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VARIATIONAL ITERATION METHOD FOR SOLVING DISCRETE KDV EQUATION

BY

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Abstract

In this paper, we apply the variational iteration method (VIM) for solving discrete KdV equation which arises in various physical phenomena related to physical and applied sciences. Numerical results show the efficiency of the proposed algorithm.

1. Introduction

This paper is devoted to the study of discrete KdV equation which arises in various physical phenomena including vibrations in lattices, currents in electrical networks, biological chains, modern physics, astrophysics and applied sciences; see [1, 5, 7, 8, 9, 10, 11] and the references therein. Several techniques including homotopy perturbation, variational iteration and exp-function have been applied to solve such problems [1, 5, 7, 8, 9, 10, 11]. It is worth mentioning that the differential difference equations were very difficult to solve but due to the formulation of He's homotopy perturbation, variational iteration and exp-function methods it is now very convenient to handle such problems. The basic motivation of this paper is to apply the variational iteration method (VIM) for solving a physical problem whose governing equation is a discrete KdV [7]. The VIM was developed and formulated by He [2, 3, 4] and has been extremely useful in solving the

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complex diversified nonlinear problems, see [2, 3, 4, 6, 8]. Numerical results show the complete reliability of the proposed technique.

2. Variational Iteration Method (VIM)

To illustrate the basic concept of the He's VIM, we consider the following general differential equation

$$Lu + Nu = g(x), \tag{1}$$

where L is a linear operator, N a nonlinear operator and $g(x)$ is the inhomogeneous term. According to variational iteration method [2, 3, 4, 6, 8], we can construct a correction functional as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(Lu_n(s) + N\tilde{u}_n(x) - g(s))ds, \tag{2}$$

where λ is a Lagrange multiplier [2, 3, 4], which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation. i.e. $\delta\tilde{u}_n = 0$; (2) is called a correction functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of variational iteration method and its applicability for various kinds of differential equations are given in [2, 3, 4]. In this method, it is required first to determine the Lagrange multiplier λ optimally. The successive approximation u_{n+1}, $n \geq 0$ of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0, consequently, the solution is given by $u = \lim_{n\to\infty} u_n$.

3. Numerical Applications

In this section, we apply He's variational iteration method (VIM) for solving the governing equation of a physical problem which is a discrete KdV and is given by [7]:

$$\frac{du_n}{dt} = u_n^2(u_{n+1} - u_{n-1}),$$

with initial conditions

$$u_n(0) = 1 - \frac{1}{n^2}.$$

The exact solution of the above problem is given by

$$u_n(t) = 1 - \frac{1}{(n+2t)^2}.$$

The correction functional is given by

$$\begin{aligned} u_{n,m+1}(t) &= \Big(1 - \frac{1}{n^2}\Big) \\ &\quad + \int_0^t \lambda(s)\Big(\frac{du_{n,m}(s)}{dt} - \tilde{u}^2_{n,m}(s)\Big(\tilde{u}_{n,m+1}(s) - \tilde{u}_{n-1,m}(s)\Big)\Big)ds. \end{aligned}$$

Making the correction functional stationary, the Lagrange multiplier can be identified as $\lambda(s) = -1$, we get the following iterative scheme:

$$u_{n,m+1}(t) = \Big(1 - \frac{1}{n^2}\Big) - \int_0^t \Big(\frac{du_{n,m}(s)}{dt} - \tilde{u}^2_{n,m}(s)\Big(\tilde{u}_{n,m+1}(s) - \tilde{u}_{n-1,m}(s)\Big)\Big)ds.$$

Consequently, following approximants are obtained

$$\begin{aligned} u_{n,0}(t) &= 1 - \frac{1}{n^2}, \\ u_{n,1}(t) &= 1 - \frac{1}{n^2} + \frac{4}{n^3}t, \\ u_{n,2}(t) &= 1 - \frac{1}{n^2} + \frac{4}{n^3}t - \frac{12}{n^4}t^2, \\ u_{n,3}(t) &= 1 - \frac{1}{n^2} + \frac{4}{n^3}t - \frac{12}{n^4}t^2 + \frac{32}{n^5}t^3, \\ &\vdots \end{aligned}$$

The series solution is given by

$$u_n(t) = 1 - \frac{1}{n^2} + \frac{4}{n^3}t - \frac{12}{n^4}t^2 + \frac{32}{n^5}t^3 + \cdots,$$

and the closed form solution is given as

$$u_n(t) = 1 - \frac{1}{(n+2t)^2}.$$

4. Conclusion

In this paper, we applied variational iteration method (VIM) for solving a physical problem related to a discrete KdV equation. The method is applied in a direct way without using linearization, transformation, discretization, perturbation or restrictive assumptions. The fact that the proposed technique solves nonlinear problems without using Adomian's polynomials is a clear advantage of this algorithm over the decomposition method.

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