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# ON THE OSCILLATION OF SEMILINEAR ELLIPTIC DIFFERENTIAL EQUATIONS

BY

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**Abstract**

Some oscillation criteria for the second order semilinear elliptic differential equation

$$\sum_{i,\,j=1}^{n} D_i[\,A_{ij}(x,y)D_jy\,]+B(x,y)=0$$

are obtained. Generalized Riccati transformation and weighted averaging technique are employed to established our results.

## 1. Introduction

We study the oscillation of the second order semilinear elliptic differential equation

$$\sum_{i,\,j=1}^{n} D_i[\,A_{ij}(x,y)D_jy\,]+B(x,y)=0, \tag{1.1}$$

in exterior domain $\Omega(r_0)=\{x:|x|\geq r_0\}$, where $r_0>0$, $x=(x_1,\cdots,x_n)\in\mathbb{R}^n$, $n\geq 2$, $D_i=\partial/\partial x_i$, and $|x|$ denotes the usual Euclidean norm in $\mathbb{R}^n$.

Throughout this paper, we always assume that the following conditions hold.

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(H1) $B \in C(\Omega(r_0) \times \mathbb{R}, \mathbb{R})$ with $B(x,-y) = -B(x,y)$ for $x \in \Omega(r_0)$ and $y > 0$;

(H2) $B(x,y) \geq p(x)f(y)$ for $x \in \Omega(r_0)$ and $y \geq 0$, where $p \in C^{\mu}_{loc}(\Omega(r_0), \mathbb{R})$, $\mu \in (0,1)$, and $f \in C(\mathbb{R}, \mathbb{R}) \cup C^1(\mathbb{R} - \{0\}, \mathbb{R})$, $yf(y) > 0$ for $y \neq 0$;

(H3) $A = (A_{ij})_{n\times n}$ is a real symmetric positive definite matrix with $A_{ij} \in C^{1+\mu}_{loc}(\Omega(r_0) \times \mathbb{R}, \mathbb{R}^+)$, and $A_{ij}(x,-y) = A_{ij}(x,y)$ for all $i, j$, $x \in \Omega(r_0)$, $y \geq 0$.

Denote by $\lambda_{\max}(x,y)$ the largest eigenvalue of the matrix $A$. Suppose that there exists a function $\lambda \in C([\,r_0, \infty) \times \mathbb{R}, \mathbb{R}^+)$ such that

$$\lambda(r,y) \geq \max_{|x|=r} \lambda_{\max}(x,y), \quad r \geq r_0.$$

Furthermore, assume that there exists a function $a \in C([r_0,\infty), \mathbb{R}^+)$ and a constant $k > 0$ such that

$$\frac{f'(y)}{\lambda(r,y)} \geq \frac{k}{a(r)} \quad \text{for all} \quad y \neq 0, \quad r \geq r_0.$$

As usual, a function $y \in C^{2+\nu}_{loc}(\Omega(r_0), \mathbb{R})$ is called a solution of (1.1) if $y(x)$ satisfies (1.1) for all $x \in \Omega(r_0)$. We restrict our attention only to the nontrivial solution of (1.1), i.e., to the solution $y(x)$ satisfying $\sup\{|y(x)| : x \in \Omega(r)\} > 0$ for every $r \geq r_0$. Regarding the question of existence of solution of (1.1) we refer the reader to the monograph [2]. A nontrivial solution $y(x)$ of (1.1) is said to be oscillatory in $\Omega(r_0)$ if the set $\{x \in \Omega(r_0) : y(x) = 0\}$ is unbounded, otherwise it is said to be non-oscillatory. (1.1) called oscillatory if all its nontrivial solutions are oscillatory. Note that not every solution of a nonlinear equation is extendible to the whole domain. Therefore, when dealing with the oscillation of (1.1), only extendible solutions are considered.

An important special case of (1.1) is the following equation

$$\sum_{i,j=1}^{n} D_i[\,a_{ij}(x) D_j y\,] + B(x,y) = 0. \tag{1.2}$$

Concerning (1.2) there exists well-elaborated oscillation theory. In 1980, by employing the vector Riccati transformation, Noussair and Swanson [6] first gave Wintner-type theorem [10] for (1.2) (Theorem 4, [6]). Swanson

[8] summarized the oscillation results for (1.2) up to 1979. Very recently, some classical oscillation theorems such as Hille [3], Kameven [4], Kong [5], Philos [7] and others for second order ordinary differential equations have been extended to (1.2) with $B(x,y) = p(x)f(y)$, see, for instance, [11, 12, 13, 14, 15] and references quoted therein.

In this paper, by using the generalized Riccati transformation ((2.6), Section 2), weighted averaging techniques [1, 9] and following the ideas of Coles [1] and Willett [9], we shall establish new oscillation criteria for (1.1). The theorems obtained here extend and improve the main results in [6, 11, 13, 14]. Finally, some examples are given to illustrate the advantages of our results.

## 2. Preliminaries

For simplicity, we introduce the following notations to be used throughout this paper. For given function $\rho \in C^1([r_0,\infty),\mathbb{R})$, we define

$$g(r) = \frac{\omega_n}{k}a(r)\varphi(r)r^{n-1}, \quad \varphi(r) = \exp\Big(-\frac{2k}{\omega_n}\int_{r_0}^{r}\rho(s)s^{1-n}ds\Big)$$

and

$$P(r) = \varphi(r)\Big\{\int_{S_r}p(x)d\sigma + \frac{k}{\omega_n}a(r)\rho^2(r)r^{1-n} - [\rho(r)a(r)]'\Big\},$$

where $S_r = \{x \in \mathbb{R}^n : |x| = r\}$ for $r > 0$, $d\sigma$ and $\omega_n$ denote the spherical integral element and the surface measure of unit sphere in $\mathbb{R}^n$, respectively.

Let $\Im$ denote the set of all non-negative locally integrable function $\phi$ on $\mathbb{R}^+$ which satisfied

$$\int^{\infty}\phi(s)\Big(\int^{s}g(u)\phi^2(u)du\Big)^{-1}\theta\Big(\int^{s}\phi(u)du\Big)ds = \infty, \tag{2.1}$$

and

$$\int^{\infty}\frac{\theta(u)}{u^2}\,du < \infty, \tag{2.2}$$

where $\theta \in C(\mathbb{R}^+,[0,\infty))$ with $\theta$ non-decreasing.

Let $\Im_0 \subset \Im$ be the set of all non-negative locally functions $\phi$ on $\mathbb{R}^+$

satisfying

$$\lim_{r\to\infty}\frac{\int^r g(u)\phi^2(u)\,du}{[\int^r \phi(u)\,du]^2}=0. \tag{2.3}$$

In order that either (2.1) and (2.3) can be satisfied by a non-negative locally integrable function $\phi$, it is necessary that $\phi$ is non-integrable on $\mathbb{R}^+$, i.e.,

$$\int^\infty \phi(s)\,ds=\infty. \tag{2.4}$$

Members of the classes $\Im$ and $\Im_0$ will be called weight functions [1, 9]. For $\phi\in\Im$, let

$$Q_\phi(s,r)=\frac{\int_s^r \phi(u)\int_s^u P(\tau)\,d\tau\,du}{\int_s^r \phi(u)\,du}. \tag{2.5}$$

**Lemma 2.1.** *Let $y=y(x)$ be a non-oscillatory solution of Eq.(1.1) in $\Omega(b)$ for $b\ge r_0$, $\rho\in C^1([b,\infty),\mathbb{R})$. Then the function $Z\in C^1([b,\infty),\mathbb{R}^+)$ defined by*

$$Z(r)=\varphi(r)\Big[\int_{S_r} W(x)\cdot\nu(x)d\sigma+\rho(r)a(r)\Big] \tag{2.6}$$

*satisfies the generalized Riccati inequality*

$$Z'(r)\le -P(r)-\frac{Z^2(r)}{g(r)}, \tag{2.7}$$

*where $W(x)=(A\nabla y)(x)/f(y(x))$, $\nabla y$ and $\nu(x)=x/|x|$, $(x\ne 0)$, denote the gradient of $y$ and the outward unit normal, respectively.*

*Proof.* Differentiation of the $i$-th component of $W(x)$ with respect to $x_i$ gives that

$$\begin{aligned} D_iW(x)_i \;=\; & -\frac{f'(y)}{f^2(y)}D_iy\Big(\sum_{i=1}^N a_{ij}(x,y)D_jy\Big)\\ & +\frac{1}{f(y)}D_i\Big(\sum_{i=1}^N a_{ij}(x,y)D_jy\Big) \text{ for all } \quad i. \end{aligned}$$

Summation over $i$, note that (1.1), leads to

$$\mathrm{div} W(x) \leq -p(x) - f'(y)(W^T A^{-1} W)(x).$$

Using Green's formula (cf [2]) in (2.6) and (H3), we get

$$\begin{aligned} Z'(r) &= \frac{\varphi'(r)}{\varphi(r)} Z(r) + \varphi(r) \Big\{ \int_{S_r} \mathrm{div} W(x) + [\rho(r)a(r)]' \Big\} \\ &\leq \frac{\varphi'(r)}{\varphi(r)} Z(r) - \varphi(r) \Big\{ \int_{S_r} p(x) d\sigma - [\rho(r)a(r)]' + \frac{k}{a(r)} \int_{S_r} |W(x)|^2 d\sigma \Big\}. \end{aligned}$$

The Cauchy-Schwartz inequality follows that

$$\int_{S_r} |W(x)|^2 \, d\sigma \geq \frac{r^{1-n}}{\omega_n} \Big[ \int_{S_r} W(x) \cdot \nu(x) \, d\sigma \Big]^2.$$

Therefore,

$$\begin{aligned} Z'(r) &\leq \frac{\varphi'(r)}{\varphi(r)} Z(r) - \varphi(r) \Big\{ \int_{S_r} p(x) d\sigma - [\rho(r)a(r)]' \\ &\quad + \frac{k r^{1-n}}{\omega_n a(r)} \Big[ \int_{S_r} W(x) \cdot \nu(x) d\sigma \Big]^2 \Big\} \\ &= \frac{\varphi'(r)}{\varphi(r)} Z(r) - \varphi(r) \Big\{ \int_{S_r} p(x) d\sigma - [\rho(r)a(r)]' \\ &\quad + \frac{k r^{1-n}}{\omega_n a(r)} \Big[ \frac{Z(r)}{\varphi(r)} - \rho(r)a(r) \Big]^2 \Big\} \\ &= -P(r) - \frac{Z^2(r)}{g(r)}. \end{aligned}$$

This completes the proof. □

We now consider some properties of solutions $Z(r)$ to (2.7). Clearly, an integral inequality with respect to (2.7) is

$$Z(r) \leq Z(b) - \int_b^r P(u) \, du - \int_b^r \frac{Z^2(u)}{g(u)} du \quad \text{for} \quad r \geq b \geq r_0. \tag{2.8}$$

**Lemma 2.2.** *Suppose that there exist $\rho \in C^1([r_0, \infty), \mathbb{R})$ and $\phi \in \Im$ such that*

$$\liminf_{r \to \infty} Q_\phi(\cdot, r) > -\infty. \tag{2.9}$$

*Let $Z(r)$ be a solution of (2.7). Then*

$$\int^{\infty} \frac{Z^2(s)}{g(s)} ds < \infty. \tag{2.10}$$

*Proof.* Let $Q(s,r) = Q_\phi(s,r)$ and assume that

$$\int^{\infty} \frac{Z^2(s)}{g(s)} ds = \infty. \tag{2.11}$$

Multiplying (2.8) by $\phi(u)$ and integrating it from $s$ to $r$, for $r \geq s \geq b$, we obtain

$$\int_s^r \phi(u)Z(u)\,du \leq [Z(s) - Q(s,r)] \int_s^r \phi(u)\,du - \int_s^r \phi(u) \int_s^u \frac{Z^2(\tau)}{g(\tau)}\,d\tau\,du. \tag{2.12}$$

Since $\phi \in \Im$, (2.4) holds. This implies that

$$\begin{aligned} Q(s,r) &= \frac{\int_b^r \phi(u)\,du}{\int_s^r \phi(u)\,du} Q(b,r) - \int_b^s P(u)\,du - \frac{\int_b^s \phi(u) \int_b^u P(\tau)\,d\tau\,du}{\int_s^r \phi(u)\,du} \\ &= \frac{\int_b^r \phi(u)\,du}{\int_s^r \phi(u)\,du} Q(b,r) - \int_b^s P(u)\,du + 0(1) \quad \text{as} \quad r \to \infty. \end{aligned}$$

Thus, by (2.8),

$$Z(s) - Q(s,r) \leq Z(b) - \frac{\int_b^r \phi(u)\,du}{\int_s^r \phi(u)\,du} Q(b,r) - \int_b^s \frac{Z^2(u)}{g(u)} du + 0(1) \quad \text{as } r \to \infty. \tag{2.13}$$

Using (2.9) and (2.11), we conclude from (2.13) that there exist constants $b_1$, $b_2$, ($b_2 > b_1 \geq b$), and $\delta > 0$ such that

$$Z(b_1) - Q(b_1, r) \leq -\delta \quad \text{for all} \quad r \geq b_2. \tag{2.14}$$

Let

$$V(r) = \int_{b_1}^r \phi(u)Z(u)\,du. \tag{2.15}$$

The Cauchy-Schwartz inequality follows that

$$V^2(r) \leq \Big[\int_{b_1}^r g(u)\phi^2(u)du\Big]\Big[\int_{b_1}^r \frac{Z^2(u)}{g(u)} du\Big]. \tag{2.16}$$

Putting (2.14), (2.15) and (2.16) into (2.12), we obtain

$$\begin{aligned} V(r) &\le -\delta\int_{b_1}^{r}\phi(u)\,du-\int_{b_1}^{r}\phi(u)\Big[\int_{b_1}^{u}g(\tau)\phi^2(\tau)\,d\tau\Big]^{-1}V^2(u)\,du \\ &=: -G(r),\quad r\ge b_1. \end{aligned}$$

Hence,

$$\begin{aligned} G'(r) &= \delta\phi(r)+\phi(r)\Big[\int_{b_1}^{r}g(\tau)\phi^2(\tau)\,d\tau\Big]^{-1}V^2(r) \\ &\ge \phi(r)\Big[\int_{b_1}^{r}g(\tau)\phi^2(\tau)d\tau\Big]^{-1}V^2(r). \end{aligned} \tag{2.17}$$

However, from the definition of $G(r)$, we get

$$0\le\delta\int_{b_1}^{r}\phi(u)\,du\le G(r)\le|V(r)|. \tag{2.18}$$

Inequalities (2.17) and (2.18) imply that

$$G^{-2}(r)\theta(G(r))G'(r)\ge\phi(r)\Big[\int_{b_1}^{r}g(u)\phi^2(u)\,du\Big]^{-1}\theta\Big(\delta\int_{b_1}^{r}\phi(u)\,du\Big).$$

Integrating the above inequality from $b_1$ to $r$, we have

$$\begin{aligned} &\int_{b_1}^{r}\phi(u)\Big(\int_{b_1}^{u}g(\tau)\phi^2(\tau)\,d\tau\Big)^{-1}\theta\Big(\delta\int_{b_1}^{u}\phi(\tau)\,d\tau\Big)du \\ &\le\int_{b_1}^{r}G^{-2}(u)\theta(G(u))G'(u)du \\ &=\int_{G(b_1)}^{G(r)}\frac{\theta(u)}{u^2}\,du\le\int_{G(b_1)}^{\infty}\frac{\theta(u)}{u^2}\,du<\infty, \end{aligned}$$

which contradicts condition (2.1). This completes the proof. □

**Lemma 2.3.** *Let $Z(r)$ be a solution of (2.7), and suppose that there exists $\rho\in C^1([r_0,\infty),\mathbb{R})$ such that $\int^{\infty}Z^2(u)/g(u)du<\infty$. Then for any $\phi\in\Im_0$, $\limsup_{r\to\infty}Q_\phi(\cdot,r)<\infty$, and*

$$\limsup_{r\to\infty}Q_\phi(s,r)\le Z(s)-\int_s^{\infty}\frac{Z^2(u)}{g(u)}du. \tag{2.19}$$

*In addition, if*

$$\int^{\infty} P(u)\,du < \infty, \tag{2.20}$$

*then*

$$Z(r) \geq \int_r^{\infty} P(u)\,du + \int_r^{\infty} \frac{Z^2(u)}{g(u)} du. \tag{2.21}$$

*Proof.* As in the proof of Lemma 2.2, we know that (2.12) holds. This implies that

$$Q(s,r) \leq Z(s) - \frac{\int_s^r \phi(u)Z(u)\,du}{\int_s^r \phi(u)\,du} - \frac{\int_s^r \phi(u)\int_s^u Z^2(\tau)/g(\tau)d\tau\,du}{\int_s^r \phi(u)\,du}. \tag{2.22}$$

Since $\phi \in \Im_0$, (2.4) holds. Thus,

$$\lim_{r\to\infty} \frac{\int_s^r \phi(u)\int_s^u Z^2(\tau)/g(\tau)d\tau du}{\int_s^r \phi(u)du} = \lim_{r\to\infty}\int_s^r \frac{Z^2(u)}{g(u)}du < \infty.$$

The Cauchy-Schwartz inequality and (2.3) imply that

$$0 \leq \lim_{r\to\infty} \frac{|\int_s^r \phi(u)Z(u)\,du|}{\int_s^r \phi(u)\,du} \leq \lim_{r\to\infty} \frac{[\int_s^r g(u)\phi^2(u)\,du]^{\frac{1}{2}}}{\int_s^r \phi(u)\,du}\left[\int_s^r \frac{Z^2(u)}{g(u)}\,du\right]^{\frac{1}{2}} = 0.$$

Hence, by (2.22), $\limsup_{r\to\infty} Q(s,r) < \infty$, and (2.19) holds.

If (2.20) holds, then for any $\varepsilon > 0$, there exists a $b^* > b$ such that $|\int_r^{\infty} P(u)\,du| < \varepsilon/2$ for all $r \geq b^*$. This implies that there is a $M > 0$ such that $|\int_r^{\infty} P(u)\,du| < M$ for all $r \geq s$. Since $\phi \in \Im_0$, (2.4) holds. Thus, there is a $b^{**} \geq b^*$ such that

$$\frac{\int_s^{b^*} \phi(u)\,du}{\int_s^r \phi(u)\,du} < \frac{\varepsilon}{2M} \quad \text{for all} \quad r \geq b^{**}.$$

Then, for all $r \geq b^{**}$,

$$\begin{aligned}\left|Q(s,r) - \int_s^{\infty} P(u)du\right| &= \left|\frac{\int_s^{b^*} \phi(u)\int_u^{\infty} P(\tau)d\tau du}{\int_s^r \phi(u)du} + \frac{\int_{b^*}^r \phi(u)\int_u^{\infty} P(\tau)\,d\tau\,du}{\int_s^r \phi(u)du}\right| \\ &\leq M\cdot\frac{\varepsilon}{2M} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

which implies that

$$\lim_{r\to\infty} Q(s,r) = \int_s^\infty P(u)\,du.$$

Thus, (2.21) holds. □

## 3. Oscillation Criteria

From Lemmas 2.2 and 2.3, the following theorem is immediate.

**Theorem 3.1.** *If there exists $\rho \in C^1([r_0,\infty),\mathbb{R})$, $\phi \in \Im$, and $\psi \in \Im_0$ such that* (2.9) *and $\limsup_{r\to\infty} Q_\psi(\cdot,r) = \infty$ hold, then* (1.1) *is oscillatory.*

**Corollary 3.1.** *If*

$$\int^\infty \frac{du}{g(u)} = \int^\infty P(u)\,du = \infty,$$

*then* (1.1) *is oscillatory.*

*Proof.* Let $\phi(r) = 1/g(r)$, $\theta(s) = 1$. Then

$$\lim_{r\to\infty} \int^r \phi(s)\Big(\int^s g(u)\phi^2(u)du\Big)^{-1} ds = \lim_{r\to\infty} \int^r \frac{1}{g(s)}\Big(\int^s \frac{du}{g(u)}\Big)^{-1} ds = \infty,$$

and

$$\lim_{r\to\infty} \frac{\int^r g(u)\phi^2(u)du}{[\int^r \phi(u)\,du]^2} = \lim_{r\to\infty} \left(\int^r \frac{du}{g(u)}\right)^{-1} = 0.$$

This implies $\phi \in \Im$. Since,

$$\lim_{r\to\infty} Q_\phi(\cdot,r) = \lim_{r\to\infty} \frac{\int^r \phi(s)\int^s P(u)\,du\,ds}{\int^r \phi(s)\,ds} = \lim_{r\to\infty} \int^r P(u)\,du = \infty,$$

thus, by Theorem 3.1, (1.1) is oscillatory. □

**Remark 3.1.** Corollary 3.1 improves Theorem 4 in [6] and Theorem 2.3 in [14].

**Theorem 3.2.** *Suppose that there exists $\rho \in C^1([r_0,\infty),\mathbb{R})$ and $\phi \in \Im_0$*

*such that*

$$\Theta(r) := \int_r^\infty P(u)\,du < \infty \quad \textit{for} \quad r \geq b. \tag{3.1}$$

*If*

$$\int^\infty \frac{\Theta_+^2(s)}{g(s)} \exp\Big(2\int^s \frac{\Theta(\tau)}{g(\tau)}\,d\tau\Big)ds = \infty, \tag{3.2}$$

*then* (1.1) *is oscillatory.*

*Proof.* Suppose that (1.1) is non-oscillatory. By Lemma 2.1, there exists a constant $b \geq r_0$ and a function $Z \in C^1([b,\infty),\mathbb{R})$ such that (2.7) holds on $[b,\infty)$, it follows from (3.1) and Lemma 2.3 that

$$Z(r) \geq \Theta(r) + \int_r^\infty \frac{Z^2(s)}{g(s)}ds \quad \text{for } r \geq b. \tag{3.3}$$

Let $u(r) = \int_r^\infty Z^2(s)/g(s)ds$, then

$$u'(r) = -\frac{Z^2(r)}{g(r)}. \tag{3.4}$$

Multiplying (3.4) by $\exp\big(2\int_r^s \Theta(\tau)/g(\tau)d\tau\big)$ and integrating it form $r$ to $r_1$, we obtain, for $r_1 \geq r \geq b$,

$$\begin{aligned} u(r) \;\geq\; & u(r_1)\exp\Big(2\int_r^{r_1} \frac{\Theta(s)}{g(s)}ds\Big) \\ & + \int_r^{r_1} \frac{Z^2(s) - 2\Theta(s)u(s)}{g(s)} \exp\Big(2\int_r^s \frac{\Theta(\tau)}{g(\tau)}\,d\tau\Big)ds. \end{aligned} \tag{3.5}$$

It follows from (3.3) that

$$Z(r) \geq \Theta(r) + u(r) \quad \text{for } r \geq b,$$

which implies that

$$Z^2(r) - 2\Theta(r)u(r) \geq \Theta_+^2(r) + u^2(r) \quad \text{for } r \geq b.$$

This and (3.5) imply that

$$u(r) \geq \int_r^\infty \frac{\Theta_+^2(s)}{g(s)} \exp\Big(2\int_r^s \frac{\Theta(\tau)}{g(\tau)}\,d\tau\Big)ds + \int_r^\infty \frac{u^2(s)}{g(s)} \exp\Big(2\int_r^s \frac{\Theta(\tau)}{g(\tau)}d\tau\Big)ds,$$

which contradicts to condition (3.2). So, the proof is complete. □

**Theorem 3.3.** *Suppose that there exists $\rho \in C^1([r_0,\infty),\mathbb{R})$ and $\phi \in \Im_0$ such that $\Theta(r) \geq 0$ holds for sufficiently large $r \geq r_0$. If there exists a locally integrable function $\eta(t) : [r_0,\infty) \to \mathbb{R}^+$ and $\xi(r) = \int_{r_0}^r \eta(u)du + \xi(r_0)$ such that*

$$\lim_{r\to\infty}\Big(\Phi^*(r)\int_{r_0}^r \frac{g(s)}{\xi(s)}ds\Big)^{-\frac{1}{2}}\int_{r_0}^r \Theta(s)ds = \infty, \tag{3.6}$$

*where $\Theta$ is defined as in Theorem 3.2, and*

$$\Phi(r) = \exp\Big(-4\int_{r_0}^r \frac{\Theta(s)}{g(s)}ds\Big), \quad \Phi^*(r) = \int_{r_0}^r \eta(s)\Phi(s)ds,$$

*then (1.1) is oscillatory.*

*Proof.* We proceed as in proof of Theorem 3.2, and obtain that (3.3) holds. So,

$$Z(r) \geq \Theta(r) + \int_r^\infty \frac{Z^2(s)}{g(s)}\,ds \geq 0 \quad \text{for} \quad r \geq b. \tag{3.7}$$

Let $u(r) = \int_r^\infty Z^2(s)/g(s)ds$. Then

$$-u'(r) = \frac{Z^2(r)}{g(r)} \geq \frac{1}{g(r)}[\Theta(r)+u(r)]^2 \geq \frac{4}{g(r)}\Theta(r)u(r),$$

which implies that

$$u(r) \leq u(b)\exp\left(-4\int_b^r \frac{\Theta(s)}{g(s)}ds\right) = u(b)\Phi(r),$$

therefore,

$$\int_r^\infty \frac{Z^2(s)}{g(s)}\,ds \leq u(b)\Phi(r).$$

Hence,

$$\int_b^r \eta(s)\left(\int_s^\infty \frac{Z^2(\tau)}{g(\tau)}\,d\tau\right)ds \leq u(b)\int_b^r \eta(s)\Phi(s)\,ds \leq C_1\Phi^*(r), \quad (C_1>0).$$

Consequently,

$$\begin{aligned}
&\int_b^r \eta(s)\Big(\int_s^r \frac{Z^2(\tau)}{g(\tau)}\,d\tau\Big)\,ds + \int_b^r \eta(s)\Big(\int_r^\infty \frac{Z^2(\tau)}{g(\tau)}d\tau\Big)\,ds \\
&= \int_b^r [\xi(s)-\xi(b)]\frac{Z^2(s)}{g(s)}ds + [\xi(r)-\xi(b)]\int_r^\infty \frac{Z^2(s)}{g(s)}ds \\
&\le C_1\Phi^*(r),
\end{aligned}$$

so,

$$\int_b^r \frac{\xi(s)}{g(s)}Z^2(s)\,ds \le C_2 + C_1\Phi^*(r), \quad (C_2>0). \tag{3.8}$$

Using (3.8) and Cauchy- Schwarz inequality, we obtain

$$\begin{aligned}
\Big(\int_b^r Z(s)ds\Big)^2 &\le \Big(\int_b^r \frac{\xi(s)}{g(s)}Z^2(s)\,ds\Big)\Big(\int_b^r \frac{g(s)}{\xi(s)}\,ds\Big) \\
&\le \Big[C_2 + C_1\Phi^*(r)\Big]\Big(\int_b^r \frac{g(s)}{\xi(s)}ds\Big).
\end{aligned}$$

It follows from (3.7) that for sufficiently large $r$,

$$\int_b^r \Theta(s)\,ds \le \int_b^r Z(\tau)\,d\tau \le C_3\Big[\Phi^*(r)\Big(\int_b^r \frac{g(s)}{\xi(s)}\,ds\Big)\Big]^{\frac{1}{2}},$$

which contradicts condition (3.6). Hence, (1.1) is oscillatory. □

**Corollary 3.2.** *If there exist* $\eta(r)$ *and* $\xi(r)$ *defined as in Theorem* 3. *such that*

$$\int^\infty \eta(s)\Phi(s)ds < \infty \quad \textit{and} \quad \lim_{r\to\infty}\Big(\int^r \frac{g(s)}{\xi(s)}ds\Big)^{-\frac{1}{2}}\int^r \Theta(s)\,ds = \infty,$$

*then* (1.1) *is oscillatory.*

**Remark 3.2.** Theorem 3.3 improves Theorem 3 in [11] and Theorem 2.5 in [13].

Using the same techniques, we may obtain a slightly different form of Theorem 3.3. Now, we state here for completeness.

**Theorem 3.4.** *Suppose that there exists* $\rho \in C^1([r_0,\infty),\mathbb{R})$ *and* $\phi \in \Im_0$ *such that* $\Theta(r) \ge 0$ *holds for sufficiently large* $r \ge r_0$. *If there exists a locally* *integrable bounded function* $\eta(t) : [r_0, \infty) \to \mathbb{R}^+$ *and* $\xi(r) = \int_{r_0}^r \eta(u)\,du + \xi(r_0)$ *such that*

$$\lim_{r\to\infty} \left[ \left( \int_{r_0}^r \Phi(s)ds \right) \left( \int_{r_0}^r \frac{g(s)}{\xi(s)}\,ds \right) \right]^{-\frac{1}{2}} \int_{r_0}^r \Theta(s)\,ds = \infty,$$

*where* $\Theta$ *and* $\Phi(r)$ *are defined as in Theorems* 3.2 *and* 3.3, *respectively, then* (1.1) *is oscillatory.*

**Corollary 3.3.** *If there exist* $\eta(r)$ *and* $\xi(r)$ *defined as in Theorem* 3.4 *such that*

$$\int^\infty \Phi(s)ds < \infty \quad \text{and} \quad \lim_{r\to\infty} \left( \int^r \frac{g(s)}{\xi(s)} ds \right)^{-\frac{1}{2}} \int^r \Theta(s)ds = \infty,$$

*then* (1.1) *is oscillatory.*

**Remark 3.3.** Some interesting corollaries can be obtained from Theorems 3.1-3.4, by choosing the appropriate function $\theta(r)$, the details are left to the reader.

**Example 3.1.** Consider the Laplace equation

$$\Delta y + \frac{1 + \alpha \sin|x|}{|x|^\beta} y = 0, \tag{3.9}$$

for $|x| > 1$, where $\alpha \in \mathbb{R}$, $0 < \beta \le 1$, $n \ge 2$, $\lambda(r, y) = 1$, $a(r) = 1$, and $k = 1$.

Let $\rho(r) = \frac{1}{2}(n-1)\omega_n r^{n-2}$. Then $\varphi(r) = r^{1-n}$, $g(r) = \omega_n$, and

$$P(r) = \omega_n \Big[ \frac{1 + \alpha \sin r}{r^\beta} - \frac{(n-1)(n-3)}{4r^2} \Big].$$

Hence,

$$\int^\infty \frac{ds}{g(s)} = \int^\infty P(s)ds = \infty.$$

By Corollary 3.1, (3.9) is oscillatory.

**Example 3.2.** Consider the elliptic differential equation

$$\frac{\partial}{\partial x_1}\Big(\frac{1}{|x| + |y|} \frac{\partial y}{\partial x_1}\Big) + \frac{\partial}{\partial x_1}\Big(\frac{1}{|x|} \frac{\partial y}{\partial x_2}\Big) + \frac{\gamma}{|x|^3}(y + \sqrt[3]{y}) = 0, \tag{3.10}$$

for $|x|>1$, where $\gamma \geq 1/2$ is a constant, $n=2$, $\lambda(r,y)=1/r$, $k=1$, and $a(r)=1/r$.

For Theorem 3.2, let $\rho(r)=0$, then we have $\varphi(r)=1$, $g(r)=2\pi$, and

$$P(r)=\frac{2\pi\gamma}{r^2},\quad \Theta(r)=\frac{2\pi\gamma}{r},\quad \int_1^r \frac{\Theta(s)}{g(s)}ds=\gamma\ln r.$$

So,

$$\int_1^r \frac{\Theta_+^2(s)}{g(s)}\exp\Big(2\int_1^s \frac{\Theta(\tau)}{g(\tau)}\,d\tau\Big)\,ds=2\pi\gamma^2\int_1^r s^{2(\gamma-1)}ds.$$

Now, we take $\phi(r)=1$ and $\theta(r)=1$, it is easy to check $\phi\in\Im_0$. Thus, all conditions of Theorem 3.2 are satisfied and hence (3.10) is oscillatory.

**Example 3.3.** Consider the semilinear elliptic equation

$$\frac{\partial}{\partial x_1}\Big(\frac{1}{|x|^{\frac14}}\frac{\partial y}{\partial x_1}\Big)+\frac{\partial}{\partial x_1}\Big(\frac{1+\cos^2 y}{|x|^{\frac14}}\frac{\partial y}{\partial x_2}\Big)+\frac{4|x|\sin|x|+4+\cos|x|}{8\,|x|^{\frac94}}(y+y^5)=0, \tag{3.11}$$

for $|x|>1$, where $n=2$, $\lambda(r,y)=1/r^{\frac14}$, $a(r)=1/r^{\frac14}$, and $k=1$.

For Corollary 3.2, let $\rho(r)=0$, we can get that $\varphi(r)=1$, $g(r)=2\pi r^{\frac34}$, and

$$P(r)=\frac{\pi(4r\sin r+4+\cos r)}{4r^{5/4}},\quad \Theta(r)=\frac{\pi(4+\cos r)}{r^{1/4}}\geq\frac{3\pi}{r^{1/4}},$$

and

$$\Phi(r)\leq\exp\Big(-6\int_1^r\frac{1}{s}ds\Big)=\frac{1}{r^6}.$$

Here, we take $\eta(r)=1$, and $\xi(r)=r$. Then

$$\int_1^\infty \eta(s)\Phi(s)ds\leq\int_1^\infty\frac{1}{s^6}ds<\infty,$$

and

$$\Big(\int_1^r\frac{g(s)}{\xi(s)}\,ds\Big)^{-\frac12}\int_1^r\Theta(s)ds\geq 3\Big(\frac{\pi}{2}\Big)^{1/2}\int_1^r s^{-1/4}ds\to\infty\quad \text{as}\, r\to\infty.$$

On the other hand, taking $\phi(r)=1$ and $\theta(r)=1$, it is easy to check $\phi\in\Im_0$. So all conditions of Corollary 3.2 are satisfied and hence (3.11) is oscillatory.

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