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A CLASS OF ANALYTIC FUNCTIONS DEFINED BY THE CARLSON-SHAFFER OPERATOR

BY

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Abstract

The Carlson-Shaffer operator $L(a,c)f = \phi(a,c) * f$, where $f(z) = z + a_2z^2 + \cdots$ is analytic in the unit disk $E = \{z : |z| < 1\}$ and $\phi(a,c;z)$ is an incomplete beta function, is used to define the class $T(a,c)$. An analytic function f belongs to $T(a,c)$ if $L(a,c)f$ is starlike in E. The object of the present paper is to derive some properties of functions f in the class $T(a,c)$.

1. Introduction

Let A be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$. A function $f \in A$ is said to be starlike of order α in E if

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > \alpha \quad (z \in E)$$

for some $\alpha\,(0 \le \alpha < 1)$. We denote this class as $S^*(\alpha)$. Also we denote by

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$S^*(0) = S^*$. A function $f \in A$ is said to be convex (univalent) in E if

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad (z \in E).$$

We denote this class as K. Clearly $f \in K$ if and only if $zf' \in S^*$.

The class A is closed under the Hadamard product or convolution

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n,$$

where

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n.$$

Let $\phi(a, c)$ be defined by

$$\phi(a, c; z) = z + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \quad (z \in E;\ c \neq 0, -1, -2, \cdots), \tag{1.2}$$

where $(\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n-1)(n \in N = \{1, 2, 3, \cdots\})$. The function $\phi(a, c)$ is an incomplete beta function. Carlson and Shaffer [1] defined a linear operator on A by the convolution as follows:

$$L(a, c)f = \phi(a, c) * f \quad (f \in A;\ c \neq 0, -1, -2, \cdots). \tag{1.3}$$

$L(a, c)$ maps A into itself. $L(c, c)$ is the identity and if $a \neq 0, -1, -2, \cdots$, then $L(a, c)$ has a continuous inverse $L(c, a)$ and is an one-to-one mapping of A onto itself. $L(a, c)$ provides a convenient representation of differentiation and integration. If $g(z) = zf'(z)$, then $g = L(2, 1)f$ and $f = L(1, 2)g$.

By using $L(a, c)$ we now introduce the subclass of A as follows.

Definition. A function $f \in A$ is said to be in the class $T(a, c)$ if

$$L(a, c)f \in S^* \quad (c \neq 0, -1, -2, \cdots). \tag{1.4}$$

Miller and Mocanu [4, Theorem 2] have proved that if $c(c \neq 0)$ and a

are real and satisfy

$$a > N(c) = \begin{cases} |c| + \frac{1}{2} & (|c| \geq \frac{1}{3}) \\ \frac{3}{2}c^2 + \frac{2}{3} & (|c| \leq \frac{1}{3}), \end{cases} \tag{1.5}$$

then the function

$$\Phi(c, a; z) = 1 + \sum_{n=1}^{\infty} \frac{(c)_n}{(a)_n} \frac{z^n}{n!} \tag{1.6}$$

is convex in E.

In [5] Noor gave the following.

Lemma A.([5, Lemma 2.1]) *If* $c(c \neq 0)$ *and* a *are real and satisfy* (1.5), *then* $\phi(c, a; z)$ *is convex in* E.

Theorem A.([5, Theorem 3.2]) *Let* $f \in T(a, c)$, *where* a *and* c *satisfy the conditions of Lemma A. Then* $f \in S^*$ *and hence* f *is univalent in* E.

Theorem B.([5, Theorem 3.3]) *Let* $f \in T(a, c)$ *with* a *and* c *satisfying* (1.5). *Then the disk* E *is mapped onto a domain that contains the disk*

$$D = \left\{ w : |w| < \frac{2(c + a)}{a} \right\}. \tag{1.7}$$

Theorem C. ([5, Theorem 3.4]) *Let* $a(a \neq 0), c$ *and* d *be real and* $c > N(d)$, *where* $N(d)$ *is defined as in* (1.5). *Then*

$$T(a, d) \subset T(a, c). \tag{1.8}$$

Theorem D.([5, Theorem 3.5]) *Let* $a(a \neq 0)$ *and* c *be real and satisfy* $c > N(a)$, *where* $N(a)$ *is defined in the similar way of* (1.5). *Let* ψ *be a convex function in* E. *If* $f \in T(a, c)$ *then* $\psi * f \in T(a, c)$.

Theorem E.([5, Theorem 3.7]) *Let* $f \in T(a, c)$ *and let* F *be defined by*

$$F(z) = \frac{\beta + 1}{z^\beta} \int_0^z t^{\beta - 1} f(t) dt \quad (\beta \in N). \tag{1.9}$$

Then

$$\operatorname{Re}\left\{\frac{z(L(a,c)F(z))'}{L(a,c)F(z)}\right\} > \alpha \quad (z \in E), \tag{1.10}$$

where

$$\alpha = \frac{-(2\beta+1)+\sqrt{4\beta^2+4\beta+9}}{4}. \tag{1.11}$$

However, we find that Lemma A is not always true for $c(c \neq 0$ real) and a satisfying (1.5).

Counterexample. Let $a = 1$ and $\frac{1}{3} \leq c < \frac{1}{2}$. Then $a > N(c) = c + \frac{1}{2}$ and

$$\phi(c,1;z) = z + \sum_{n=1}^{\infty} \frac{(c)_n}{n!} z^{n+1} = \frac{z}{(1-z)^c}. \tag{1.12}$$

For $z = \rho e^{i\theta} (0 < \rho < 1)$ and $1 - \frac{c}{2} < \cos\theta < 1 (0 < \theta < \frac{\pi}{2})$, we have

$$1 + \frac{z\phi''(c,1;z)}{\phi'(c,1;z)} = 1 + \frac{(c+1)\rho e^{i\theta}}{1-\rho e^{i\theta}} + \frac{(c-1)\rho e^{i\theta}}{1+(c-1)\rho e^{i\theta}}.$$

Hence

$$\begin{aligned}
\lim_{\rho\to 1} \operatorname{Re}\left\{1 + \frac{\rho e^{i\theta}\phi''(c,1;\rho e^{i\theta})}{\phi'(c,1;\rho e^{i\theta})}\right\} &= \frac{1-c}{2} + (c-1)\operatorname{Re}\left\{\frac{e^{i\theta}}{1+(c-1)e^{i\theta}}\right\} \\
&= \frac{1-c}{2} + (c-1)\frac{c-1+\cos\theta}{|1+(c-1)e^{i\theta}|^2} \\
&= -\frac{(1-c)(2-c)(2\cos\theta+c-2)}{2|1+(c-1)e^{i\theta}|^2} \\
&< 0,
\end{aligned} \tag{1.13}$$

which implies that the function $\phi(c,1;z)(\frac{1}{3} \leq c < \frac{1}{2})$ is not convex in E.

In view of $\frac{z}{(1-z)^2} \in S^*$, we see that

$$f_c(z) = \phi(c,1;z) * \frac{z}{(1-z)^2} \in T(1,c).$$

But $f_c(z) = z\phi'(c,1;z)(\frac{1}{3} \leq c < \frac{1}{2})$ is not starlike in E. Thus the counterexample shows that Theorem A is not true when $a = 1$ and $\frac{1}{3} \leq c < \frac{1}{2}$. In [5], the proof of Theorem B used Lemma A, and so its validity is not justified.

Similarly the proof of Theorem C in [5] is not valid. Also the result from Theorem E is not sharp.

In this paper we discuss similar problems and obtain useful results for the class $T(a, c)$.

2. Preliminary Results

To prove our results, we need the following lemmas.

Lemma 2.1.([4, Corollary 4.1]) *If a, b and c are real and satisfy $-1 \leq a \leq 1$, $b \geq 0$ and $c > 1 + \max\{2 + |a + b - 2|, 1 - (a - 1)(b - 1)\}$, then*

$$zF(a, b; c; z) \in S^*, \tag{2.1}$$

where

$$F(a, b; c; z) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \tag{2.2}$$

is the Gaussian hypergeometric function.

Applying Lemma 2.1, we derive the following result.

Lemma 2.2. *If a and c are real and satisfy $-1 \leq a \leq 1$ and $c > 3 + |a|$, then $\phi(a, c; z)$ defined by* (1.2)*is convex in E.*

Proof. From (1.2) we have

$$\begin{aligned} z\phi'(a, c; z) &= z + \sum_{n=1}^{\infty} \frac{(n+1)(a)_n}{(c)_n} z^{n+1} \\ &= z + \sum_{n=1}^{\infty} \frac{(a)_n (2)_n}{(c)_n} \frac{z^{n+1}}{n!} \\ &= zF(a, 2; c; z). \end{aligned} \tag{2.3}$$

Since $-1 \leq a \leq 1$ and $c > 3 + |a|$, it follows from (2.3) and Lemma 2.1 (with $b = 2$) that $z\phi'(a, c; z)$ is starlike in E, which leads to $\phi(a, c) \in K$. $\square$

Lemma 2.3.([6]) *If* $f \in K$ *and* $g \in S^*$, *then* $f * g \in S^*$.

Let f and g be analytic in E. The function f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if g is univalent in $E, f(0) = g(0)$ and $f(E) \subset g(E)$.

Lemma 2.4.([2]) *Let* $\alpha(\alpha \neq 0)$ *and* β *be complex numbers and let* p *and* h *be analytic in* E *with* $p(0) = h(0)$. *If* $Q(z) = \alpha h(z) + \beta$ *is convex and* $\operatorname{Re} Q(z) > 0$ *in* E, *then*

$$p(z) + \frac{zp'(z)}{\alpha p(z) + \beta} \prec h(z)$$

implies that $p(z) \prec h(z)$.

Lemma 2.5.([3]) *Let* $\alpha(\alpha \neq 0)$ *and* β *be complex numbers and let* h *be analytic and univalent in* E *and* $Q(z) = \alpha h(z) + \beta$. *Let* p *be analytic in* E *and satisfy*

$$p(z) + \frac{zp'(z)}{\alpha p(z) + \beta} \prec h(z) \quad (p(0) = h(0)). \tag{2.4}$$

If

(i) $\operatorname{Re} Q(z) > 0$ *for* $z \in E$, *and*
(ii) Q *and* $\frac{1}{Q}$ *are convex in* E,

then the solution of the differential equation

$$q(z) + \frac{zq'(z)}{\alpha q(z) + \beta} = h(z) \quad (q(0) = h(0)) \tag{2.5}$$

is univalent in E *and is the best dominant of* (2.4).

Lemma 2.6.([7]) *Let* μ *be a positive measure on the unit interval* $[0, 1]$. *Let* $g(t, z)$*be a function analytic in* E *for each* $t \in [0, 1]$ *and integrable in* t *for each* $z \in E$ *and for almost all* $t \in [0, 1]$, *and suppose that* $\operatorname{Re} g(t, z) > 0$ *in* E, $g(t, -\rho)$ *is real and*

$$\operatorname{Re} \frac{1}{g(t, z)} \geq \frac{1}{g(t, -\rho)} \quad (|z| \leq \rho < 1;\ t \in [0, 1]).$$

If $g(z)=\int_0^1 g(t,z)d\mu(t)$, *then*

$$\mathrm{Re}\frac{1}{g(z)} \geq \frac{1}{g(-\rho)} \quad (|z|\leq\rho). \tag{2.6}$$

3. The Class $T(a,c)$

Theorem 3.1. *Let* a *and* c *be real and satisfy*

$$c\neq 0, -1<c\leq 1 \ \ \textit{and} \ \ a>3+|c|. \tag{3.1}$$

Then $T(a,c)\subset S^*$.

Proof. If $f\in T(a,c)$, then $L(a,c)f=\phi(a,c)*f\in S^*$. Since a and c satisfy (3.1), we have from Lemma 2.2 that $\phi(c,a)\in K$. Therefore an application of Lemma 2.3 leads to

$$f=\phi(c,a)*(\phi(a,c)*f)\in S^*.$$

This completes the proof of the theorem. □

Theorem 3.2. *Let* a *and* c *satisfy* (3.1). *If* $f\in T(a,c)$, *then* $f(E)$ *contains the disk*

$$D=\left\{w:|w|<\frac{a}{2(|c|+a)}\right\}. \tag{3.2}$$

Proof. Let

$$f(z)=z+\sum_{n=1}^{\infty}a_{n+1}z^{n+1}\in T(a,c),$$

where a and c satisfy (3.1), and $w_0 (w_0\neq 0)$ be any complex number such that $f(z)\neq w_0$ for $z\in E$. Then the function

$$g(z)=\frac{w_0 f(z)}{w_0-f(z)}=z+\left(a_2+\frac{1}{w_0}\right)z^2+\cdots$$

is analytic and univalent in E by Theorem 3.1, and hence

$$\frac{1}{|w_0|}-|a_2|\leq\left|a_2+\frac{1}{w_0}\right|\leq 2. \tag{3.3}$$

Since

$$L(a,c)f(z) = z + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} a_{n+1} z^{n+1} \in S^*,$$

we have $|\frac{aa_2}{c}| \leq 2$, and it follows from (3.3) that

$$|w_0| \geq \frac{1}{2+|a_2|} \geq \frac{a}{2(|c|+a)}. \tag{3.4}$$

This gives the desired result. □

Theorem 3.3. *Let a, c and d be real. If*

$$d \neq 0,\ -1 < d \leq 1 \quad and \quad c > 3 + |d|, \tag{3.5}$$

then $T(a,d) \subset T(a,c)$.

Proof. If $f \in T(a,d)$, then $L(a,d)f = \phi(a,d) * f \in S^*$. Since c and d satisfy (3.5), $\phi(d,c) \in K$ by Lemma 2.2. Hence it follows from Lemma 2.3 that

$$L(a,c)f = \phi(a,c) * f = (\phi(a,d) * \phi(d,c)) * f$$

$$= \phi(d,c) * (\phi(a,d) * f) \in S^*,$$

that is, $f \in T(a,c)$. The proof is complete. □

Theorem 3.4. *Let* $f \in T(a,c)$ *and* $\psi \in K$. *Then* $\psi * f \in T(a,c)$.

Proof. Since $L(a,c)f \in S^*$ and $\psi \in K$, it follows from Lemma 2.3 that

$$L(a,c)(\psi * f) = \psi * L(a,c)f \in S^*.$$

Hence $\psi * f \in T(a,c)$. □

In view of Theorem 3.4, we see that the assumption "$a(a \neq 0)$ and c are real and satisfy $c > N(a)$, where $N(a)$ is defined in the similar way of (1.5)" in Theorem D is redundant.

Theorem 3.5. *If* $a \geq 1$, *then*

$$T(a+1,c) \subset T(a,c). \tag{3.6}$$

Proof. It is known that for $f \in A$,

$$z(L(a,c)f(z))' = aL(a+1,c)f(z) - (a-1)L(a,c)f(z). \tag{3.7}$$

Let us put

$$g(z) = \frac{z(L(a,c)f(z))'}{L(a,c)f(z)}. \tag{3.8}$$

Then $g(0) = 1$ and from (3.7) and (3.8) we get

$$\frac{aL(a+1,c)f(z)}{L(a,c)f(z)} = g(z) + a - 1. \tag{3.9}$$

Differentiating both sides of (3.9) logarithmically and using (3.8) we have

$$\frac{z(L(a+1,c)f(z))'}{L(a+1,c)f(z)} = g(z) + \frac{zg'(z)}{g(z)+a-1}. \tag{3.10}$$

If $f \in T(a+1,c)$, then (3.10) leads to

$$g(z) + \frac{zg'(z)}{g(z)+a-1} \prec \frac{1+z}{1-z}. \tag{3.11}$$

Since $Q(z) = \frac{1+z}{1-z} + a - 1$ is convex in E and $\operatorname{Re} Q(z) > a-1 \geq 0 (z \in E)$, it follows from (3.11) and Lemma 2.4 that $g(z) \prec \frac{1+z}{1-z}$, which is equivalent to $f \in T(a,c)$. This proves (3.6). □

Theorem 3.6. *Let $f \in T(a,c)$ and*

$$F(z) = \frac{\beta+1}{z^\beta}\int_0^z t^{\beta-1} f(t)dt \quad (\beta > 0). \tag{3.12}$$

Then $L(a,c)F \in S^*(\sigma(\beta))$, where

$$\sigma(\beta) = \left(4\int_0^1 \frac{t^\beta}{(1+t)^2}dt\right)^{-1} - \beta. \tag{3.13}$$

The result is sharp, that is, the order $\sigma(\beta)$ cannot be increased.

Proof. From (3.12) we have $F \in A$ and

$$\beta L(a,c)F(z) + z(L(a,c)F(z))' = (\beta+1)L(a,c)f(z)$$

or

$$\frac{z(L(a,c)F(z))'}{L(a,c)F(z)}+\beta=(\beta+1)\frac{L(a,c)f(z)}{L(a,c)F(z)}. \tag{3.14}$$

Differentiating both sides of (3.14) logarithmically we deduce that

$$p(z)+\frac{zp'(z)}{p(z)+\beta}=\frac{z(L(a,c)f(z))'}{L(a,c)f(z)}, \tag{3.15}$$

where

$$p(z)=\frac{z(L(a,c)F(z))'}{L(a,c)F(z)}. \tag{3.16}$$

Since $f\in T(a,c)$, it follows from (3.15) that

$$p(z)+\frac{zp'(z)}{p(z)+\beta}\prec\frac{1+z}{1-z}\quad (p(0)=1). \tag{3.17}$$

Taking $\alpha=1,\beta>0,h(z)=\frac{1+z}{1-z}$ and $Q(z)=\frac{1+z}{1-z}+\beta$, it is clear that the conditions (i) and (ii) in Lemma 2.5 are satisfied. Thus, by Lemma 2.5, the differential equation

$$q(z)+\frac{zq'(z)}{q(z)+\beta}=\frac{1+z}{1-z}\quad (q(0)=1) \tag{3.18}$$

has a univalent solution $q(z)$,

$$p(z)\prec q(z)\prec\frac{1+z}{1-z}, \tag{3.19}$$

and $q(z)$ is the best dominant of (3.17). It is easy to verify that the solution $q(z)$ of (3.18) is

$$\begin{aligned} q(z)&=\frac{z^{\beta+1}}{(1-z)^2\int_0^z\frac{u^\beta}{(1-u)^2}du}-\beta\\ &=\left((1-z)^2\int_0^1\frac{t^\beta}{(1-tz)^2}dt\right)^{-1}-\beta. \end{aligned} \tag{3.20}$$

It is well known that for $c_1>b_1>0$ and $z\in E$, the Gaussian hypergeometric function defined by (2.2) satisfies

$$F(a_1,b_1;c_1;z)=\frac{\Gamma(c_1)}{\Gamma(b_1)\Gamma(c_1-b_1)}\int_0^1\frac{t^{b_1-1}(1-t)^{c_1-b_1-1}}{(1-tz)^{a_1}}dt \tag{3.21}$$

and

$$F(a_1,b_1;c_1;z)=F(b_1,a_1;c_1;z)=(1-z)^{c_1-a_1-b_1}F(c_1-b_1,c_1-a_1;c_1;z). \tag{3.22}$$

By using (3.21) and (3.22), $q(z)$ given by (3.20) can be expressed as

$$\begin{aligned} q(z) &= \frac{\beta+1}{(1-z)^2F(2,\beta+1;\beta+2;z)}-\beta \\ &= \frac{\beta+1}{(1-z)F(1,\beta;\beta+2;z)}-\beta. \end{aligned} \tag{3.23}$$

From (3.21) and (3.23) we have

$$q(z)=\frac{1}{g(z)}-\beta, \tag{3.24}$$

where

$$g(z)=\int_0^1 g(t,z)d\mu(t),$$

$$g(t,z)=\frac{1}{\beta+1}\left(\frac{1-z}{1-tz}\right),\quad d\mu(t)=\beta(\beta+1)t^{\beta-1}(1-t)dt\quad(\beta>0). \tag{3.25}$$

Note that for $|z|\le\rho<1$ and $0\le t\le 1$,

$$\mathrm{Re}\frac{1}{g(t,z)}\ge(\beta+1)\left(\frac{1+t\rho}{1+\rho}\right)=\frac{1}{g(t,-\rho)}>0.$$

Now applying Lemma 2.6, it follows from (3.24) and (3.25) that

$$\mathrm{Re}\,q(z)\ge\frac{1}{g(-\rho)}-\beta=\frac{\beta+1}{\int_0^1\frac{1+\rho}{1+t\rho}d\mu(t)}-\beta\quad(|z|\le\rho),$$

which leads to

$$\inf_{z\in E}\mathrm{Re}\,q(z)=\frac{1}{g(-1)}-\beta=q(-1). \tag{3.26}$$

Since $q(z)$ is the best dominant of (3.17), from (3.16)-(3.20) and (3.26) we conclude that

$$\mathrm{Re}\frac{z(L(a,c)F(z))'}{L(a,c)F(z)} > q(-1) = \left(4\int_0^1 \frac{t^\beta}{(1+t)^2}dt\right)^{-1} - \beta = \sigma(\beta)$$

and the bound $\sigma(\beta)$ cannot be increased. The proof is now complete. □

We note that Theorem 3.6 is better than Theorem E by Noor.

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