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MAGNETOHYDRODYNAMIC APPROACHES TO MEASURE-VALUED SOLUTIONS OF THE TWO-DIMENSIONAL STATIONARY EULER EQUATIONS

BY

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Abstract

It is proved that Galerkin approximations of nonstationary magnetohydrodynamic equations with artificial terms generate measure-valued solutions to the stationary Euler equations in two dimensions.

1. Introduction

Let us introduce a nonstationary system of equations for a viscous and perfectly conductive magneto-fluid in a domain Ω ($\subset \mathbf{R}^2$) with artificial terms:

$$\left.\begin{aligned} &\boldsymbol{v}_t + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \alpha\omega J\boldsymbol{v}_t - \beta\Delta\boldsymbol{v}_t \\ &\quad = -\nabla q + \boldsymbol{B}\cdot\nabla\boldsymbol{B} - \nabla(|\boldsymbol{B}|^2/2) + \gamma\Delta\boldsymbol{v}, \\ &\boldsymbol{B}_t = J\nabla((\boldsymbol{v}+\alpha\boldsymbol{v}_t)\cdot(J\boldsymbol{B})), \\ &\nabla\cdot\boldsymbol{v} = \nabla\cdot\boldsymbol{B} = 0. \end{aligned}\right\} \tag{1}$$

Here $(\boldsymbol{v}, \boldsymbol{B}, q) : \Omega \times \{t > 0\} \to \mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R}$ is the triple of the velocity, magnetic field and pressure of the magneto-fluid with unit density. The operator J represents $\pi/2$ counterclockwise rotation around the origin on $\mathbf{R}^2$-plane. It corresponds to the three-dimensional vector product symbol $\times$ (see §2) and $\omega = (J\nabla)\cdot\boldsymbol{v}$ means vorticity. The constants α and β denote

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the magnitudes of the artificial terms, while the constant γ denotes viscosity. The second equation of (1) is rewritten in the form

$$\Psi_t + (\boldsymbol{v} + \alpha \boldsymbol{v}_t) \cdot \nabla \Psi = 0, \tag{2}$$

where Ψ is a flux function of $\boldsymbol{B}$, that is, $\boldsymbol{B} = -J\nabla\Psi = -J\nabla(\Psi + c(t))$ with an arbitrary function $c(t)$. On the boundary Γ, we assume that

$$\boldsymbol{v}|_\Gamma = \boldsymbol{0}, \qquad \boldsymbol{B} \cdot \boldsymbol{n}|_\Gamma = 0 \tag{3}$$

are satisfied, where $\boldsymbol{n}$ is the unit outward normal vector on Γ.

The reason for introducing (1) is that the author is interested in justifying Moffatt's magnetic relaxation approach to stationary Euler flows somehow in a rigorous sense. If we set $\alpha = \beta = 0$ and $\gamma > 0$, then (1) corresponds to the two-dimensional version of Moffatt's system in [6]. In [7, §5], he asserted its relaxation to an equilibrium

$$\boldsymbol{B} \cdot \nabla \boldsymbol{B} = \nabla(q + |\boldsymbol{B}|^2/2), \qquad \nabla \cdot \boldsymbol{B} = 0, \qquad \boldsymbol{v} = \boldsymbol{B}_t = \boldsymbol{0}$$

at $t = \infty$, and moreover, the "topological accessibility" of $\boldsymbol{B}|_{t=\infty}$ from the initial data $\boldsymbol{B}|_{t=0}$. Namely, $\boldsymbol{u} = \boldsymbol{B}|_{t=\infty}$ with $p = -(q + |\boldsymbol{B}|^2/2)|_{t=\infty}$ is a solution to the two-dimensional stationary Euler equations for an inviscid incompressible fluid:

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p, \qquad \nabla \cdot \boldsymbol{u} = 0, \tag{4}$$

and the topology of field lines of $\boldsymbol{u} = \boldsymbol{B}|_{t=\infty}$ (or streamlines) is similar to that of $\boldsymbol{B}|_{t=0}$. This is remarkable because his approach to (4) with nonvanishing vorticity is completely different from usual variational approaches (e.g., [13]) and others ([12], [15]). However, to justify his assertion in its entirety in a rigorous sense, we need the existence of a temporally global solution to his system in the sense of distribution in space, at the weakest, which seems difficult to prove. In addition, even if it is proved and the decay $\boldsymbol{v} \to \boldsymbol{0}$ (as $t \to \infty$) in $\boldsymbol{L}^2(\Omega) = (L^2(\Omega))^2$ is obtained, the rigorous derivation of $\boldsymbol{v}_t \to \boldsymbol{0}$ in $\boldsymbol{L}^2(\Omega)$ does not seem easy.

If we set $\alpha > 0$ and $\beta = \gamma = 0$, then (1) corresponds to the system of Vallis et al. [14, §5]. They asserted its relaxation to a steady state as $t \to \infty$. However, their assertion is also difficult to prove rigorously.

Two essential features of Moffatt's magnetic relaxation theory are the monotone decrease of energy and the frozen-in phenomenon of magnetic field lines to the fluid. Although (1) with $\alpha > 0$ and $\beta > 0$ may seem too artificial, it retains these two features. Indeed, the monotone decrease of energy can be shown as (16) below, and (2) implies that magnetic field lines are advected with the pseudo-velocity $\boldsymbol{v} + \alpha \boldsymbol{v}_t$.

Our main aim in this paper is to prove that a sequence of solutions to a Galerkin approximation of (1) with (3) for $\alpha > 0$, $\beta \geq 0$ and $\gamma > 0$ converges to a measure-valued solution to (4). It is done in §3 by letting both t and the number of basis functions go to infinity simultaneously. By this simultaneous limiting, we can evade the problem of temporally global solvability of (1). The positiveness of α enables us to have (19) below. If (19), or weaklier (28) can be proved for $\alpha = 0$, then Theorem 1 will remain valid for $\alpha = 0$.

In §4, we prove three lemmas on the measure-valued solution. Particularly, Lemmas 2 and 3 are obtained when $\beta > 0$. Their validity for $\beta = 0$ is open. If we can prove Theorem 1 for $\alpha = 0$, that is, (28) for $\alpha = 0$, then all of Lemmas 1–3 will be valid for $\alpha = \beta = 0$. In other words, the only reason for adding the artificial terms in (1) is that we do not know the validity of (28) for $\alpha = 0$.

Solvability in the sense of measure is a generalization of usual weak solvability. It was first used in the discussion of equations of conservation laws by DiPerna [1]. Generalizing his results, DiPerna and Majda [2], [3] applied the concept of a measure-valued solution to the nonstationary Euler equations. They justified its utility for the description of fluid motion accompanied with singularity such as vortex sheets (see also [5]).

For the three-dimensional stationary Euler equations, the author [9] deduced a result analogous to Theorem 1 by considering a Galerkin approximation of a three-dimensional version of (1) with $\beta = 0$. To prove the nontriviality of the obtained measure-valued solution, he used magnetic helicity, that is, $(L^2)^3$-product of a magnetic field and its vector potential. In the two-dimensional case, however, magnetic helicity is not useful because it is always equal to zero. Instead, we can use (2).

It is interesting that the magnetic field $\boldsymbol{B}$ in (1) tends to the fluid velocity $\boldsymbol{u}$ in (4). However, it may seem indirect to consider five quantities $\boldsymbol{B} = (B_1, B_2)$, $\boldsymbol{v} = (v_1, v_2)$ and q in order to obtain three quantities $\boldsymbol{u} = (u_1, u_2)$

and p. For this reason, the author proposed a simpler equation in three-dimensional context in [11] and its two-dimensional version is

$$\boldsymbol{B}_t = J\nabla(P_\sigma(\zeta J\boldsymbol{B}) \cdot (J\boldsymbol{B})). \tag{5}$$

Here $\zeta = (J\nabla) \cdot \boldsymbol{B}$ and P_σ is a solenoidal projection operator, that is, $P_\sigma \boldsymbol{f} = \boldsymbol{f} + \nabla Q$ with Q determined by $\Delta Q = -\nabla \cdot \boldsymbol{f}$ and $(\boldsymbol{f} + \nabla Q) \cdot \boldsymbol{n}|_\Gamma = 0$. The equation (5) is similar to the second equation of (1), in other words, $\boldsymbol{B}$ is advected with the pseudo-velocity $P_\sigma(\zeta J\boldsymbol{B})$. In §5, we prove that a measure-valued solution to (4) is generated from a Galerkin approximation of (5).

Lastly, it should be noted that solutions to (4) are also generated by considering nonstationary *non*-magnetohydrodynamic equations ([8], [10]).

2. Preliminaries

First, to make clear the relation between two-dimensional vector calculations with J and three-dimensional ones with the vector product symbol $\times$, we review the following equalities for vector functions $\boldsymbol{f} = (f_1(x,y), f_2(x,y))$, $\boldsymbol{g} = (g_1(x,y), g_2(x,y))$, a scalar function $h(x,y)$ and $\nabla = (\partial/\partial x,\, \partial/\partial y)$:

$$\begin{gathered}
(\boldsymbol{f}\,;0) \times (\boldsymbol{g}\,;0) = (\mathbf{0}\,;(J\boldsymbol{f}) \cdot \boldsymbol{g}) = (\mathbf{0}\,;-\boldsymbol{f} \cdot (J\boldsymbol{g})),\\
(\mathbf{0}\,;h) \times (\boldsymbol{f}\,;0) = (hJ\boldsymbol{f}\,;0),\\
\nabla_3 \times (\boldsymbol{f}\,;0) = (\mathbf{0}\,;(J\nabla) \cdot \boldsymbol{f}) = (\mathbf{0}\,;-\nabla \cdot (J\boldsymbol{f})),\\
\nabla_3 \times (\mathbf{0}\,;h) = (-J\nabla h\,;0).
\end{gathered}$$

Here $(\cdot\,;\cdot)$ is defined by $(\boldsymbol{f}\,;h) = (f_1, f_2, h)$, and $\nabla_3 = (\partial/\partial x,\, \partial/\partial y,\, 0)$. In particular, it is convenient to note that the above equalities yield

$$(J\nabla(\boldsymbol{f} \cdot (J\boldsymbol{g}))\,;0) = \nabla_3 \times ((\boldsymbol{f}\,;0) \times (\boldsymbol{g}\,;0)) = (\boldsymbol{g} \cdot \nabla\boldsymbol{f} - \boldsymbol{f} \cdot \nabla\boldsymbol{g}\,;0) \tag{6}$$

if $\boldsymbol{f}$ and $\boldsymbol{g}$ are divergence-free.

Next, let us introduce our notation. We assume that Ω is an arbitrary, open, bounded and simply connected domain in $\mathbf{R}^2$ and its boundary Γ is sufficiently smooth. By $C_0^\infty(\Omega)$, we denote the set of all infinite-times continuously differentiable functions on Ω whose supports are inside Ω. By

$\boldsymbol{C}_{0,\sigma}^{\infty}(\Omega)$, we denote the set of all two-dimensional divergence-free functions whose components belong to $C_0^{\infty}(\Omega)$. The product $((\cdot,\cdot))$ means

$$((f,g)) = \int_{\Omega} f(x,y)\, g(x,y)\, dxdy \quad \text{or} \quad ((\boldsymbol{f},\boldsymbol{g})) = \int_{\Omega} \boldsymbol{f}(x,y)\cdot \boldsymbol{g}(x,y)\, dxdy$$

for scalar functions f, g or vector functions $\boldsymbol{f}$, $\boldsymbol{g}$. The norm $\|\cdot\|$ is defined by

$$\|f\| = ((f,f))^{1/2} \quad \text{or} \quad \|\boldsymbol{f}\| = ((\boldsymbol{f},\boldsymbol{f}))^{1/2}.$$

The spaces $\boldsymbol{H}_\sigma$ and $\boldsymbol{H}_\sigma^1$ represent the closures of $\boldsymbol{C}_{0,\sigma}^{\infty}(\Omega)$ with respect to $\|\cdot\|$ and $\|\cdot\|_1$, respectively. Here $\|\cdot\|_j$ ($j \in \mathbf{N}$) is the norm in $\boldsymbol{W}_2^j(\Omega) = (W_2^j(\Omega))^2$ and $W_2^j(\Omega)$ is the Sobolev space of the j-th order.

By $M(G)$, we denote the space of Radon measures on $G = \bar{\Omega}, S$ or $\mathbf{R}^2$, where $\bar{\Omega} = \Omega \cup \Gamma$ and S is the unit circle. The total variation of $\mu \in M(G)$ is defined by the supremum of $|\int_G \chi\, d\mu|$ for all $\chi \in C(\bar{\Omega}), C(S)$ or $C_0(\mathbf{R}^2)$ such that $|\chi| \le 1$. Here $C(G)$ is the set of all continuous functions on G and $C_0(\mathbf{R}^2)$ is the subset of $C(\mathbf{R}^2)$ whose elements have compact supports. The subspace of nonnegative measures in $M(G)$ is denoted by $M^+(G)$. The subspace of measures with unit total variation in $M^+(G)$ is represented by $\mathrm{Prob}M(G)$.

Let $\boldsymbol{w}_1^k \in \boldsymbol{H}_\sigma^1 \cap \boldsymbol{W}_2^l(\Omega)$ ($k = 1,2,3,\ldots$) with a large integer l be eigenfunctions for the problem

$$\Delta \boldsymbol{w} = \nabla s - \lambda \boldsymbol{w}, \qquad \nabla\cdot \boldsymbol{w} = 0, \qquad \boldsymbol{w}|_\Gamma = \boldsymbol{0}, \qquad \|\boldsymbol{w}\| = 1$$

with some function $s \in W_2^{l-1}(\Omega)$ and eigenvalues $\lambda = \lambda_{1,k}$ such that $0 < \lambda_{1,k} \le \lambda_{1,k+1}$ and $\lim_{k\to\infty} \lambda_{1,k} = \infty$ (the two-dimensional version of the result in [4, Chapter 2, §4]). Then $\{\boldsymbol{w}_1^k\}_{k=1}^{\infty}$ is a complete orthonormal system in $\boldsymbol{H}_\sigma$.

By $\psi^{(k)} \in W_2^l(\Omega)$ ($k = 1,2,3,\ldots$) with a large l, we define eigenfunctions for $\Delta\psi = -\lambda\psi$ and $\psi|_\Gamma = 0$ with eigenvalues $\lambda = \lambda_{2,k}$ such that $0 < \lambda_{2,k} \le \lambda_{2,k+1}$ and $\lim_{k\to\infty} \lambda_{2,k} = \infty$. It is well known that $\lambda_{2,1} < \lambda_{2,2}$ and we can suppose that $\psi^{(1)} > 0$ everywhere in Ω. The set $\{\psi^{(k)}\}_{k=1}^{\infty}$ constitutes a complete orthogonal system in $L^2(\Omega)$. Set

$$\boldsymbol{w}_2^k = \frac{-J\nabla\psi^{(k)}}{\|\nabla\psi^{(k)}\|} = \frac{-J\nabla\psi^{(k)}}{\lambda_{2,k}^{1/2}\|\psi^{(k)}\|}. \tag{7}$$

Then $\{\boldsymbol{w}_2^k\}_{k=1}^{\infty}$ is a complete orthonormal system in $\boldsymbol{H}_\sigma$ such that $\boldsymbol{w}_2^k \in \boldsymbol{H}_\sigma \cap \boldsymbol{W}_2^{l-1}(\Omega)$. It should be noted that $\boldsymbol{u} = c\,\boldsymbol{w}_2^k$ with any constant c is of a smooth solution to (4) with $\boldsymbol{u}\cdot\boldsymbol{n}|_\Gamma = 0$. Indeed, it is easy to verify that

$$(c\,\boldsymbol{w}_2^k)\cdot\nabla(c\,\boldsymbol{w}_2^k) = \nabla\frac{c^2\big(|\nabla\psi^{(k)}|^2 + \lambda_{2,k}(\psi^{(k)})^2\big)}{2\|\nabla\psi^{(k)}\|^2}.$$

For $\boldsymbol{f} = (f_1(x,y), f_2(x,y))$ and a 2×2-matrix $\mathfrak{g} = (g_{ij})$, we define

$$\boldsymbol{f}\otimes\boldsymbol{f} = \begin{pmatrix} f_1^2 & f_1 f_2 \\ f_1 f_2 & f_2^2 \end{pmatrix},$$

$$\nabla\boldsymbol{f} : \mathfrak{g} = f_{1x}g_{11} + f_{1y}g_{12} + f_{2x}g_{21} + f_{2y}g_{22}.$$

3. A Galerkin Approximation and Its Limit

We consider the initial value problem for the ordinary differential equations of $\{a_{n,k}(t)\}_{k=1}^n$ and $\{b_{n,k}(t)\}_{k=1}^n$ with a fixed $n\in\mathbf{N}$ and constants α, β, γ:

$$\begin{aligned}(1+\beta\lambda_{1,k})\frac{da_{n,k}}{dt} &+ \alpha\sum_{j=1}^n((\omega^{(n)}\boldsymbol{J}\boldsymbol{w}_1^j,\, \boldsymbol{w}_1^k))\frac{da_{n,j}}{dt} \\ &= ((\boldsymbol{B}^n\cdot\nabla\boldsymbol{B}^n - \boldsymbol{v}^n\cdot\nabla\boldsymbol{v}^n + \gamma\Delta\boldsymbol{v}^n,\, \boldsymbol{w}_1^k)),\end{aligned} \tag{8}$$

$$\frac{db_{n,k}}{dt} = ((J\nabla((\boldsymbol{v}^n + \alpha\boldsymbol{v}_t^n)\cdot(J\boldsymbol{B}^n)),\, \boldsymbol{w}_2^k)), \tag{9}$$

$$\boldsymbol{v}^n = \sum_{j=1}^n a_{n,j}\boldsymbol{w}_1^j, \qquad \omega^{(n)} = (J\nabla)\cdot\boldsymbol{v}^n, \qquad \boldsymbol{B}^n = \sum_{j=1}^n b_{n,j}\boldsymbol{w}_2^j. \tag{10}$$

It is easy to see that this system is an approximation of (1) with (3) by the $2n$ basis functions $\{\boldsymbol{w}_1^k\}_{k=1}^n$ and $\{\boldsymbol{w}_2^k\}_{k=1}^n$. The initial conditions are

$$a_{n,k}(0) = ((\boldsymbol{v}_0, \boldsymbol{w}_1^k)), \qquad b_{n,k}(0) = ((\boldsymbol{B}_0, \boldsymbol{w}_2^k)), \tag{11}$$

where $\boldsymbol{v}_0\in\boldsymbol{H}_\sigma^1$ and $\boldsymbol{B}_0\in\boldsymbol{H}_\sigma$ are given arbitrarily but $\boldsymbol{B}_0\not\equiv\boldsymbol{0}$. We denote $\boldsymbol{v}^n|_{t=0}$, $\omega^{(n)}|_{t=0}$ and $\boldsymbol{B}^n|_{t=0}$ by $\boldsymbol{v}_0^n$, $\omega_0^{(n)}$ and $\boldsymbol{B}_0^n$, respectively.

Note that the function Ψ_0 defined by

$$\Psi_0 = \sum_{j=1}^{\infty}((\boldsymbol{B}_0, \boldsymbol{w}_2^j))\frac{\psi^{(j)}}{\lambda_{2,j}^{1/2}\|\psi^{(j)}\|} \tag{12}$$

is a flux function of $\boldsymbol{B}_0$, that is, $\boldsymbol{B}_0 = -J\nabla\Psi_0$ (see (7)) and $\Psi_0|_\Gamma = 0$. Then we have the following theorem.

Theorem 1. *Let $\alpha > 0$, $\beta \geq 0$ and $\gamma > 0$. Then the system* (8)–(11) *has a unique smooth solution globally in time. There exist sequences $\{n_m \in \mathbf{N} \mid n_m < n_{m+1}\}$ and $\{t_m \mid 2^{m-1} < t_m < 2^m\}$ $(m = 1, 2, 3, \ldots)$ such that $\left\|\boldsymbol{v}^{n_m}|_{t=t_m}\right\|$, $\left\|\omega^{(n_m)}|_{t=t_m}\right\|$ and $\left\|\boldsymbol{v}_t^{n_m}|_{t=t_m}\right\|$ (and moreover, $\left\|\omega_t^{(n_m)}|_{t=t_m}\right\|$ if $\beta > 0$) tend to zero and $\left|\boldsymbol{B}^{n_m}|_{t=t_m}\right|^2$ (or a subsequence of it) converges weakly-* to a measure μ in $M(\bar{\Omega})$ satisfying*

$$\lambda_{2,1}\|\Psi_0\|^2 \leq \int_{\bar{\Omega}} d\mu \leq I_0. \tag{13}$$

Here $I_0 = \|\boldsymbol{v}_0\|^2 + (\alpha\gamma + \beta)\|(J\nabla)\cdot\boldsymbol{v}_0\|^2 + \|\boldsymbol{B}_0\|^2$. Furthermore, the system (8)–(11) *with $n = n_m$ and $t = t_m$ yields the existence of a μ-measurable map (x, y) $(\in \Omega) \mapsto \{\nu_{(x,y)}, \xi_{(x,y)}\}$ $(\in M^+(\mathbf{R}^2) \oplus \mathrm{Prob}M(S))$ such that $\{\mu, \nu_{(x,y)}, \xi_{(x,y)}\}$ is a measure-valued solution of* (4) *in the sense of DiPerna–Majda* [2]:

$$\int_\Omega \nabla\boldsymbol{\Phi} : \left(\int_S \frac{\boldsymbol{u}}{|\boldsymbol{u}|} \otimes \frac{\boldsymbol{u}}{|\boldsymbol{u}|} d\xi_{(x,y)}\right) d\mu \;=\; 0, \tag{14}$$

$$\int_\Omega \nabla\phi \cdot \left(\int_{\mathbf{R}^2} \frac{\boldsymbol{u}}{1 + |\boldsymbol{u}|^2} d\nu_{(x,y)}\right)(1 + \rho)dx\,dy \;=\; 0 \tag{15}$$

for all $\boldsymbol{\Phi} \in \boldsymbol{C}_{0,\sigma}^\infty(\Omega)$ and all $\phi \in C_0^\infty(\Omega)$. Here ρ is the Radon–Nikodym derivative of the absolutely continuous part of μ with respect to the Lebesgue measure.

Proof. The $n\times n$-matrix whose (k, j)-component is defined by $((\omega^{(n)}J\boldsymbol{w}_1^j, \boldsymbol{w}_1^k))$ (contained in (8)) is anti-symmetric. This implies that the real parts of its eigenvalues are all zero and the problem (8)–(11) has a unique smooth solution at least locally in time.

Multiplying (8) by $a_{n,k}$ or $(d/dt)a_{n,k}$ and summing up the products from $k=1$ to n, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\boldsymbol{v}^n\|^2+\beta\|\omega^{(n)}\|^2) &= ((-\alpha\omega^{(n)}J\boldsymbol{v}_t^n+\boldsymbol{B}^n\cdot\nabla\boldsymbol{B}^n,\,\boldsymbol{v}^n))-\gamma\|\omega^{(n)}\|^2,\\ \|\boldsymbol{v}_t^n\|^2+\beta\|\omega_t^{(n)}\|^2 &= ((\boldsymbol{B}^n\cdot\nabla\boldsymbol{B}^n-\boldsymbol{v}^n\cdot\nabla\boldsymbol{v}^n,\,\boldsymbol{v}_t^n))-\frac{\gamma}{2}\frac{d}{dt}\|\omega^{(n)}\|^2.\end{aligned}$$

Here we used $((\boldsymbol{f}\cdot\nabla\boldsymbol{g},\,\boldsymbol{h}))=-((\boldsymbol{f}\cdot\nabla\boldsymbol{h},\,\boldsymbol{g}))$ ($=0$ if $\boldsymbol{g}=\boldsymbol{h}$) for smooth $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$ such that one of them vanishes on Γ and $\boldsymbol{f}$ is divergence-free. Summing up the products of (9) and $b_{n,k}$ from $k=1$ to n and noting (6), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{B}^n\|^2 &= ((J\nabla((\boldsymbol{v}^n+\alpha\boldsymbol{v}_t^n)\cdot(J\boldsymbol{B}^n)),\,\boldsymbol{B}^n))\\ &= ((\boldsymbol{B}^n\cdot\nabla(\boldsymbol{v}^n+\alpha\boldsymbol{v}_t^n),\,\boldsymbol{B}^n)).\end{aligned}$$

These equalities and $((\boldsymbol{v}^n\cdot\nabla\boldsymbol{v}^n,\,\boldsymbol{v}_t^n))=((\omega^{(n)}J\boldsymbol{v}^n,\,\boldsymbol{v}_t^n))=-((\omega^{(n)}J\boldsymbol{v}_t^n,\,\boldsymbol{v}^n))$ yield

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\boldsymbol{v}^n\|^2+(\alpha\gamma+\beta)\|\omega^{(n)}\|^2+\|\boldsymbol{B}^n\|^2\right)\\ &\quad= -\alpha\|\boldsymbol{v}_t^n\|^2-\gamma\|\omega^{(n)}\|^2-\alpha\beta\|\omega_t^{(n)}\|^2,\end{aligned}\tag{16}$$

which leads to

$$\begin{aligned}&\big\|\boldsymbol{v}^n|_{t=t'}\big\|^2+(\alpha\gamma+\beta)\big\|\omega^{(n)}|_{t=t'}\big\|^2+\big\|\boldsymbol{B}^n|_{t=t'}\big\|^2\\ &\qquad+2\int_0^{t'}(\alpha\|\boldsymbol{v}_t^n\|^2+\gamma\|\omega^{(n)}\|^2+\alpha\beta\|\omega_t^{(n)}\|^2)\,dt\\ &\quad= \|\boldsymbol{v}_0^n\|^2+(\alpha\gamma+\beta)\|\omega_0^{(n)}\|^2+\|\boldsymbol{B}_0^n\|^2\le I_0\end{aligned}\tag{17}$$

for any $t'>0$. Therefore, the problem (8)–(11) has a unique smooth solution globally in time.

Since (17) means that $\|\boldsymbol{v}_t^n\|$ and $\|\omega^{(n)}\|$ (and moreover, $\|\omega_t^{(n)}\|$ if $\beta>0$) are square integrable over $(0,\infty)$, there exists a sequence $\{t_m^{(n)}\mid 2^{m-1}<t_m^{(n)}<2^m\}$ ($m=1,2,3,\ldots$) such that

$$\alpha\big\|\boldsymbol{v}_t^n|_{t=t_m^{(n)}}\big\|^2+\gamma\big\|\omega^{(n)}|_{t=t_m^{(n)}}\big\|^2+\alpha\beta\big\|\omega_t^{(n)}|_{t=t_m^{(n)}}\big\|^2\le 2^{-m}I_0.\tag{18}$$

Indeed, if we suppose the nonexistence of this $\{t_m^{(n)}\}$, then we have

$$\int_{2^{m-1}}^{2^m}(\alpha\|\boldsymbol{v}_t^n\|^2+\gamma\|\omega^{(n)}\|^2+\alpha\beta\|\omega_t^{(n)}\|^2)\,dt>2^{-m}I_0(2^m-2^{m-1})=I_0/2,$$

which contradicts (17). From (18), we deduce the existence of $\{n_m \in \mathbf{N} \mid n_m < n_{m+1}\}$ and $\{t_m \mid t_m = t_m^{(n_m)}\}$ such that

$$\left\|\boldsymbol{v}_t^{n_m}|_{t=t_m}\right\| \to 0, \tag{19}$$

$$\left\|\boldsymbol{v}^{n_m}|_{t=t_m}\right\| \le \lambda_{1,1}^{-1/2}\left\|\omega^{(n_m)}|_{t=t_m}\right\| \to 0, \tag{20}$$

and in addition, $$\left\|\omega_t^{(n_m)}|_{t=t_m}\right\| \to 0 \quad \text{if } \beta > 0, \tag{21}$$

as $m \to \infty$. Furthermore, since (17) implies that $|((|\boldsymbol{B}^n|^2, \chi))|$ for any $\chi \in C(\bar{\Omega})$ satisfying $|\chi| \le 1$ is bounded uniformly in n and t, we have the weak-* convergence of $\left|\boldsymbol{B}^{n_m}|_{t=t_m}\right|^2$ (or a subsequence of it, if necessary) to a measure μ in $M(\bar{\Omega})$. The limit

$$\begin{aligned}&\lim_{m\to\infty}\int_\Omega \nabla\mathbf{\Phi} : (\boldsymbol{B}^{n_m}\otimes\boldsymbol{B}^{n_m})|_{t=t_m}\,dxdy\\ &= \int_\Omega \nabla\mathbf{\Phi} : \left(\int_S \frac{\boldsymbol{u}}{|\boldsymbol{u}|}\otimes\frac{\boldsymbol{u}}{|\boldsymbol{u}|}\,d\xi_{(x,y)}\right)d\mu\end{aligned} \tag{22}$$

is derived from theorems in [2, §4] for a μ-measurable $\xi_{(\cdot,\cdot)} : \Omega \to \mathrm{Prob}M(S)$ and an arbitrary $\mathbf{\Phi} \in \boldsymbol{C}_{0,\sigma}^\infty(\Omega)$.

Defining $\mathbf{\Phi}^n \in \boldsymbol{H}_\sigma^1 \cap \boldsymbol{W}_2^l(\Omega)$ with a large l by

$$\mathbf{\Phi}^n = \sum_{j=1}^n ((\mathbf{\Phi}, \boldsymbol{w}_1^j))\boldsymbol{w}_1^j$$

and summing up the products of (8) and $((\mathbf{\Phi}, \boldsymbol{w}_1^k))$ from $k = 1$ to n, we obtain

$$\begin{aligned}&((\boldsymbol{v}_t^n + \alpha\omega^{(n)}J\boldsymbol{v}_t^n,\, \mathbf{\Phi}^n)) - \beta((\boldsymbol{v}_t^n,\, \Delta\mathbf{\Phi}^n))\\ &\quad = -\int_\Omega \nabla\mathbf{\Phi} : \boldsymbol{B}^n\otimes\boldsymbol{B}^n\,dxdy + \int_\Omega \nabla(\mathbf{\Phi}-\mathbf{\Phi}^n) : \boldsymbol{B}^n\otimes\boldsymbol{B}^n\,dxdy\\ &\qquad +((\boldsymbol{v}^n\cdot\nabla\mathbf{\Phi}^n,\, \boldsymbol{v}^n)) + \gamma((\boldsymbol{v}^n,\, \Delta\mathbf{\Phi}^n)).\end{aligned} \tag{23}$$

Therefore, we derive (14) by using (19), (20) and (22). The estimate $\int_{\bar{\Omega}} d\mu \le I_0$ in (13) is obtained from (17).

Noting that $\boldsymbol{w}_2^k \in \boldsymbol{H}_\sigma$ means $((\nabla\phi,\, \boldsymbol{B}^n)) = 0$ for any $\phi \in C_0^\infty(\Omega)$, we get (15) by theorems in [2, §4].

Lastly, let us prove $\int_{\bar{\Omega}} d\mu \geq \lambda_{2,1}\|\Psi_0\|^2$. It is convenient to define

$$\Psi^{(n)} = \sum_{j=1}^{n} \frac{b_{n,j}\psi^{(j)}}{\lambda_{2,j}^{1/2}\|\psi^{(j)}\|}, \tag{24}$$

which is a flux function of $\boldsymbol{B}^n$, that is, $\boldsymbol{B}^n = -J\nabla\Psi^{(n)}$. The initial state $\Psi^{(n)}|_{t=0}$ is denoted by $\Psi_0^{(n)}$. Noting (7), we can rewrite (9) as

$$\frac{db_{n,k}}{dt} = \Big(\Big((\boldsymbol{v}^n + \alpha\boldsymbol{v}_t^n)\cdot\nabla\Psi^{(n)},\ \frac{\Delta\psi^{(k)}}{\lambda_{2,k}^{1/2}\|\psi^{(k)}\|}\Big)\Big). \tag{25}$$

Multiplying this by $\lambda_{2,k}^{-1}b_{n,k}$ and summing up the products from $k = 1$ to n, we have

$$\frac{1}{2}\frac{d}{dt}\|\Psi^{(n)}\|^2 = (((\boldsymbol{v}^n + \alpha\boldsymbol{v}_t^n)\cdot\nabla\Psi^{(n)},\ -\Psi^{(n)})) = 0.$$

This yields

$$\|\Psi^{(n)}\|^2 = \|\Psi_0^{(n)}\|^2 \to \|\Psi_0\|^2 \quad \text{for all } t > 0, \text{ as } n \to \infty. \tag{26}$$

Since

$$\|\Psi^{(n)}\|^2 \leq \lambda_{2,1}^{-1}\sum_{j=1}^{n} b_{n,j}^2 = \lambda_{2,1}^{-1}\|\boldsymbol{B}^n\|^2, \tag{27}$$

we obtain $\|\Psi_0\|^2 \leq \lambda_{2,1}^{-1}\int_{\bar{\Omega}} d\mu$. □

Remark 1. The sequence $\{t_m \mid 2^{m-1} < t_m < 2^m\}$ in Theorem 1 can be generalized as $\{t_m \mid 0 < t_m < t_{m+1},\ \lim_{m\to\infty}(t_{m+1} - t_m) = \infty\}$.

Remark 2. If we can prove (19) for $\alpha = 0$, or weaklier

$$((\boldsymbol{v}_t^{n_m}|_{t=t_m}, \boldsymbol{\Phi})) \to 0 \qquad \text{with any } \boldsymbol{\Phi} \in \boldsymbol{C}_{0,\sigma}^{\infty}(\Omega) \tag{28}$$

for $\alpha = 0$, then Theorem 1 will be valid for $\alpha = 0$. Indeed, we can deduce (14) from (28) and (23) with $\alpha = 0$ by using (20) and (22), which hold for $\alpha = 0$, and noting that $((\boldsymbol{v}_t^n,\ (1 - \beta\Delta)\boldsymbol{\Phi}^n)) = ((\boldsymbol{v}_t^n,\ (1 - \beta\Delta)\boldsymbol{\Phi}))$.

4. Lemmas on the Measure-Valued Solution

For the measure-valued solution in Theorem 1, we have the following:

Lemma 1. *Let $\boldsymbol{v}_0 \equiv \boldsymbol{0}$. Assume that $\boldsymbol{B}_0 \in \boldsymbol{H}_\sigma$ is not a solution to* (4) *in the sense of distribution, in other words,*

$$\int_\Omega \nabla \boldsymbol{w}_1^k : \boldsymbol{B}_0 \otimes \boldsymbol{B}_0 \, dxdy \neq 0 \qquad \text{for some } k \in \mathbf{N}. \tag{29}$$

Then μ in Theorem 1 satisfies $\int_{\bar{\Omega}} d\mu < \|\boldsymbol{B}_0\|^2$.

Proof. Assume that $\int_{\bar{\Omega}} d\mu = \|\boldsymbol{B}_0\|^2$. Then

$$\lim_{m\to\infty} \int_0^{t_m} (\|\boldsymbol{v}_t^{n_m}\|^2 + \|\omega^{(n_m)}\|^2 + \beta\|\omega_t^{(n_m)}\|^2)\, dt = 0$$

follows from (17), (19) and (20). It leads to

$$\lim_{m\to\infty} \int_0^{T} (\|\boldsymbol{v}_t^{n_m}\|^2 + \|\omega^{(n_m)}\|^2 + \beta\|\omega_t^{(n_m)}\|^2)\, dt = 0 \tag{30}$$

for any finite $T > 0$. Integrating (17) over $(0, T)$ and letting $n = n_m \to \infty$, we obtain

$$\lim_{m\to\infty} \int_0^{T} (\|\boldsymbol{B}^{n_m}\|^2 - \|\boldsymbol{B}_0\|^2)\, dt = 0$$

by (30). Note that (9) with (30) yields

$$|((\boldsymbol{B}^{n_m} - \boldsymbol{B}_0^{n_m}, \boldsymbol{w}_2^k))| \leq c_{0,k} \int_0^T \|\boldsymbol{B}^{n_m}\| \|\boldsymbol{v}^{n_m} + \alpha \boldsymbol{v}_t^{n_m}\|\, dt \to 0$$

uniformly in $t \in (0, T)$, where $c_{0,k}$ is a positive constant determined by the derivatives of $\boldsymbol{w}_2^k$. Namely, $\boldsymbol{B}^{n_m}$ converges to $\boldsymbol{B}_0$ weakly in $\boldsymbol{H}_\sigma$, uniformly in t. Therefore,

$$\begin{aligned}\lim_{m\to\infty} &\int_0^T \|\boldsymbol{B}^{n_m} - \boldsymbol{B}_0\|^2\, dt \\ &= \lim_{m\to\infty} \int_0^T (\|\boldsymbol{B}^{n_m}\|^2 - 2((\boldsymbol{B}^{n_m}, \boldsymbol{B}_0)) + \|\boldsymbol{B}_0\|^2)\, dt = 0\end{aligned}$$

holds. Integrating (8) over $(0, T)$ and letting $n = n_m \to \infty$, we have

$$\int_\Omega \nabla \boldsymbol{w}_1^k : \boldsymbol{B}_0 \otimes \boldsymbol{B}_0 \, dxdy = 0$$

for any k. This contradicts (29). □

Since the function $\Psi^{(n)}$ defined by (24) is uniformly bounded from above in $W_2^1(\Omega)$ (see (17) and (27)) and $\Psi^{(n)}|_\Gamma = 0$, it converges to a function $\Theta \in W_2^1(\Omega)$ strongly in $L^2(\Omega)$ as $n = n_m$ and $t = t_m$ (or subsequences of them) go to ∞, and $\Theta|_\Gamma = 0$. The relation

$$\|\Theta\| = \|\Psi_0\| \tag{31}$$

follows from (26). Furthermore, we have

Lemma 2. *If $\beta > 0$, then*

$$\int_\Omega \Theta \, dx dy = \int_\Omega \Psi_0 \, dx dy. \tag{32}$$

Proof. Multiplying (25) by $\lambda_{2,k}^{-1/2}((1, \|\psi^{(k)}\|^{-1}\psi^{(k)}))$ and summing up the products from $k = 1$ to n, we get

$$\begin{aligned} ((1, \Psi_t^{(n)})) &= (((\boldsymbol{v}^n + \alpha \boldsymbol{v}_t^n) \cdot \nabla \Psi^{(n)}, \, -P_n(1))) \\ &= (((\boldsymbol{v}^n + \alpha \boldsymbol{v}_t^n) \cdot \nabla \Psi^{(n)}, \, 1 - P_n(1))), \end{aligned}$$

where $P_n(1) = \sum_{j=1}^n ((1, \|\psi^{(j)}\|^{-1}\psi^{(j)}))\|\psi^{(j)}\|^{-1}\psi^{(j)}$, that is, the projection of unit onto the space spanned by $\{\psi^{(j)}\}_{j=1}^n$. Since

$$\int_0^T \|\boldsymbol{v}^n + \alpha \boldsymbol{v}_t^n\|_1^2 dt \le (\lambda_{1,1}^{-1} + 1) \int_0^T \|\omega^{(n)} + \alpha \omega_t^{(n)}\|^2 dt$$

and $L^2(0, T; \boldsymbol{W}_2^1(\Omega)) \subset L^2(0, T; \boldsymbol{L}^\tau(\Omega))$ with arbitrary $T > 0$ and $\tau > 2$, we have

$$\begin{aligned} &\left| \int_\Omega (\Psi^{(n)}|_{t=T} - \Psi_0^{(n)}) dx dy \right| \\ &\quad \le \int_0^T dt \int_\Omega |\boldsymbol{v}^n + \alpha \boldsymbol{v}_t^n| |\boldsymbol{B}^n| |1 - P_n(1)| dx dy \\ &\quad \le c_1 T^{1/2} \left(\int_\Omega |1 - P_n(1)|^{2\tau/(\tau-2)} dx dy \right)^{(\tau-2)/(2\tau)} \end{aligned} \tag{33}$$

by Hölder's inequality and (17). Here c_1 is a positive constant which depends on I_0 and τ, but neither on n nor T.

Let $E_n(\epsilon) = \{(x, y) \in \Omega \mid |1 - P_n(1)| \ge \epsilon\}$. Then, for arbitrary positive ϵ and η, there exists a number $N_0(\epsilon, \eta) \in \mathbf{N}$ such that the Lebesgue measure of $E_n(\epsilon)$ is less than η for any $n > N_0(\epsilon,\eta)$. We have

$$\int_\Omega |1-P_n(1)|^{2\tau/(\tau-2)}\,dxdy = \Big(\int_{E_n(\epsilon)} + \int_{\Omega-E_n(\epsilon)}\Big)|1-P_n(1)|^{2\tau/(\tau-2)}\,dxdy \\ \le c_2^{2\tau/(\tau-2)}\eta + c_3\epsilon^{2\tau/(\tau-2)}$$

for $n > N_0(\epsilon,\eta)$, where $c_2 = \sup_{\Omega,n}|1-P_n(1)|$ and $c_3 = \int_\Omega dxdy$. Choose $\{n_m\}_{m=1}^\infty$ in §3 so that $n_m > N_0(2^{-m}, 2^{-2m\tau/(\tau-2)})$ is satisfied. This implies that n_m can go to infinity faster than $t_m \in (2^{m-1}, 2^m)$. Then, from (33), we derive

$$\left|\int_\Omega (\Psi^{(n_m)}|_{t=t_m} - \Psi_0^{(n_m)})\,dxdy\right| \le c_4 t_m^{1/2} 2^{-m} \le c_4 2^{-m/2},$$

where $c_4 > 0$ depends only on c_1, c_2, c_3 and τ. Letting $m\to\infty$, we obtain (32). □

Because of (31), we restrict $\boldsymbol{H}_\sigma$ to

$$\boldsymbol{Y} = \{\boldsymbol{f}\in\boldsymbol{H}_\sigma \mid \boldsymbol{f} = -J\nabla h,\ h|_\Gamma = 0,\ \|h\| = 1\} \\ = \Big\{\boldsymbol{f}\in\boldsymbol{H}_\sigma \mid \boldsymbol{f} = -J\nabla\sum_{j=1}^\infty r_j\frac{\psi^{(j)}}{\|\psi^{(j)}\|} = \sum_{j=1}^\infty r_j\lambda_{2,j}^{1/2}\boldsymbol{w}_2^j,\quad \sum_{j=1}^\infty r_j^2 = 1\Big\}.$$

According to §2,

$$\boldsymbol{U} = \Big\{\pm\lambda_{2,j}^{1/2}\boldsymbol{w}_2^j\Big\}_{j=1}^\infty$$

is a subset of $\boldsymbol{Y}$ such that each element is a smooth solution to (4). For any $\boldsymbol{f}\in\boldsymbol{Y}$, the inequality $\|\boldsymbol{f}\| \ge \lambda_{2,1}^{1/2}$ is valid and $\|\boldsymbol{f}\| = \lambda_{2,1}^{1/2}$ means $\boldsymbol{f} = \pm\lambda_{2,1}^{1/2}\boldsymbol{w}_2^1$.

If all our measure-valued solutions to (4) for $\boldsymbol{B}_0\in\boldsymbol{Y}-\boldsymbol{U}$ corresponded to elements of $\boldsymbol{U}$, that is, they had the form

$$d\mu = \lambda_{2,j}|\boldsymbol{w}_2^j|^2\,dxdy,\ \nu_{(x,y)} = \delta_{\kappa\lambda_{2,j}^{1/2}\boldsymbol{w}_2^j(x,y)},\ \xi_{(x,y)} = \delta_{\kappa\boldsymbol{w}_2^j(x,y)/|\boldsymbol{w}_2^j(x,y)|}, \quad (34)$$

where $\kappa = 1$ or -1, and $\delta_{\boldsymbol{f}}$ is the Dirac measure at $\boldsymbol{f}$, then our discussion would be trivial. This is denied by the following lemma.

Lemma 3. *Let $\beta > 0$ and $\boldsymbol{v}_0 \equiv \boldsymbol{0}$. Assume that $\boldsymbol{B}_0$ is an element of $\boldsymbol{Y}-\boldsymbol{U}$ satisfying (29) and $\lambda_{2,1} < \|\boldsymbol{B}_0\|^2 \le \lambda_{2,2}$. Then, for at least one*

$\boldsymbol{B}_0$, *the measure-valued solution of* (4) *in Theorem 1 does not have the form* (34). *It satisfies* $\lambda_{2,1} \leq \int_{\bar{\Omega}} d\mu < \|\boldsymbol{B}_0\|^2$.

Proof. The last statement is clear by (13) and Lemma 1. Therefore, if the measure-valued solution has the form (34), then it is with $j = 1$.

Let us suppose that every measure-valued solution for the above conditions has the form (34) with $j = 1$ and $\Theta = \kappa\|\psi^{(1)}\|^{-1}\psi^{(1)}$. From this, we will derive a contradiction.

A flux function for $\boldsymbol{B}_0 \in \boldsymbol{Y} - \boldsymbol{U}$ is given by (12) with $\sum_{j=1}^{\infty} r_j^2 = 1$ and $|r_k| \neq 1$ for any $k \in \mathbf{N}$, where $r_j = \lambda_{2,j}^{-1/2}((\boldsymbol{B}_0,\, \boldsymbol{w}_2^j))$. Therefore, from (32), we deduce

$$\int_{\Omega} \frac{\psi^{(1)}}{\|\psi^{(1)}\|}\, dx\, dy = \frac{1}{r_1'} \int_{\Omega} \sum_{j=2}^{\infty} r_j \frac{\psi^{(j)}}{\|\psi^{(j)}\|}\, dx\, dy,$$

where $r_1' = -r_1 + \kappa$. The value of the right member changes according to $\{r_j\}_{j=1}^{\infty}$, while the left member is equal to a fixed positive number. Hence we reach a contradiction. □

Remark 3. If $\lambda_{2,1} < \int_{\bar{\Omega}} d\mu$ holds for $\mu = \mu^1$ in Lemma 3, then we have another measure-valued solution $\{\mu^2, \nu^2_{(x,y)}, \xi^2_{(x,y)}\}$ by taking a new $\boldsymbol{B}_0$ such that (29) and $\lambda_{2,1} < \|\boldsymbol{B}_0\|^2 \leq \int_{\bar{\Omega}} d\mu^1$ are satisfied. We can repeat this process N-times ($N \leq \infty$) until $\lambda_{2,1} = \int_{\bar{\Omega}} d\mu^N$ is obtained.

Remark 4. If we can prove Theorem 1 for $\alpha = 0$, that is, (28) for $\alpha = 0$, then Lemmas 2 and 3 will be valid for $\alpha = \beta = 0$. This is because Lemma 1 and (33) will hold for $\alpha = \beta = 0$.

5. Another Magnetohydrodynamic Approach

In this section, we investigate a Galerkin approximation of (5) with $\boldsymbol{B} \cdot \boldsymbol{n}|_{\Gamma} = 0$:

$$\frac{db_{n,k}}{dt} = ((J\nabla(P_\sigma(\zeta^{(n)} J\boldsymbol{B}^n) \cdot (J\boldsymbol{B}^n)),\, \boldsymbol{w}_2^k)), \tag{35}$$

$$\boldsymbol{B}^n = \sum_{j=1}^{n} b_{n,j}(t)\boldsymbol{w}_2^j, \quad \zeta^{(n)} = (J\nabla) \cdot \boldsymbol{B}^n, \quad b_{n,k}(0) = ((\boldsymbol{B}_0, \boldsymbol{w}_2^k)). \tag{36}$$

For this system, the following theorem is deduced.

Theorem 2. *Assume that* $\boldsymbol{B}_0 \in \boldsymbol{H}_\sigma$ *satisfies* (29) (*or* (29) *with* $\boldsymbol{w}_1^k$ *replaced by* $\boldsymbol{w}_2^k$). *Then* (35) *with* (36) *has a unique smooth solution globally in time. There exist sequences* $\{n_m \in \mathbf{N} \mid n_m < n_{m+1}\}$ *and* $\{t_m \mid 2^{m-1} < t_m < 2^m\}$ $(m = 1, 2, 3, \dots)$ *such that* $\left|\boldsymbol{B}^{n_m}|_{t=t_m}\right|^2$ (*or a subsequence of it*) *converges weakly-* to a measure* μ *in* $M(\bar{\Omega})$ *satisfying*

$$\lambda_{2,1}\|\Psi_0\|^2 \leq \int_{\bar{\Omega}} d\mu < \|\boldsymbol{B}_0\|^2$$

with Ψ_0 *given by* (12). *Furthermore,* (35) *with* (36) *for* $n = n_m$ *and* $t = t_m$ *yields the existence of a* μ*-measurable map* (x, y) $(\in \Omega) \mapsto \{\nu_{(x,y)}, \xi_{(x,y)}\}$ $(\in M^+(\mathbf{R}^2) \oplus \mathrm{Prob}M(S))$ *such that* $\{\mu, \nu_{(x,y)}, \xi_{(x,y)}\}$ *is a measure-valued solution of* (4) *in DiPerna–Majda's sense, which satisfies* (14) *and* (15).

Proof. Sum up the products of (35) and $b_{n,k}$ from $k = 1$ to n and note that $P_\sigma(\zeta^{(n)} J\boldsymbol{B}^n) \cdot \boldsymbol{n}|_\Gamma = \boldsymbol{B}^n \cdot \boldsymbol{n}|_\Gamma = 0$, that is, $P_\sigma(\zeta^{(n)} J\boldsymbol{B}^n) \cdot (J\boldsymbol{B}^n)|_\Gamma = 0$. Then we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\boldsymbol{B}^n\|^2 &= ((J\nabla(P_\sigma(\zeta^{(n)} J\boldsymbol{B}^n) \cdot (J\boldsymbol{B}^n)),\, \boldsymbol{B}^n)) \\
&= -((\nabla(P_\sigma(\zeta^{(n)} J\boldsymbol{B}^n) \cdot (J\boldsymbol{B}^n)),\, J\boldsymbol{B}^n)) \\
&= ((P_\sigma(\zeta^{(n)} J\boldsymbol{B}^n) \cdot (J\boldsymbol{B}^n),\, \nabla \cdot (J\boldsymbol{B}^n))) \\
&= -((P_\sigma(\zeta^{(n)} J\boldsymbol{B}^n),\, \zeta^{(n)} J\boldsymbol{B}^n)) \\
&= -\|P_\sigma(\zeta^{(n)} J\boldsymbol{B}^n)\|^2.
\end{aligned}$$

Therefore, $\left\|P_\sigma(\zeta^{(n_m)} J\boldsymbol{B}^{n_m})|_{t=t_m}\right\| \to 0$ is derived for some $\{n_m\}$ and $\{t_m\}$, analogously to (19)–(21). Using this fact, we can prove the theorem in the same way as Theorem 1 and Lemma 1. For example, (14) is obtained from (22) and

$$((P_\sigma(\zeta^{(n)} J\boldsymbol{B}^n),\, \boldsymbol{\Phi})) = -\int_\Omega \nabla\boldsymbol{\Phi} : \boldsymbol{B}^n \otimes \boldsymbol{B}^n \, dx\,dy$$

for any $\boldsymbol{\Phi} \in \boldsymbol{C}^\infty_{0,\sigma}(\Omega)$. □

Remark 5. It is open whether facts like Lemmas 2 and 3 can be proved for (35).

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