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SOME RESULTS ON HARMONIC BOEHMIANS

BY

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Abstract

Some classical theorems for harmonic functions, such as Liouville's theorem, are extended to the space of Boehmians. Also, existence and uniqueness theorems for Poisson's equation $\Delta u = F$, with F a Boehmian, are established.

1. Introduction

Solutions to Laplace's equation $\Delta u = 0$, where $\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_d^2}$, are called harmonic functions and appear in many areas of mathematics, as well as in physics.

By expanding our domain to the space of Schwartz distributions [9], no new solutions of Laplace's equation occur. However, if we consider the space of generalized functions known as Boehmians, new solutions to Laplace's equation, called harmonic Boehmians, exist (see [2], [5]). In this note, we will extend some classical theorems for harmonic functions to the space of harmonic Boehmians. More specifically, a uniqueness theorem and a Liouville type theorem will be established. Also, in the last section we consider Poisson's equation $\Delta u = F$, where F is a Boehmian.

2. Preliminaries

Let $C(\mathbb{R}^d)$ denote the space of all continuous functions on $\mathbb{R}^d$, and let $\mathcal{D}(\mathbb{R}^d)$ denote the subspace of $C(\mathbb{R}^d)$ of all infinitely differentiable func-

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tions with compact support. Let $\alpha = (\alpha_1, \ldots, \alpha_d)$, where α_j is a non-negative integer, be a multi-index. Then, $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_d$ and $D^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1} \ldots (\frac{\partial}{\partial x_d})^{\alpha_d}$. If $x, y \in \mathbb{R}^d$, then $x = (x_1, x_2, \ldots, x_d)$, $y = (y_1, y_2, \ldots, y_d)$, $x \cdot y = x_1 y_1 + x_2 y_2 + \ldots + x_d y_d$, and $||x|| = \sqrt{x \cdot x}$.

A sequence $\varphi_n \in \mathcal{D}(\mathbb{R}^d)$ is called a *delta sequence* provided:

(i) $\int \varphi_n = 1$ for all $n \in \mathbb{N}$,

(ii) $\int |\varphi_n| \leq M$ for some constant M and all $n \in \mathbb{N}$,

(iii) For every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $\varphi_n(x) = 0$ for $||x|| > \varepsilon$ and $n > n_\varepsilon$.

A pair of sequences (f_n, φ_n) is called a *quotient of sequences* if $f_n \in C(\mathbb{R}^d)$ for $n \in \mathbb{N}$, $\{\varphi_n\}$ is a delta sequence, and $f_k * \varphi_m = f_m * \varphi_k$ for all $k, m \in \mathbb{N}$, where $*$ denotes convolution:

$$(f * \varphi)(x) = \int_{\mathbb{R}^d} f(x-u)\varphi(u)du, \tag{2.1}$$

Two quotients of sequences (f_n, φ_n) and (g_n, ψ_n) are said to be equivalent if $f_k * \psi_m = g_m * \varphi_k$ for all $k, m \in \mathbb{N}$. A straightforward calculation shows that this is an equivalence relation. The equivalence classes are called Boehmians. The space of all Boehmians will be denoted by $\beta(\mathbb{R}^d)$ and a typical element of $\beta(\mathbb{R}^d)$ will be written as $F = \left[\frac{f_n}{\varphi_n}\right]$.

The operations of addition, scalar multiplication, and differentiation are defined as follows:

$$\left[\frac{f_n}{\varphi_n}\right] + \left[\frac{g_n}{\psi_n}\right] = \left[\frac{f_n * \psi_n + g_n * \varphi_n}{\varphi_n * \psi_n}\right] \tag{2.2}$$

$$\gamma\left[\frac{f_n}{\varphi_n}\right] = \left[\frac{\gamma f_n}{\varphi_n}\right], \text{ where } \gamma \in \mathbb{C} \tag{2.3}$$

$$D^\alpha\left[\frac{f_n}{\varphi_n}\right] = \left[\frac{f_n * D^\alpha \varphi_n}{\varphi_n * \varphi_n}\right] \tag{2.4}$$

Define the map $\iota : C(\mathbb{R}^d) \to \beta(\mathbb{R}^d)$ by

$$\iota(f) = \left[\frac{f * \varphi_n}{\varphi_n}\right], \tag{2.5}$$

where $\{\varphi_n\}$ is any fixed delta sequence.

It is not difficult to show that the mapping ι is an injection which preserves the algebraic properties of $C(\mathbb{R}^d)$. Thus, $C(\mathbb{R}^d)$ can be identified with a proper subspace of $\beta(\mathbb{R}^d)$. Likewise, the space of Schwartz distributions $\mathcal{D}'(\mathbb{R}^d)$ can be identified with a proper subspace of $\beta(\mathbb{R}^d)$. Using this identification, the Dirac measure δ corresponds to the Boehmian $\left[\frac{\varphi_n}{\varphi_n}\right]$, where $\{\varphi_n\}$ is any delta sequence.

For $\psi \in \mathcal{D}(\mathbb{R}^d)$ and $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta(\mathbb{R}^d)$, $F * \psi$ is defined as

$$F * \psi = \left[\frac{f_n * \psi}{\varphi_n}\right]. \tag{2.6}$$

Let Ω be an open subset of $\mathbb{R}^d$. A Boehmian F is said to *vanish on* Ω, provided that there exists a delta sequence $\{\varphi_n\}$ such that $F * \varphi_n \in C(\mathbb{R}^d)$ for all $n \in \mathbb{N}$, and $F * \varphi_n \to 0$ uniformly on compact subsets of Ω as $n \to \infty$.

The *support of a Boehmian* F is the complement of the largest open set on which F vanishes. The space of all Boehmians with compact support will be denoted by $\beta_c(\mathbb{R}^d)$.

Since $\beta_c(\mathbb{R}^d)$ will be important later on, we present Theorem 2.1 below, which, among other things, will show that $\beta_c(\mathbb{R}^d)\backslash\mathcal{E}'(\mathbb{R}^d)$ is nonempty ($\mathcal{E}'(\mathbb{R}^d)$ is the space of distributions with compact support).

The series $\sum_{\alpha\in\mathbb{N}^d} c_\alpha D^\alpha \delta(x_1, x_2, \ldots, x_d)$ is said to *converge* in $\beta(\mathbb{R}^d)$ if for some delta sequence $\{\varphi_n\}$, for each $k \in \mathbb{N}$, $\sum_{|\alpha|\le n} c_\alpha D^\alpha \varphi_k(x_1, x_2, \ldots, x_d)$ converges uniformly on compact sets of $\mathbb{R}^d$ as $n \to \infty$.

Theorem 2.1.(See [7]) *The series $\sum_{\alpha\in\mathbb{N}^d} c_\alpha D^\alpha \delta(x_1, x_2, \ldots, x_d)$ converges in $\beta(\mathbb{R}^d)$ provided that $C_I\{\frac{1}{\nu_n}\}$ is not quasi-analytic, where*

$$\nu_n = \max\{|c_{\alpha_1\alpha_2\ldots\alpha_d}| : \sum_{j=1}^{d} \alpha_j = n\}.$$

For an introduction to the notion of quasi-analytic classes, the reader is referred to [8].

By using the definition of support, it is clear that if

$$F = \sum_{\alpha \in \mathbb{N}^d} c_\alpha D^\alpha \delta(x_1, x_2, \ldots, x_d), \tag{2.7}$$

then the support of F is the origin. If F is a distribution with support the origin, then F has a representation similar to equation (2.7). However, only finitely many of the c_α's are nonzero.

3. Harmonic Boehmians

Both Burzyk [2] and Mikusiński [5] showed that there exist Boehmian solutions to Laplace's equation that are not classical solutions. In this section we will extend some of the classical theorems for harmonic functions to harmonic Boehmians.

Let $L^2(\mathbb{R}^d)$ denote the class of all complex measurable functions f on $\mathbb{R}^d$ such that $\int |f(x)|^2 dx < \infty$. A Boehmian $F = \left[\frac{f_n}{\varphi_n}\right]$ is an element of $\beta_{L^2}(\mathbb{R}^d)$ provided that $f_n \in L^2(\mathbb{R}^d)$, $n \in \mathbb{N}$. By using the mapping (2.5), $L^2(\mathbb{R}^d)$ can be considered a subspace of $\beta_{L^2}(\mathbb{R}^d)$.

A complex-valued function f is called *slowly increasing* if there exists a polynomial p on $\mathbb{R}^d$ such that $\frac{f(x)}{p(x)}$ is bounded. The space of all slowly increasing continuous functions on $\mathbb{R}^d$ is denoted by $\mathcal{T}(\mathbb{R}^d)$. A Boehmian $F = \left[\frac{f_n}{\varphi_n}\right]$ is an element of $\beta_{\mathcal{T}}(\mathbb{R}^d)$ provided that $f_n \in \mathcal{T}(\mathbb{R}^d)$, $n \in \mathbb{N}$. Elements of $\beta_{\mathcal{T}}(\mathbb{R}^d)$ are called *tempered Boehmians*. By using the mapping (2.5), the space $\mathcal{T}(\mathbb{R}^d)$ as well as the space of tempered distributions [9] can both be considered subspaces of $\beta_{\mathcal{T}}(\mathbb{R}^d)$.

Both $\beta_{L^2}(\mathbb{R}^d)$ and $\beta_{\mathcal{T}}(\mathbb{R}^d)$ contain objects which are neither functions nor distributions. Indeed, $\beta_c(\mathbb{R}^d)$ is a subspace of both spaces. This can be seen by using the fact that if $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta_c(\mathbb{R}^d)$, then for each $n \in \mathbb{N}$ the support of f_n is compact.

Theorem 3.1. *If $u \in \beta_{L^2}(\mathbb{R}^d)$ and $\Delta u = 0$, then $u = 0$.*

Proof. Suppose $f \in L^2(\mathbb{R}^d)$ such that $\Delta f = 0$. Since $L^2(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d)$, f is a classical harmonic function. By applying Hölder's inequality to the mean-value property with respect to volume measure [1, p.6] and then taking the limit, we see that $f = 0$. Thus, the theorem is valid for any harmonic L^2 function.

Now, let $u = \left[\frac{f_n}{\varphi_n}\right] \in \beta_{L^2}(\mathbb{R}^d)$ such that $\Delta u = 0$. Thus, $f_n \in L^2(\mathbb{R}^d)$ such that $\Delta f_n = 0$, $n \in \mathbb{N}$. From above, $f_n = 0$ for all $n \in \mathbb{N}$. Hence $u = 0$. □

The Fourier transform may be defined for each tempered Boehmian. The following will be needed for the proof of Theorem 3.5.

A complex-valued infinitely differentiable function f is called *rapidly decreasing* if

$$\sup_{|\alpha| \le m} \sup_{x \in \mathbb{R}^d} (1 + x_1^2 + \ldots + x_d^2)^m |D^\alpha f(x)| < \infty \tag{3.1}$$

for every nonnegative integer m. The space of all rapidly decreasing functions on $\mathbb{R}^d$ is denoted by $\mathcal{S}(\mathbb{R}^d)$.

Let $f \in \mathcal{T}(\mathbb{R}^d)$. The Fourier transform of f, denoted $\widehat{f}$, is the distribution defined by $\widehat{f}(\varphi) = f(\widehat{\varphi})$ where $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $\widehat{\varphi}(x) = \int_{\mathbb{R}^d} \varphi(t) e^{-ix\cdot t} dt$.

Definition 3.2. (See [6]) Let $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta_{\mathcal{T}}(\mathbb{R}^d)$. The Fourier transform of F, denoted $\widehat{F}$, is defined as $\widehat{F} = lim_{n\to\infty} \widehat{f_n}$, where the limit is taken in $\mathcal{D}'(\mathbb{R}^d)$.

The above limit exists and is independent of the representative. The Fourier transform of a tempered Boehmian has similar properties as the Fourier transform of a tempered distribution.

Lemma 3.3. *Suppose* $F = \left[\frac{q_n}{\varphi_n}\right] \in \beta(\mathbb{R}^d)$, *where* q_n *is a polynomial. Then,* F *is a polynomial.*

Proof. Since $F = \left[\frac{q_n}{\varphi_n}\right] \in \beta(\mathbb{R}^d)$,

$$\deg q_n = \deg q_n * \varphi_k = \deg q_k * \varphi_n = \deg q_k, \text{ for all } k, n \in \mathbb{N}.$$

By using the Fourier transform and the above, there exist $m \in \mathbb{N}$, and for each $n \in \mathbb{N}$, a set of complex numbers $\{c_{\alpha n}\}_{\alpha \in \mathbb{N}^d}$ $(|\alpha| \le m)$ such that, for each $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$\sum_{|\alpha|\leq m} c_{\alpha n} D^\alpha \varphi(\mathbf{0}) = \widehat{q}_n(\varphi) \to \widehat{F}(\varphi) \text{ as } n \to \infty. \tag{3.2}$$

Since (3.2) is valid for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, we obtain that $c_{\alpha n} \to a_\alpha$ as $n \to \infty$, where a_α is a set of complex numbers ($|\alpha| \leq m$). Thus, $\widehat{F}(\varphi) = \sum_{|\alpha|\leq m} a_\alpha D^\alpha \varphi(\mathbf{0})$, $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Hence, F is a polynomial. □

A sequence of tempered Boehmians $\{F_n\}$ is said to be δ-*convergent* to a tempered Boehmian F, denoted $F_n \xrightarrow{\delta} F$, if there exists a delta sequence $\{\varphi_n\}$ such that $F_n * \varphi_k$, $F * \varphi_k \in \mathcal{T}(\mathbb{R}^d)$ $(k, n \in \mathbb{N})$, and for each $k \in \mathbb{N}$, there exists a polynomial p_k such that $\frac{F_n*\varphi_k - F*\varphi_k}{p_k} \to 0$ uniformly on $\mathbb{R}^d$ as $n \to \infty$.

It can be shown that if $F_n \xrightarrow{\delta} F$, then $\widehat{F}_n \to \widehat{F}$ in $\mathcal{D}'(\mathbb{R}^d)$. Thus, by examining the proof of the previous lemma, we obtain the following proposition. Although the proposition is not about harmonic Boehmians, it is an interesting result about Boehmians.

Proposition 3.4. *Let $m \in \mathbb{N}$ and $\{q_n\}$ be a sequence of polynomials such that* $\deg q_n \leq m$. *If $q_n \xrightarrow{\delta} F$ for some $F \in \beta_{\mathcal{T}}(\mathbb{R}^d)$, then F is a polynomial. Moreover, $q_n \to F$ uniformly on compact subsets of $\mathbb{R}^d$ as $n \to \infty$.*

Theorem 3.5. *Let F be a harmonic Boehmian. Then, F is tempered if and only if F is a polynomial.*

Proof. Suppose that

$$F = \left[\frac{f_n}{\varphi_n}\right] \in \beta_{\mathcal{T}}(\mathbb{R}^d) \tag{3.3}$$

such that

$$\Delta F = 0 \tag{3.4}$$

By (3.3) and (3.4),

$$\Delta f_n = 0 \tag{3.5}$$

and

$$|f_n(x)| \leq |p_n(x)|, \tag{3.6}$$

where $p_n(x)$ is a polynomial.

From the classical theory of harmonic functions, we obtain

$$f_n = q_n, \tag{3.7}$$

where q_n is a polynomial.

Thus, by the previous lemma, F is a polynomial.

The other direction is routine. $\square$

Let $L^\infty(\mathbb{R}^d)$ denote the class of all essentially bounded (with respect to Lebesgue measure) functions on $\mathbb{R}^d$. A Boehmian $F = \left[\frac{f_n}{\varphi_n}\right]$ is an element of $\beta_{L^\infty}(\mathbb{R}^d)$ provided that $f_n \in L^\infty(\mathbb{R}^d), n \in \mathbb{N}$. Elements of $\beta_{L^\infty}(\mathbb{R}^d)$ are said to be *bounded*. By using the natural mapping (2.5), $L^\infty(\mathbb{R}^d)$ can be viewed as a subspace of $\beta_{L^\infty}(\mathbb{R}^d)$. Moreover, the space of bounded distributions on $\mathbb{R}^d$ [9] can be thought of as a subspace of $\beta_{L^\infty}(\mathbb{R}^d)$.

Theorem 3.6. (Liouville) *Every bounded harmonic Boehmian is constant.*

Proof. Let $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta_{L^\infty}(\mathbb{R}^d)$ such that $\Delta F = 0$. Thus, $f_n \in L^\infty(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d)$ and $\Delta f_n = 0$, $n \in \mathbb{N}$. By the classical Liouville's theorem, there exists a sequence of complex numbers $\{c_n\}$ such that $f_n(x) = c_n$, for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}^d$.

Now, for all $k, n \in \mathbb{N}$,

$$\begin{aligned} c_n &= c_n \int_{\mathbb{R}^d} \varphi_k(t)dt \\ &= \int_{\mathbb{R}^d} f_n(x-t)\varphi_k(t)dt \\ &= \int_{\mathbb{R}^d} f_k(x-t)\varphi_n(t)dt \\ &= c_k \int_{\mathbb{R}^d} \varphi_n(t)dt \quad = \quad c_k. \end{aligned}$$

Thus, $F = \left[\frac{f_n}{\varphi_n}\right] = \left[\frac{c*\varphi_n}{\varphi_n}\right]$, for some constant c. This completes the proof. $\square$

4. Poisson's Equation

In this section we investigate solutions to Poisson's equation $\Delta u = F$, where F is a Boehmian. For convenience, we consider $\beta(\mathbb{R}^3)$. However, the results in this section are valid for $\beta(\mathbb{R}^d)$, $d \geq 3$.

A Boehmian $F = \left[\frac{f_n}{\varphi_n}\right]$ is an element of $\beta_{\mathcal{S}}(\mathbb{R}^3)$ provided that $f_n \in \mathcal{S}(\mathbb{R}^3)$, $n \in \mathbb{N}$. Elements of $\beta_{\mathcal{S}}(\mathbb{R}^3)$ are called *rapidly decreasing Boehmians*. By the natural mapping (2.5), $\mathcal{S}(\mathbb{R}^3)$ can be viewed as a subspace of $\beta_{\mathcal{S}}(\mathbb{R}^3)$. The space of rapidly decreasing distributions [9] can also be identified with a subspace of $\beta_{\mathcal{S}}(\mathbb{R}^3)$ by using the mapping (2.5).

Convolution can be extended to $\beta_{\mathcal{S}}(\mathbb{R}^3) \times \beta_{\mathcal{T}}(\mathbb{R}^3)$. Let $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta_{\mathcal{S}}(\mathbb{R}^3)$ and $G = \left[\frac{g_n}{\psi_n}\right] \in \beta_{\mathcal{T}}(\mathbb{R}^3)$. Then $F * G = \left[\frac{f_n * g_n}{\varphi_n * \psi_n}\right]$, where f_n and g_n are viewed as elements of $\mathcal{S}(\mathbb{R}^3)$ and $\mathcal{S}'(\mathbb{R}^3)$, respectively, and the convolution is defined as in distribution theory.

Let $F, G \in \beta_{\mathcal{S}}(\mathbb{R}^3)$ and $H, J \in \beta_{\mathcal{T}}(\mathbb{R}^3)$. Then,

(i) $F * H \in \beta_{\mathcal{T}}(\mathbb{R}^3)$ (4.1)

(ii) $F * (G * H) = (F * G) * H$ (4.2)

(iii) $F * (H + J) = F * H + F * J$ (4.3)

(iv) $D^\alpha(F * H) = D^\alpha F * H = F * D^\alpha H.$ (4.4)

Theorem 4.1. *Let $F \in \beta_{\mathcal{S}}(\mathbb{R}^3)$. Then*

(a) *There exists a tempered Boehmian u such that $\Delta u = F$.*

(b) *If $u_1, u_2 \in \beta_{\mathcal{T}}(\mathbb{R}^3)$ such that $\Delta u_j = F$ (for $j = 1, 2$), then there exists a unique harmonic polynomial p such that $u_1 = u_2 + p$.*

Proof. Let E be the fundamental solution of Laplace's equation. That is, $E(x) = \frac{-1}{4\pi||x||} \in \mathcal{S}'(\mathbb{R}^3)$ and $\Delta E = \delta$. Now, let $F \in \beta_{\mathcal{S}}(\mathbb{R}^3)$ and $u = F * E$. Then, $u \in \beta_{\mathcal{T}}(\mathbb{R}^3)$ and $\Delta u = \Delta(F * E) = F * \Delta E = F * \delta = F$. This completes the proof for part (a).

For part (b), let $F \in \beta_{\mathcal{S}}(\mathbb{R}^3)$. Suppose that $u_j \in \beta_{\mathcal{T}}(\mathbb{R}^3)$ such that $\Delta u_j = F$, for j = 1, 2. Then, $\Delta(u_1 - u_2) = 0$ and $u_1 - u_2 \in \beta_{\mathcal{T}}(\mathbb{R}^3)$. So, by Theorem 3.5, $u_1 - u_2 = p$, for some polynomial p. Thus, $u_1 = u_2 + p$, where p is a harmonic polynomial. Clearly, the polynomial p is unique. This completes the proof of the theorem. $\square$

The class of all continuous functions on $\mathbb{R}^3$ which vanish at infinity is denoted by $C_0(\mathbb{R}^3)$. A Boehmian $F = \left[\frac{f_n}{\varphi_n}\right]$ is an element of $\beta_0(\mathbb{R}^3)$ provided that $f_n \in C_0(\mathbb{R}^3)$, $n \in \mathbb{N}$. Elements of $\beta_0(\mathbb{R}^3)$ are said to *vanish at infinity*. $C_0(\mathbb{R}^3)$ can be thought of as a subspace of $\beta_0(\mathbb{R}^3)$.

Comparing the various spaces used in this paper, we have

$$\beta_c(\mathbb{R}^d) \subset \beta_{\mathcal{S}}(\mathbb{R}^d) \subset \beta_0(\mathbb{R}^d) \subset \beta_{L^\infty}(\mathbb{R}^d) \subset \beta_{\mathcal{T}}(\mathbb{R}^d) \subset \beta(\mathbb{R}^d)$$

where the inclusions are proper.

Theorem 4.2. *Let $F \in \beta_c(\mathbb{R}^3)$. Then there exists a unique $u \in \beta_0(\mathbb{R}^3)$ such that $\Delta u = F$.*

Proof. Let $F \in \beta_c(\mathbb{R}^3)$. Then, $u = F * E$, where E is the fundamental solution of Laplace's equation, is the desired solution.

Suppose $u_j \in \beta_0(\mathbb{R}^3)$ and $\Delta u_j = F$, for $j = 1, 2$. Thus, $\Delta(u_1 - u_2) = 0$ and $u_1 - u_2 \in \beta_0(\mathbb{R}^3) \subset \beta_{L^\infty}(\mathbb{R}^3)$. By Liouville's theorem, $u_1 - u_2 = c$, for some constant c. Since $c = u_1 - u_2 \in \beta_0(\mathbb{R}^3), c = 0$. This shows uniqueness and completes the proof. $\square$

Let $F \in \beta_c(\mathbb{R}^3)$. When does the equation $\Delta u = F$ have a solution in $\beta_c(\mathbb{R}^3)$? Notice that $\Delta E = \delta, E(x) = \frac{-1}{4\pi|x||} \notin \beta_c(\mathbb{R}^3)$. $E(x)$ is the unique solution in $\beta_0(\mathbb{R}^d)$.

For distributions, we have the following theorem.

Theorem 4.3.(See [4]) *If $f \in \mathcal{E}'(\mathbb{R}^3)$ then the equation $\Delta u = f$ has a solution $u \in \mathcal{E}'(\mathbb{R}^3)$ if and only if $\frac{\widehat{f}(z_1,z_2,z_3)}{z_1^2+z_2^2+z_3^2}$ is an entire function. The solution is then uniquely determined and ch supp u = ch supp f. (ch A denotes the closed convex hull of A.)*

Conjecture: Let $F \in \beta_c(\mathbb{R}^3)$. There exists $u \in \beta_c(\mathbb{R}^3)$ such that $\Delta u = F$ if and only if $\frac{\widehat{F}(z_1,z_2,z_3)}{z_1^2+z_2^2+z_3^2}$ is an entire function. In this case, ch supp u = ch supp F .

Burzyk [3] proved a Paley-Wiener type theorem for $\beta_c(\mathbb{R}^d)$, where $d = 1$. What is needed to prove the above conjecture is the Paley-Wiener-Burzyk theorem for $\beta_c(\mathbb{R}^3)$. The proof would then follow by using the Paley-Wiener-Burzyk theorem for $\beta_c(\mathbb{R}^3)$ in conjunction with Lemma 7.3.3 in [4].

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