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ON A CLASS OF MULTIVALENT ANALYTIC FUNCTIONS ASSOCIATED WITH AN INTEGRAL OPERATOR

BY

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Abstract

The object of the present paper is to introduce and study a new class of multivalent analytic functions associated with an integral operator Q_β^α which was investigated recently by Jung, Kim and Srivastava [J.Math.Anal.Appl. 176(1993), 138-147].

1. Introduction and Preliminaries

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{n+p} \quad (p \in N = \{1, 2, 3, \cdots\}), \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$.

Suppose that $f(z)$ and $g(z)$ are analytic in U. We say that the function $f(z)$ is subordinate to $g(z)$ in U, and we write $f(z) \prec g(z) \quad (z \in U)$, if there exists an analytic function $w(z)$ in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z)) \quad (z \in U)$. If $g(z)$ is univalent in U, then the following equivalence relationship holds true.

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

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For functions $f_j(z) \in A(p) \quad (j = 1, 2)$ given by

$$f_j(z) = z^p + \sum_{n=1}^{\infty} a_{n,j} z^{n+p} \quad (j = 1, 2),$$

we define the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = z^p + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} z^{n+p} = (f_2 * f_1)(z).$$

Recently, Jung, Kim and Srivastava [3] introduced the following integral operator $Q_\beta^\alpha : A(1) \to A(1)$:

$$Q_\beta^\alpha f(z) = \binom{\alpha+\beta}{\alpha} \frac{\alpha}{z^\beta} \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t) dt$$

$$(\alpha > 0, \beta > -1; f(z) \in A(1)). \tag{1.2}$$

Some interesting subclasses of analytic functions, associated with the operator Q_β^α, have been considered by Jung et al. [3], Aouf et al. [1], Liu [4, 6, 7], Liu and Owa [5] and others.

Motivated by Jung, Kim and Srivastava's work [3]. we consider a linear operator $Q_\beta^\alpha : A(p) \to A(p)$ as following:

$$Q_\beta^\alpha f(z) = \binom{p+\alpha+\beta-1}{p+\beta-1} \frac{\alpha}{z^\beta} \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t) dt$$

$$(\alpha \geq 0, \beta > -1; f(z) \in A(p)). \tag{1.3}$$

It is easily verified from the definition (1.3) that

$$z(Q_\beta^\alpha f(z))' = (\alpha+\beta+p-1) Q_\beta^{\alpha-1} f(z) - (\alpha+\beta-1) Q_\beta^\alpha f(z). \tag{1.4}$$

Let P be the class of functions $h(z)$ with $h(0) = 1$, which are analytic and convex univalent in U.

Now we introduce the following subclass of $A(p)$ associated with the operator Q^{α}_{β}.

Definition. A function $f(z) \in A(p)$ is said to be in the class $M_{p,\alpha,\beta}(\lambda; h)$ if it satisfies the subordination condition

$$(1-\lambda)z^{-p}Q^{\alpha}_{\beta}f(z) + \frac{\lambda}{p}z^{-p+1}(Q^{\alpha}_{\beta}f(z))' \prec h(z), \tag{1.5}$$

where λ is a complex number and $h(z) \in P$.

A function $f(z) \in A(1)$ is said to be in the class $S^*(\rho)$ if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \rho \quad (z \in U) \tag{1.6}$$

for some $\rho(\rho < 1)$. When $0 \leq \rho < 1$, $S^*(\rho)$ is the class of starlike functions of order ρ in U. A function $f(z) \in A(1)$ is said to be prestarlike of order ρ in U if

$$\frac{z}{(1-z)^{2(1-\rho)}} * f(z) \in S^*(\rho) \quad (\rho < 1). \tag{1.7}$$

We note this class by $R(\rho)$ (see [10]). Clearly a function $f(z) \in A(1)$ is in the class $R(0)$ if and only if $f(z)$ is convex univalent in U and

$$R\left(\frac{1}{2}\right) = S^*\left(\frac{1}{2}\right).$$

We need the following lemmas in order to derive our main results for the class $M_{p,\alpha,\beta}(\lambda; h)$.

Lemma 1. *Let $g(z)$ be analytic in U and $h(z)$ be analytic and convex univalent in U with $h(0) = g(0)$. If*

$$g(z) + \frac{1}{\mu}zg'(z) \prec h(z), \tag{1.8}$$

where $Re\mu \geq 0$ and $\mu \neq 0$, then

$$g(z) \prec \widetilde{h}(z) = \mu z^{-\mu}\int_0^z t^{\mu-1}h(t)dt \prec h(z)$$

and $\widetilde{h}(z)$ is the best dominant of (1.8).

Lemma 2. *Let $\rho < 1$, $f(z) \in S^*(\rho)$ and $g(z) \in R(\rho)$. Then, for any analytic function $F(z)$ in U,*

$$\frac{g * (fF)}{g * f}(U) \subset \overline{co}(F(U)),$$

where $\overline{co}(F(U))$ denotes the closed convex hull of $F(U)$.

Lemma 1 is due to Miller and Mocanu [9] (see also [2]) and Lemma 2 can be found in Ruscheweyh [10].

Lemma 3.(see [8]) *Let $g(z) = 1 + \sum_{n=k}^{\infty} b_n z^n \quad (k \in N)$ be analytic in U. If $Re\{g(z)\} > 0 \quad (z \in U)$, then*

$$\mathrm{Re}\{g(z)\} \geq \frac{1 - |z|^k}{1 + |z|^k} \quad (k \in N; z \in U).$$

2. Main Results

Theorem 1. *Let $0 \leq \lambda_1 < \lambda_2$. Then*

$$M_{p,\alpha,\beta}(\lambda_2; h) \subset M_{p,\alpha,\beta}(\lambda_1; h).$$

Proof. Let $0 \leq \lambda_1 < \lambda_2$ and suppose that

$$g(z) = z^{-p} Q_\beta^\alpha f(z) \tag{2.1}$$

for $f(z) \in M_{p,\alpha,\beta}(\lambda_2; h)$. Then the function $g(z)$ is analytic in U with $g(0) = 1$. Differentiating both sides of (2.1) with respect to z and using (1.5), we have

$$\begin{aligned} &(1 - \lambda_2) z^{-p} Q_\beta^\alpha f(z) + \frac{\lambda_2}{p} z^{-p+1} (Q_\beta^\alpha f(z))' \\ &= g(z) + \frac{\lambda_2}{p} z g'(z) \prec h(z). \end{aligned} \tag{2.2}$$

Hence an application of Lemma 1 yields

$$g(z) \prec h(z). \tag{2.3}$$

Noting that $0 \le \frac{\lambda_1}{\lambda_2} < 1$ and that $h(z)$ is convex univalent in U, it follows from (2.1) to (2.3) that

$$\begin{aligned}&(1-\lambda_1)z^{-p}Q_\beta^\alpha f(z)+\frac{\lambda_1}{p}z^{-p+1}(Q_\beta^\alpha f(z))'\\&=\frac{\lambda_1}{\lambda_2}\left((1-\lambda_2)z^{-p}Q_\beta^\alpha f(z)+\frac{\lambda_2}{p}z^{-p+1}(Q_\beta^\alpha f(z))'\right)+\left(1-\frac{\lambda_1}{\lambda_2}\right)g(z)\\&\prec h(z).\end{aligned}$$

Thus $f(z) \in M_{p,\alpha,\beta}(\lambda_1; h)$ and the proof of Theorem 1 is completed.

Theorem 2. *Let $\lambda > 0, \gamma > 0$ and $f(z) \in M_{p,\alpha,\beta}(\lambda; \gamma h + 1 - \gamma)$. If $\gamma \le \gamma_0$, where*

$$\gamma_0=\frac{1}{2}\left(1-\frac{p}{\lambda}\int_0^1\frac{u^{\frac{p}{\lambda}-1}}{1+u}du\right)^{-1}, \tag{2.4}$$

then $f(z) \in M_{p,\alpha,\beta}(0; h)$. The bound γ_0 is sharp when $h(z) = \frac{1}{1-z}$.

Proof. Let us define

$$g(z)=z^{-p}Q_\beta^\alpha f(z) \tag{2.5}$$

for $f(z) \in M_{p,\alpha,\beta}(\lambda; \gamma h + 1 - \gamma)$ with $\lambda > 0$ and $\gamma > 0$. Then we have

$$\begin{aligned}g(z)+\frac{\lambda}{p}zg'(z) &= (1-\lambda)z^{-p}Q_\beta^\alpha f(z)+\frac{\lambda}{p}z^{-p+1}(Q_\beta^\alpha f(z))'\\&\prec \gamma h(z)+1-\gamma.\end{aligned}$$

Hence an application of Lemma 1 yields

$$g(z)\prec\frac{\gamma p}{\lambda}z^{-\frac{p}{\lambda}}\int_0^z t^{\frac{p}{\lambda}}h(t)dt+1-\gamma=(h*\psi)(z), \tag{2.6}$$

where

$$\psi(z)=\frac{\gamma p}{\lambda}z^{-\frac{p}{\lambda}}\int_0^z\frac{t^{\frac{p}{\lambda}-1}}{1-t}dt+1-\gamma. \tag{2.7}$$

If $0 < \gamma \le \gamma_0$, where $\gamma_0 > 1$ is given by (2.4), then it follows from (2.7) that

$$\begin{aligned}\text{Re}\psi(z) &= \frac{\gamma p}{\lambda}\int_0^1 u^{\frac{p}{\lambda}-1}\text{Re}\left(\frac{1}{1-uz}\right)du+1-\gamma \\ &> \frac{\gamma p}{\lambda}\int_0^1 \frac{u^{\frac{p}{\lambda}-1}}{1+u}du+1-\gamma \\ &\geq \frac{1}{2} \quad (z\in U).\end{aligned}$$

Now,by using the Herglotz representation for $\psi(z)$, from (2.5) and (2.6) we arrive at

$$z^{-p}Q_\beta^\alpha f(z) \prec (h*\psi)(z) \prec h(z)$$

because $h(z)$ is convex univalent in U. This shows that $f(z)\in M_{p,\alpha,\beta}(0;h)$.

For $h(z)=\frac{1}{1-z}$ and $f(z)\in A(p)$ defined by

$$z^{-p}Q_\beta^\alpha f(z) = \frac{\gamma p}{\lambda}z^{-\frac{p}{\lambda}}\int_0^z \frac{t^{\frac{p}{\lambda}-1}}{1-t}dt+1-\gamma,$$

it is easy to verify that

$$(1-\lambda)z^{-p}Q_\beta^\alpha f(z)+\frac{\lambda}{p}z^{-p+1}(Q_\beta^\alpha f(z))' = \gamma h(z)+1-\gamma.$$

Thus $f(z)\in M_{p,\alpha,\beta}(\lambda;\gamma h+1-\gamma)$. Also, for $\gamma>\gamma_0$, we have

$$\text{Re}\{z^{-p}Q_\beta^\alpha f(z)\} \to \frac{\gamma p}{\lambda}\int_0^1 \frac{u^{\frac{p}{\lambda}-1}}{1+u}du+1-\gamma < \frac{1}{2} \quad (z\to -1),$$

which implies that $f(z)\notin M_{p,\alpha,\beta}(0;h)$. Hence the bound γ_0 cannot be increased when $h(z)=\frac{1}{1-z}$.

Theorem 3. *Let* $f(z)\in M_{p,\alpha,\beta}(\lambda;h)$,

$$g(z)\in A(p) \text{ and } Re\{z^{-p}g(z)\} > \frac{1}{2} \quad (z\in U). \tag{2.8}$$

Then

$$(f*g)(z)\in M_{p,\alpha,\beta}(\lambda;h)).$$

Proof. For $f(z) \in M_{p,\alpha,\beta}(\lambda; h)$ and $g(z) \in A(p)$, we have

$$\begin{aligned}
&(1-\lambda)z^{-p}Q_\beta^\alpha(f*g)(z) + \frac{\lambda}{p}z^{-p+1}(Q_\beta^\alpha(f*g)(z))' \\
&= (1-\lambda)(z^{-p}g(z)) * (z^{-p}Q_\beta^\alpha f(z)) + \frac{\lambda}{p}(z^{-p}g(z)) * (z^{-p+1}(Q_\beta^\alpha f(z))') \\
&= (z^{-p}g(z)) * \psi(z), \qquad (2.9)
\end{aligned}$$

where

$$\psi(z) = (1-\lambda)z^{-p}Q_\beta^\alpha f(z) + \frac{\lambda}{p}z^{-p+1}(Q_\beta^\alpha f(z))' \prec h(z). \qquad (2.10)$$

In view of (2.8), the function $z^{-p}g(z)$ has the Herglotz representation

$$z^{-p}g(z) = \int_{|x|=1} \frac{d\mu(x)}{1-xz} \quad (z \in U), \qquad (2.11)$$

where $\mu(x)$ is a probability measure defined on the unit circle $|x| = 1$ and

$$\int_{|x|=1} d\mu(x) = 1.$$

Since $h(z)$ is convex univalent in U, it follows from (2.9) to (2.11) that

$$\begin{aligned}
&(1-\lambda)z^{-p}Q_\beta^\alpha(f*g)(z) + \frac{\lambda}{p}z^{-p+1}(Q_\beta^\alpha(f*g)(z))' \\
&= \int_{|x|=1} \psi(xz)d\mu(x) \prec h(z).
\end{aligned}$$

This shows that $(f*g)(z) \in M_{p,\alpha,\beta}(\lambda; h)$ and the theorem is proved.

Corollary 1. *Let $f(z) \in M_{p,\alpha,\beta}(\lambda; h)$ be given by (1.1) and let*

$$s_m(z) = z^p + \sum_{n=1}^{m-1} a_n z^{n+p} \quad (m \in N \setminus \{1\}).$$

Then the function

$$\sigma_m(z) = \int_0^1 t^{-p}s_m(tz)dt$$

is also in the class $M_{p,\alpha,\beta}(\lambda; h)$.

Proof. We have

$$\sigma_m(z) = z^p + \sum_{n=1}^{m-1} \frac{a_n}{n+1} z^{n+p} = (f * g_m)(z) \quad (m \in N \setminus \{1\}), \tag{2.12}$$

where

$$f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{n+p} \in M_{p,\alpha,\beta}(\lambda; h)$$

and

$$g_m(z) = z^p + \sum_{n=1}^{m-1} \frac{z^{n+p}}{n+1} \in A(p).$$

Also, for $m \in N \setminus \{1\}$, it is known from [11] that

$$\mathrm{Re}\{z^{-p} g_m(z)\} = \mathrm{Re}\left\{1 + \sum_{n=1}^{m-1} \frac{z^n}{n+1}\right\} > \frac{1}{2} \quad (z \in U). \tag{2.13}$$

In view of (2.12) and (2.13), an application of Theorem 3 leads to $\sigma_m(z) \in M_{p,\alpha,\beta}(\lambda; h)$.

Theorem 4. *Let* $f(z) \in M_{p,\alpha,\beta}(\lambda; h)$,

$$g(z) \in A(p) \text{and } z^{-p+1} g(z) \in R(\rho) \quad (\rho < 1).$$

Then

$$(f * g)(z) \in M_{p,\alpha,\beta}(\lambda; h).$$

Proof. For $f(z) \in M_{p,\alpha,\beta}(\lambda; h)$ and $g(z) \in A(p)$, from (2.9) (used in the proof of Theorem 3) we can write

$$\begin{aligned} &(1-\lambda) z^{-p} Q_\beta^\alpha (f * g)(z) + \frac{\lambda}{p} z^{-p+1} (Q_\beta^\alpha (f * g)(z))' \\ &= \frac{(z^{-p+1} g(z)) * (z\psi(z))}{(z^{-p+1} g(z)) * z} \quad (z \in U), \end{aligned} \tag{2.14}$$

where $\psi(z)$ is defined as in (2.10).

Since $h(z)$ is convex univalent in U,

$$\psi(z) \prec h(z), z^{-p+1} g(z) \in R(\rho) \text{ and } z \in S^*(\rho) \quad (\rho < 1),$$

it follows from (2.14) and Lemma 2 the desired result.

Taking $\rho = 0$ and $\rho = \frac{1}{2}$, Theorem 4 reduces to the following.

Corollary 2. *Let $f(z) \in M_{p,\alpha,\beta}(\lambda; h)$ and let $g(z) \in A(p)$ satisfy either of the following conditions:*

(i) *$z^{-p+1}g(z)$ is convex univalent in U*

or

(ii) *$z^{-p+1}g(z) \in S^*(\frac{1}{2})$.*

Then

$$(f * g)(z) \in M_{p,\alpha,\beta}(\lambda; h).$$

Theorem 5. *Let $\lambda \geq 0$ and*

$$f_j(z) = z^p + \sum_{n=1}^{\infty} a_{n,j} z^{n+p} \in M_{p,\alpha,\beta}(\lambda; h_j) \quad (j = 1, 2), \tag{2.15}$$

where

$$h_j(z) = \beta_j + (1 - \beta_j)\frac{1+z}{1-z} \quad \text{and} \quad \beta_j < 1. \tag{2.16}$$

If $f(z) \in A(p)$ is defined by

$$Q_\beta^\alpha f(z) = Q_\beta^\alpha f_1(z) * Q_\beta^\alpha f_2(z), \tag{2.17}$$

then $f(z) \in M_{p,\alpha,\beta}(\lambda; h)$, where

$$h(z) = \beta_3 + (1 - \beta_3)\frac{1+z}{1-z} \tag{2.18}$$

and the parameter β_3 is given by

$$\beta_3 = \begin{cases} 1 - 4(1-\beta_1)(1-\beta_2)(1 - \frac{p}{\lambda}\int_0^1 \frac{u^{\frac{p}{\lambda}-1}}{1+u}du) & (\lambda > 0), \\ 1 - 2(1-\beta_1)(1-\beta_2) & (\lambda = 0). \end{cases} \tag{2.19}$$

The bound β_3 is the best possible.

Proof. We consider the case when $\lambda > 0$. By setting

$$F_j(z) = (1-\lambda)z^{-p}Q^{\alpha}_{\beta}f_j(z) + \frac{\lambda}{p}z^{-p+1}(Q^{\alpha}_{\beta}f_j(z))' \quad (j = 1, 2)$$

for $f_j(z) \quad (j = 1, 2)$ given by (2.15), we find that

$$F_j(z) = 1 + \sum_{n=1}^{\infty} b_{n,j}z^n \prec \beta_j + (1-\beta_j)\frac{1+z}{1-z} \quad (j = 1, 2) \tag{2.20}$$

and

$$Q^{\alpha}_{\beta}f_j(z) = \frac{p}{\lambda}z^{-\frac{p(1-\lambda)}{\lambda}}\int_0^z t^{\frac{p}{\lambda}-1}F_j(t)dt \quad (j = 1, 2). \tag{2.21}$$

Now, if $f(z) \in A(p)$ is defined by (2.17), we find from (2.21) that

$$\begin{aligned} Q^{\alpha}_{\beta}f(z) &= Q^{\alpha}_{\beta}f_1(z) * Q^{\alpha}_{\beta}f_2(z) \\ &= \left(\frac{p}{\lambda}z^p\int_0^1 u^{\frac{p}{\lambda}-1}F_1(uz)du\right) * \left(\frac{p}{\lambda}z^p\int_0^1 u^{\frac{p}{\lambda}-1}F_2(uz)du\right) \\ &= \frac{p}{\lambda}z^p\int_0^1 u^{\frac{p}{\lambda}-1}F(uz)du, \end{aligned} \tag{2.22}$$

where

$$F(z) = \frac{p}{\lambda}\int_0^1 u^{\frac{p}{\lambda}-1}(F_1 * F_2)(uz)du. \tag{2.23}$$

Also, by using (2.20) and the Herglotz theorem, we see that

$$\mathrm{Re}\left\{\left(\frac{F_1(z)-\beta_1}{1-\beta_1}\right) * \left(\frac{1}{2} + \frac{F_2(z)-\beta_2}{2(1-\beta_2)}\right)\right\} > 0 \quad (z \in U),$$

which leads to

$$\mathrm{Re}\{(F_1 * F_2)(z)\} > \beta_0 = 1 - 2(1-\beta_1)(1-\beta_2) \quad (z \in U).$$

According to Lemma 3, we have

$$\mathrm{Re}\{(F_1 * F_2)(z)\} \geq \beta_0 + (1-\beta_0)\frac{1-|z|}{1+|z|} \quad (z \in U). \tag{2.24}$$

Now it follows from (2.22) to (2.24) that

$$\begin{aligned}
&\mathrm{Re}\left\{(1-\lambda)z^{-p}Q_\beta^\alpha f(z)+\frac{\lambda}{p}z^{-p+1}(Q_\beta^\alpha f(z))'\right\}=\mathrm{Re}\{F(z)\}\\
&=\frac{p}{\lambda}\int_0^1 u^{\frac{p}{\lambda}-1}\mathrm{Re}\left\{(F_1*F_2)(uz)\right\}du\\
&\geq\frac{p}{\lambda}\int_0^1 u^{\frac{p}{\lambda}-1}\left(\beta_0+(1-\beta_0)\frac{1-u|z|}{1+u|z|}\right)du\\
&>\beta_0+\frac{p(1-\beta_0)}{\lambda}\int_0^1 u^{\frac{p}{\lambda}-1}\frac{1-u}{1+u}du\\
&=1-4(1-\beta_1)(1-\beta_2)\left(1-\frac{p}{\lambda}\int_0^1\frac{u^{\frac{p}{\lambda}-1}}{1+u}du\right)\\
&=\beta_3\quad(z\in U),
\end{aligned}$$

which proves that $f(z)\in M_{p,\alpha,\beta}(\lambda;h)$ for the function $h(z)$ given by (2.18).

In order to show that the bound β_3 is sharp, we take the functions $f_j(z)\in A(p)\quad(j=1,2)$ defined by

$$Q_\beta^\alpha f_j(z)=\frac{p}{\lambda}z^{-\frac{p(1-\lambda)}{\lambda}}\int_0^z t^{\frac{p}{\lambda}-1}\left(\beta_j+(1-\beta_j)\frac{1+t}{1-t}\right)dt\quad(j=1,2),\quad(2.25)$$

for which we have

$$\begin{aligned}
F_j(z)&=(1-\lambda)z^{-p}Q_\beta^\alpha f_j(z)+\frac{\lambda}{p}z^{-p+1}(Q_\beta^\alpha f_j(z))'\\
&=\beta_j+(1-\beta_j)\frac{1+z}{1-z}\quad(j=1,2)
\end{aligned}$$

and

$$(F_1*F_2)(z)=1+4(1-\beta_1)(1-\beta_2)\frac{z}{1-z}.$$

Hence, for $f(z)\in A(p)$ given by (2.17), we obtain

$$\begin{aligned}
&(1-\lambda)z^{-p}Q_\beta^\alpha f(z)+\frac{\lambda}{p}z^{-p+1}(Q_\beta^\alpha f(z))'\\
&=\frac{p}{\lambda}\int_0^1 u^{\frac{p}{\lambda}-1}\left(1+4(1-\beta_1)(1-\beta_2)\frac{uz}{1-uz}\right)du\\
&\to\beta_3\quad(\text{ as } z\to -1).
\end{aligned}$$

Finally, for the case when $\lambda = 0$, the proof of Theorem 5 is simple, and so we choose to omit the details involved.

Theorem 6. *Let $f(z) \in M_{p,\alpha,\beta}(\lambda; h)$. Then the function $F(z)$ defined by*

$$F(z) = \frac{\mu + p}{z^{\mu}} \int_0^z t^{\mu-1} f(t) dt \quad (Re\mu > -p) \tag{2.26}$$

is in the class $M_{p,\alpha,\beta}(\lambda; \widetilde{h})$, where

$$\widetilde{h}(z) = (\mu + p) z^{-(\mu+p)} \int_0^z t^{\mu+p-1} h(t) dt \prec h(z).$$

Proof. For $f(z) \in A(p)$ and $\mathrm{Re}\mu > -p$, we find from (2.26) that $F(z) \in A(p)$ and

$$(\mu + p) f(z) = \mu F(z) + z F'(z). \tag{2.27}$$

Define $G(z)$ by

$$z^p G(z) = (1 - \lambda) Q_{\beta}^{\alpha} F(z) + \frac{\lambda}{p} z (Q_{\beta}^{\alpha} F(z))'. \tag{2.28}$$

Differentiating both sides of (2.28) with respect to z, we get

$$zG'(z) + pG(z) = (1 - \lambda) z^{-p} Q_{\beta}^{\alpha}(zF'(z)) + \frac{\lambda}{p} z^{-p+1} (Q_{\beta}^{\alpha}(zF'(z)))'. \tag{2.29}$$

Furthermore, it follows from (2.27) to (2.29) that

$$\begin{aligned}
&(1-\lambda) z^{-p} Q_{\beta}^{\alpha} f(z) + \frac{\lambda}{p} z^{-p+1} (Q_{\beta}^{\alpha} f(z))' \\
&= (1-\lambda) z^{-p} Q_{\beta}^{\alpha} \left(\frac{\mu F(z) + zF'(z)}{\mu + p} \right) + \frac{\lambda}{p} z^{-p+1} \left(Q_{\beta}^{\alpha} \left(\frac{\mu F(z) + zF'(z)}{\mu + p} \right) \right)' \\
&= \frac{\mu}{\mu + p} G(z) + \frac{1}{\mu + p} (zG'(z) + pG(z)) \\
&= G(z) + \frac{zG'(z)}{\mu + p}. \qquad (2.30)
\end{aligned}$$

Let $f(z) \in M_{p,\alpha,\beta}(\lambda; h)$. Then, by (2.30),

$$G(z) + \frac{zG'(z)}{\mu + p} \prec h(z) \quad (\mathrm{Re}\mu > -p),$$

and so it follows from Lemma 1 that

$$G(z) \prec \widetilde{h}(z) = (\mu + p)z^{-(\mu+p)} \int_0^z t^{\mu+p-1} h(t)dt \prec h(z).$$

Therefore we conclude that

$$F(z) \in M_{p,\alpha,\beta}(\lambda; \widetilde{h}) \subset M_{p,\alpha,\beta}(\lambda; h).$$

Theorem 7. *Let $f(z) \in A(p)$ and $F(z)$ be defined as in Theorem 6. If*

$$(1-\gamma)z^{-p}Q^{\alpha}_{\beta}F(z) + \gamma z^{-p}Q^{\alpha}_{\beta}f(z) \prec h(z) \quad (\gamma > 0), \tag{2.31}$$

then $F(z) \in M_{p,\alpha,\beta}(0; \widetilde{h})$, where $Re\mu > -p$ and

$$\widetilde{h}(z) = \frac{\mu + p}{\gamma} z^{-\frac{\mu+p}{\gamma}} \int_0^z t^{\frac{\mu+p}{\gamma}-1} h(t)dt \prec h(z).$$

Proof. Let us define

$$G(z) = z^{-p}Q^{\alpha}_{\beta}F(z). \tag{2.32}$$

Then $G(z)$ is analytic in U, with $G(0) = 1$, and

$$zG'(z) = -pG(z) + z^{-p+1}(Q^{\alpha}_{\beta}F(z))'. \tag{2.33}$$

Making use of (2.27), (2.31), (2.32) and (2.33), we deduce that

$$\begin{aligned}
&(1-\gamma)z^{-p}Q^{\alpha}_{\beta}F(z) + \gamma z^{-p}Q^{\alpha}_{\beta}f(z) \\
&= (1-\gamma)z^{-p}Q^{\alpha}_{\beta}F(z) + \frac{\gamma}{\mu+p}(\mu z^{-p}Q^{\alpha}_{\beta}F(z) + z^{-p+1}(Q^{\alpha}_{\beta}F(z))') \\
&= G(z) + \frac{\gamma}{\mu+p} zG'(z) \prec h(z)
\end{aligned}$$

for $\mathrm{Re}\mu > -p$ and $\gamma > 0$. Therefore an application of Lemma 1 yields the assertion of Theorem 7.

Theorem 8. *Let $F(z) \in M_{p,\alpha,\beta}(\lambda; h)$. If the function $f(z)$ is defined by*

$$F(z) = \frac{\mu+p}{z^{\mu}} \int_0^z t^{\mu-1} f(t)dt \quad (\mu > -p), \tag{2.34}$$

then

$$\sigma^{-p} f(\sigma z) \in M_{p,\alpha,\beta}(\lambda; h),$$

where

$$\sigma = \sigma_p(\mu) = \frac{\sqrt{1+(\mu+p)^2}-1}{\mu+p} \in (0,1). \tag{2.35}$$

The bound σ is sharp when

$$h(z) = \gamma + (1-\gamma)\frac{1+z}{1-z} \quad (\gamma \neq 1). \tag{2.36}$$

Proof. For $F(z) \in A(p)$, it is easy to verify that

$$F(z) = F(z) * \frac{z^p}{1-z} \text{ and } zF'(z) = F(z) * \left(\frac{z^p}{(1-z)^2} + (p-1)\frac{z^p}{1-z}\right).$$

Hence, by (2.34), we have

$$f(z) = \frac{\mu F(z) + zF'(z)}{\mu+p} = (F * g)(z) \quad (z \in U; \mu > -p), \tag{2.37}$$

where

$$g(z) = \frac{1}{\mu+p}\left((\mu+p-1)\frac{z^p}{1-z} + \frac{z^p}{(1-z)^2}\right) \in A(p). \tag{2.38}$$

Next we show that

$$\text{Re}\{z^{-p} g(z)\} > \frac{1}{2} \quad (|z| < \sigma), \tag{2.39}$$

where $\sigma = \sigma_p(\mu)$ is given by (2.35). Setting

$$\frac{1}{1-z} = Re^{i\theta} \quad (R > 0) \quad \text{ and } \quad |z| = r < 1,$$

we see that

$$\cos\theta = \frac{1+R^2(1-r^2)}{2R} \quad \text{and} \quad R \geq \frac{1}{1+r}. \tag{2.40}$$

For $\mu > -p$ it follows from (2.38) and (2.40) that

$$\begin{aligned}
2\mathrm{R}e\{z^{-p}g(z)\} &= \frac{2}{\mu+p}[(\mu+p-1)R\cos\theta + R^2(2\cos^2\theta - 1)] \\
&= \frac{1}{\mu+p}[(\mu+p-1)(1+R^2(1-r^2))+(1+R^2(1-r^2))^2-2R^2] \\
&= \frac{R^2}{\mu+p}[R^2(1-r^2)^2+(\mu+p+1)(1-r^2)-2]+1 \\
&\geq \frac{R^2}{\mu+p}[(1-r)^2+(\mu+p+1)(1-r^2)-2]+1 \\
&= \frac{R^2}{\mu+p}(\mu+p-2r-(\mu+p)r^2)+1.
\end{aligned}$$

This evidently gives (2.39), which is equivalent to

$$\mathrm{R}e\{z^{-p}\sigma^{-p}g(\sigma z)\} > \frac{1}{2} \quad (z \in U). \tag{2.41}$$

Let $F(z) \in M_{p,\alpha,\beta}(\lambda; h)$. Then, by using (2.37) and (2.41), an application of Theorem 3 yields

$$\sigma^{-p}f(\sigma z) = F(z) * (\sigma^{-p}g(\sigma z)) \in M_{p,\alpha,\beta}(\lambda; h).$$

For $h(z)$ given by (2.36), we consider the function $F(z) \in A(p)$ defined by

$$\begin{aligned}
&(1-\lambda)z^{-p}Q_\beta^\alpha F(z) + \frac{\lambda}{p}z^{-p+1}(Q_\beta^\alpha F(z))' \\
&\quad = \gamma + (1-\gamma)\frac{1+z}{1-z} \quad (\gamma \neq 1).
\end{aligned} \tag{2.42}$$

Then, by (2.42), (2.28) and (2.30) (used in the proof of Theorem 6), we find that

$$\begin{aligned}
&(1-\lambda)z^{-p}Q_\beta^\alpha f(z) + \frac{\lambda}{p}z^{-p+1}(Q_\beta^\alpha f(z))' \\
&\quad = \gamma + (1-\gamma)\frac{1+z}{1-z} + \frac{z}{\mu+p}\left(\gamma + (1-\gamma)\frac{1+z}{1-z}\right)'
\end{aligned}$$

$$= \gamma + \frac{(1-\gamma)(\mu+p+2z-(\mu+p)z^2)}{(\mu+p)(1-z)^2}$$
$$= \gamma \quad (z = -\sigma).$$

Therefore we conclude that the bound $\sigma = \sigma_p(\mu)$ cannot be increased for each $\mu(\mu > -p)$.

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