.*

Proof. If possible, suppose that (1.1) admits a nonoscillatory solution.

By Wang [17], the associated characteristic equation

$$G(\mu) = -\mu^n (1 - pe^{\mu\tau}) + \sum_{i=1}^{m} p_i e^{\mu\sigma_i} = 0$$

admits a real root, say, μ_0. Clearly, $G(\mu) > 0$ for $\mu \le 0$ and hence $\mu_0 > 0$ and

$$G(\mu_0) = -\mu_0^n (1 - pe^{\mu_0\tau}) + \sum_{i=1}^{m} p_i e^{\mu_0\sigma_i} = 0.$$

Consequently,

$$\sum_{i=1}^{m} p_i e^{\mu_0\sigma_i} = \mu_0^n (1 - pe^{\mu_0\tau}). \tag{2.27}$$

Dividing (2.27) throughout by μ_0^n and using

$$\min_{x>0} \frac{e^x}{x^n} = \left(\frac{e}{n}\right)^n,$$

we obtain

$$\left(\frac{e}{n}\right)^n \sum_{i=1}^{m} p_i \sigma_i^n \le \sum_{i=1}^{m} p_i \frac{e^{\mu_0\sigma_i}}{\mu_0^n} = 1 - pe^{\mu_0\tau} < (1-p)e^{\mu_0\tau} \tag{2.28}$$

Further, dividing (2.27) throughout by $e^{\mu_0\tau}$ and rearranging the terms we obtain

$$\Big(\sum_{i=1}^{m} p_i\Big) > \sum_{i=1}^{m} p_i e^{\mu_0(\sigma_i - \tau)} = \mu_0^n \left(e^{-\mu_0\tau} - p\right) > \mu_0^n(-p).$$

Hence

$$\mu_0 < \Big[\frac{1}{-p} \sum_{i=1}^{m} p_i\Big]^{1/n}. \tag{2.29}$$

From (2.28) and (2.29) it follows that

$$\begin{aligned}\left(\frac{e}{n}\right)^n \Big(\sum_{i=1}^{m} p_i \sigma_i^n\Big) &< (1-p)(e^{\mu_0\tau}) \\ &< (1-p)\exp\Big\{\Big[\frac{1}{(-p)} \sum_{i=1}^{m} p_i\Big]^{1/n} \tau\Big\},\end{aligned}$$

which is a contradiction to (2.26). This completes the proof. □

Example 1. Consider the neutral delay differential equation

$$\left(x(t)-\frac{1}{2}x(t-2)\right)'+x(t-1)+3x(t-3)=0.$$

By Corollary 1 of this paper, all solutions of this equation are oscillatory. But Theorem 1 of Zhang [19] does not apply because (1.8) fails to hold.

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