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ON CERTAIN SUBCLASSES OF MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY USING A DIFFERENTIAL OPERATOR

BY

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Abstract

Making use of a differential operator the author investigate the various impartant properties and characteristics of the subclass $T_j(n, m, p, q, \alpha)$ $(p, j, m \in N = \{1, 2, \ldots\}, q, n \in N_0 = N \cup \{0\}, 0 \leq \alpha < p - q)$ of p-valently analytic functions with negative coefficients. Finally, several applications involving an integral operator and certain fractional calculus operators are also considered.

1. Introduction

Let $T(j, p)$ denote the class of functions of the form :

$$f(z) = z^p - \sum_{k=j+p}^{\infty} a_k z^k \quad (a_k \geq 0; p, j \in N = \{1, 2, \ldots.\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $U = \{z : |z| < 1\}$. A function $f(z) \in T(j, p)$ is said to be p-valently starlike of order α if it satisfies the inequality :

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U;\ \ 0 \leq \alpha < p; p \in N). \tag{1.2}$$

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We denote by $T_j^*(p,\alpha)$ the class of all p-valently starlike functions of order α. Also a function $f(z) \in T(j,p)$ is said to be p-valently convex of order α if it satisfies the inequality :

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in U;\ 0 \le \alpha < p; p \in N). \tag{1.3}$$

We denote by $C_j(p,\alpha)$ the class of all p-valently convex functions of order α. We note that (see for example Duren [6] and Goodman [7])

$$f(z) \in C_j(p,\alpha) \Longleftrightarrow \frac{zf'(z)}{p} \in T_j^*(p,\alpha) \quad (0 \le \alpha < p; p \in N). \tag{1.4}$$

The classes $T_j^*(p,\alpha)$ and $C_j(p,\alpha)$ are studied by Owa [13].

For each $f(z) \in T(j,p)$, we have (see [5])

$$f^{(q)}(z) = \frac{p!}{(p-q)!}z^{p-q} - \sum_{k=j+p}^{\infty}\frac{k!}{(k-q)!}a_k z^{k-q} \quad (q \in N_0 = N \cup \{0\}; p > q). \tag{1.5}$$

For a function in $T(j,p)$, we have

$$\begin{aligned} D_p^0 f^{(q)}(z) &= f^{(q)}(z) \\ D_p^1 f^{(q)}(z) &= Df^{(q)}(z) = \frac{z}{(p-q)}(f^{(q)}(z))' = \frac{z}{(p-q)}f^{(1+q)}(z) \\ &= \frac{p!}{(p-q)!}z^{p-q} - \sum_{k=j+p}^{\infty}\frac{k!}{(k-q)!}(\frac{k-q}{p-q})a_k z^{k-q}, \qquad (1.6) \\ D_p^2 f^{(q)}(z) &= D(D_p^1 f^{(q)}(z)) \\ &= \frac{p!}{(p-q)!}z^{p-q} - \sum_{k=j+p}^{\infty}\frac{k!}{(k-q)!}(\frac{k-q}{p-q})^2 a_k z^{k-q}, \qquad (1.7) \end{aligned}$$

and

$$\begin{aligned} D_p^n f^{(q)}(z) &= D(D_p^{n-1} f^{(q)}(z)) \qquad (n \in N) \\ &= \frac{p!}{(p-q)!}z^{p-q} - \sum_{k=j+p}^{\infty}\frac{k!}{(k-q)!}(\frac{k-q}{p-q})^n a_k z^{k-q} \\ &\qquad (p,j,n \in N; q \in N_0; p > q). \qquad (1.8) \end{aligned}$$

We note that by taking $q = 0$ and $p = 1$, the differential operator $D_1^n = D^n$

was introduced by Salagean [14].

It is easy to see that

$$\frac{z}{(p-q)}\left(D_p^n f^{(q)}(z)\right)' = D_p^{n+1} f^{(q)}(z). \tag{1.9}$$

With the help of the differential operator D_p^n, we say that a function $f(z)$ belonging to $T(j,p)$ is in the class $T_j(n,m,p,q,\alpha)$ if and only if

$$\operatorname{Re}\left\{\frac{(p-q)D_p^{n+m} f^{(q)}(z)}{D_p^n f^{(q)}(z)}\right\} > \alpha \quad (p,m \in N; q,n \in N_0) \tag{1.10}$$

for some $\alpha\,(0 \le \alpha < p-q, p > q)$ and for all $z \in U$.

We note that , by specializing the parameters j,p,n,m,q and α, we obtain the following subclasses studied by various authors :

(i) $T_j(n,m,1,0,\alpha) = T_j(n,m,\alpha)\,(0 \le \alpha < 1)$ (Sekine [17], Hossen et al. [8] and Aouf [1]);

(ii) $T_j(0,1,p,q,\alpha) = S_j(p,q,\alpha)$ and $T_j(1,1,p,q,\alpha) = C_j(p,q,\alpha)$ (Chen et al. [4]);

(iii) $T_j(0,1,p,0,\alpha) = \begin{cases} T_j^*(p,\alpha) & \text{(Owa) [13])} \\ T_\alpha(p,\alpha) & \text{(Yamakawa[22]);} \end{cases}$

(iv) $T_j(1,1,p,0,\alpha) = \begin{cases} C_j(p,\alpha) & \text{(Owa) [13])} \\ CT_\alpha(p,j) & \text{(Yamakawa [22]);} \end{cases}$

(v) $T_1(0,1,p,0,\alpha) = T^*(p,\alpha)$ and $T_1(1,1,p,0,\alpha) = C(p,\alpha)(p \in N; 0 \le \alpha < p)$ (Owa [12] and Salagean et al. [15]);

(vi) $T_j(0,1,1,0,\alpha) = T_\alpha(j)$ and $T_j(1,1,1,0,\alpha) = C_\alpha(j)(j \in N; 0 \le \alpha < 1)$ (Srivastava et al. [21]);

(vii) $T_j(n,1,1,0,\alpha) = P(j,\alpha,n)(j \in N; n \in N_0; 0 \le \alpha < 1)$ (Aouf and Srivastava [2]).

In this paper, we shall make use of the familiar operator $J_{c,p}$ defined by (cf. [3], [9] and [10] ; see also [20])

$$(J_{c,p}f)(z) = \frac{c+p}{z^c}\int_0^z t^{c-1} f(t)dt \tag{1.11}$$

$$(f \in T(j,p);\ c > -p; p \in N)$$

as well as the fractional calculus operator D_z^μ for which it is well known that (see , for details, [11] and [18]; see also Section 5 below)

$$D_z^\mu\{z^\rho\} = \frac{\Gamma(\rho+1)}{\Gamma(\rho+1-\mu)} z^{\rho-\mu} \quad (\rho > -1; \mu \in R) \tag{1.12}$$

in terms of Gamma functions.

2. Coefficient Estimates

Theorem 1. *Let the function* $f(z)$ *defined by* (1.1). *Then* $f(z) \in T_j(n,m,p,q,\alpha)$ *if and only if*

$$\sum_{k=j+p}^{\infty} (\frac{k-q}{p-q})^n \left[(p-q)(\frac{k-q}{p-q})^m - \alpha\right] \delta(k,q) a_k \leq (p-q-\alpha)\delta(p,q)$$
$$(0 \leq \alpha < p-q; p,j,m \in N; q,n \in N_0; p > q), \tag{2.1}$$

where

$$\delta(p,q) = \frac{p!}{(p-q)!} = \begin{cases} p(p-1)\cdots(p-q+1) & (q \neq 0) \\ 1 & (q = 0)\,. \end{cases} \tag{2.2}$$

Proof. Assume that the inequality (2.1) holds true. Then we find that

$$\begin{aligned} &\left| \frac{(p-q)D_p^{n+m} f^{(q)}(z)}{D_p^n f^{(q)}(z)} - (p-q) \right| \\ &\leq \frac{(p-q)\sum_{k=j+p}^{\infty} (\frac{k-q}{p-q})^n \left[(\frac{k-q}{p-q})^m - 1\right] \delta(k,q) a_k |z|^{k-p}}{\delta(p,q) - \sum_{k=j+p}^{\infty} (\frac{k-q}{p-q})^n \delta(k,q) a_k |z|^{k-p}} \\ &\leq \frac{(p-q)\sum_{k=j+p}^{\infty} (\frac{k-q}{p-q})^n \left[(\frac{k-q}{p-q})^m - 1\right] \delta(k,q) a_k}{\delta(p,q) - \sum_{k=j+p}^{\infty} (\frac{k-q}{p-q})^n \delta(k,q) a_k} \\ &\leq p-q-\alpha. \end{aligned}$$

This shows that the values of the function

$$\Phi(z) = \frac{(p-q)D_p^{n+m}f^{(q)}(z)}{D_p^n f^{(q)}(z)} \tag{2.3}$$

lie in a circle which is centered at $w = (p-q)$ and whose radius is $(p-q-\alpha)$. Hence $f(z)$ satisfies the condition (1.10).

Conversely , assume that the function $f(z)$ is in the class $T_j(n, m, p, q, \alpha)$. Then we have

$$\begin{aligned} &\text{Re}\left\{\frac{(p-q)D_p^{n+m}f^{(q)}(z)}{D_p^n f^{(q)}(z)}\right\} \\ &= \text{Re}\left\{\frac{(p-q)\delta(p,q) - \sum\limits_{k=j+p}^{\infty}(p-q)(\frac{k-q}{p-q})^{n+m}\delta(k,q)a_k z^{k-p}}{\delta(p,q) - \sum\limits_{k=j+p}^{\infty}(\frac{k-q}{p-q})^{n}\delta(k,q)a_k z^{k-p}}\right\} > \alpha, \end{aligned} \tag{2.4}$$

for some α $(0 \le \alpha < p-q)$, $p, j, m \in N, q, n \in N_0, p > q$ and $z \in U$. Choose values of z on the real axis so that $\Phi(z)$ given by (2.3) is real. Upon clearing the denominator in (2.4) and letting $z \to 1^-$ through real values, we can see that

$$\begin{aligned} &(p-q)\delta(p,q) - \sum_{k=j+p}^{\infty}(p-q)(\frac{k-q}{p-q})^{n+m}\delta(k,q)a_k \\ &\quad \ge \alpha\left\{\delta(p,q) - \sum_{k=j+p}^{\infty}(\frac{k-q}{p-q})^{n}\delta(k,q)a_k\right\}. \end{aligned} \tag{2.5}$$

Thus we have the inequality (2.1).

Corollary 1. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $T_j(n, m, p, q, \alpha)$. *Then*

$$a_k \le \frac{(p-q-\alpha)\delta(p,q)}{(\frac{k-q}{p-q})^n\left[(p-q)(\frac{k-q}{p-q})^m - \alpha\right]\delta(k,q)}$$

$$(k \ge j+p; p, j, m \in N; q, n \in N_0; p > q). \tag{2.6}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(p-q-\alpha)\delta(p,q)}{(\frac{k-q}{p-q})^n\left[(p-q)(\frac{k-q}{p-q})^m-\alpha\right]\delta(k,q)}z^k$$
$$(k \geq j+p; p,j,m \in N; q,n \in N_0; p>q). \tag{2.7}$$

3. Distortion Theorem

Theorem 2. *If a function $f(z)$ defined by* (1.1) *is in the class $T_j(n,m,p,$*

$q,\alpha)$, then

$$\left\{\frac{p!}{(p-\sigma)!} - \frac{(p-q-\alpha)\delta(p,q)(j+p-q)!}{(\frac{j+p-q}{p-q})^n\left[(p-q)(\frac{j+p-q}{p-q})^m-\alpha\right](j+p-\sigma)!}|z|^j\right\}|z|^{p-\sigma}$$
$$\leq \left|f^{(\sigma)}(z)\right|$$
$$\leq \left\{\frac{p!}{(p-\sigma)!} + \frac{(p-q-\alpha)\delta(p,q)(j+p-q)!}{(\frac{j+p-q}{p-q})^n\left[(p-q)(\frac{j+p-q}{p-q})^m-\alpha\right](j+p-\sigma)!}|z|^j\right\}|z|^{p-\sigma}$$
$$(z \in U; 0 \leq \alpha < p-q; p,j,m \in N; q,n,\sigma \in N_0; p > \max\{q,\sigma\}). \tag{3.1}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(p-q-\alpha)\delta(p,q)}{(\frac{j+p-q}{p-q})^n\left[(p-q)(\frac{j+p-q}{p-q})^m-\alpha\right]\delta(j+p,q)}z^{j+p}$$
$$(p,j,m \in N; q,n \in N_0; p>q). \tag{3.2}$$

Proof. In view of Theorem 1, we have

$$\frac{(\frac{j+p-q}{p-q})^n\left[(p-q)(\frac{j+p-q}{p-q})^m-\alpha\right]\delta(j+p,q)}{(p-q-\alpha)\delta(p,q)(j+p)!}\sum_{k=j+p}^{\infty} k!a_k$$
$$\leq \sum_{k=j+p}^{\infty}\frac{(\frac{k-q}{p-q})^n[(p-q)(\frac{k-q}{p-q})^m-\alpha]\delta(k,q)}{(p-q-\alpha)\delta(p,q)}a_k \leq 1$$

which readily yields

$$\sum_{k=j+p}^{\infty} k!a_k \leq \frac{(p-q-\alpha)\delta(p,q)(j+p-q)!}{(\frac{j+p-q}{p-q})^n \left[(p-q)(\frac{j+p-q}{p-q})^m - \alpha\right]} \ . \tag{3.3}$$

Now, by differentiating both sides of (1.1) σ times, we have

$$f^{(\sigma)}(z) = \frac{p!}{(p-\sigma)!} z^{p-\sigma} - \sum_{k=j+p}^{\infty} \frac{k!}{(k-\sigma)!} a_k z^{k-\sigma}$$
$$(k \geq j+p; p, j \in N; q, \sigma \in N_0; p > \max\{q, \sigma\}). \tag{3.4}$$

Theorem 2 would follow from (3.3) and (3.4).

Finally, it is easy to see that the bounds in (3.1) are attained for the function $f(z)$ given by (3.2). □

Remark 1. (i) Putting $\sigma = q = 0$ and $p = 1$ in Theorem 2, we obtain the result obtained by Sekine [17, Corollary 3];

(ii) Putting $q = 0$ and $\sigma = p = 1$ in Theorem 2, we obtain the result obtained by Sekine [17, Corollary 4].

4. Radii of Close-to-Convexity, Starlikeness and Convexity

Theorem 3. *Let the function $f(z)$ defined by* (1.1) *be in the class $T_j(n, m, p, q, \alpha)$, then*

(i) *$f(z)$ is p-valently close - to - convex of order $\varphi (0 \leq \varphi < p)$ in $|z| < r_1$, where*

$$r_1 = \inf_k \left\{ \frac{(\frac{k-q}{p-q})^n \left[(p-q)(\frac{k-q}{p-q})^m - \alpha\right] \delta(k,q)}{(p-q-\alpha)\delta(p,q)} (\frac{p-\varphi}{k}) \right\}^{\frac{1}{k-p}}$$
$$(k \geq j+p; p, j, m \in N; q, n \in N_0; p > q), \tag{4.1}$$

(ii) *$f(z)$ is p-valently starlike of order $\varphi (0 \leq \varphi < p)$ in $|z| < r_2$, where*

$$r_2 = \inf_k \left\{ \frac{(\frac{k-q}{p-q})^n \left[(p-q)(\frac{k-q}{p-q})^m - \alpha\right] \delta(k,q)}{(p-q-\alpha)\delta(p,q)} (\frac{p-\varphi}{k-\varphi}) \right\}^{\frac{1}{k-p}}$$
$$(k \geq j+p; p, j, m \in N; q, n \in N_0; p > q), \tag{4.2}$$

(iii) *$f(z)$ is p-valently convex of order $\varphi(0 \le \varphi < p)$ in $|z| < r_3$, where*

$$r_3 = \inf_k \left\{ \frac{(\frac{k-q}{p-q})^n \left[(p-q)(\frac{k-q}{p-q})^m - \alpha\right] \delta(k,q)}{(p-q-\alpha)\delta(p,q)} . \frac{p(p-\varphi)}{k(k-\varphi)} \right\}^{\frac{1}{k-p}}$$

$$(k \ge j+p; p, j, m \in N; q, n \in N_0; p > q). \tag{4.3}$$

Each of these results is sharp for the function $f(z)$ given by (2.7).

Proof. It is sufficient to show that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \le p - \varphi \quad (|z| < r_1;\ 0 \le \varphi < p; p \in N), \tag{4.4}$$

$$\left|\frac{zf'(z)}{f(z)} - p\right| \le p - \varphi \quad (|z| < r_2;\ 0 \le \varphi < p; p \in N), \tag{4.5}$$

and that

$$\left|1 + \frac{zf''(z)}{f'(z)} - p\right| \le p - \varphi \quad (|z| < r_3;\ 0 \le \varphi < p; p \in N) \tag{4.6}$$

for a function $f(z) \in T_j(n, m, p, q, \alpha)$, where r_1, r_2 and r_3 are defined by (4.1), (4.2) and (4.3), respectively. □

Remark 2. Putting $q = 0$ and $p = 1$ Theorem 3, we obtain the results obtained by Hossen et al. [8, Theorems 8 and 9 and Corollary 3, respectively].

5. Modified Hadamard Products

For the functions $f_\nu(z) = (\nu = 1, 2)$ given by

$$f_\nu(z) = z^p - \sum_{k=j+p}^{\infty} a_{k,\nu} z^k \quad (a_{k,\nu} \ge 0; \nu = 1, 2) \tag{5.1}$$

we denote by $(f_1 \circledast f_2)(z)$ the modified Hadamard product (or convolution) of the functions $f_1(z)$ and $f_2(z)$ by

$$(f_1 \circledast f_2)(z) = z^p - \sum_{k=j+p}^{\infty} a_{k,1}.a_{k,2} z^k. \tag{5.2}$$

Theorem 4. *Let the functions* $f_\nu(z)(\nu=1,2)$ *defined by* (5.1) *be in the class* $T_j(n,m,p,q,\alpha)$. *Then* $(f_1 \circledast f_2)(z) \in T_j(n,m,p,q,\gamma)$, *where*

$$\gamma=(p-q)\left\{1-\frac{(p-q-\alpha)^2\left[(\frac{j+p-q}{p-q})^m-1\right]\delta(p,q)}{(\frac{j+p-q}{p-q})^n\left[(p-q)(\frac{j+p-q}{p-q})^m-\alpha\right]^2\delta(j+p,q)-(p-q-\alpha)^2\delta(p,q)}\right\}. \tag{5.3}$$

The result is sharp for the functions $f_\nu(z)(\nu=1,2)$ *given by*

$$f_\nu(z)=z^p-\frac{(p-q-\alpha)\delta(p,q)}{(\frac{j+p-q}{p-q})^n\left[(p-q)(\frac{j+p-q}{p-q})^m-\alpha\right]\delta(j+p,q)}z^{j+p}\ \ (\nu=1,2). \tag{5.4}$$

Proof. Emloying the technique used earlier by Schild and Silverman [16], we need to find the largest γ such that

$$\sum_{k=j+p}^{\infty}\frac{(\frac{k-q}{p-q})^n\left[(p-q)(\frac{k-q}{p-q})^m-\gamma\right]\delta(k,q)}{(p-q-\gamma)\delta(p,q)}a_{k,1}.a_{k,2}\le 1$$
$$(f_\nu(z)\in T_j(n,m,p,q,\alpha)\ \ (\nu=1,2)). \tag{5.5}$$

Since $f_\nu(z)\in T_j(n,m,p,q,\alpha)(\nu=1,2)$, we readily see that

$$\sum_{k=j+p}^{\infty}\frac{(\frac{k-q}{p-q})^n\left[(p-q)(\frac{k-q}{p-q})^m-\alpha\right]\delta(k,q)}{(p-q-\alpha)\delta(p,q)}a_{k,\nu}\le 1\ \ (\nu=1,2). \tag{5.6}$$

Therefore, by the Cauchy - Schwarz inequality, we obtain

$$\sum_{k=j+p}^{\infty}\frac{(\frac{k-q}{p-q})^n\left[(p-q)(\frac{k-q}{p-q})^m-\alpha\right]\delta(k,q)}{(p-q-\alpha)\delta(p,q)}\sqrt{a_{k,1}.a_{k,2}}\le 1. \tag{5.7}$$

Thus we only need to show that

$$\frac{\left[(p-q)(\frac{k-q}{p-q})^m-\gamma\right]}{(p-q-\gamma)}a_{k,1}.a_{k,2}\ \le\ \frac{\left[(p-q)(\frac{k-q}{p-q})^m-\alpha\right]}{(p-q-\alpha)}\sqrt{a_{k,1}.a_{k,2}}$$
$$(k\ge j+p;p,j\in N), \tag{5.8}$$

or, equivalently , that

$$\sqrt{a_{k,1}.a_{k,2}} \le \frac{(p-q-\gamma)\left[(p-q)(\frac{k-q}{p-q})^m-\alpha\right]}{(p-q-\alpha)\left[(p-q)(\frac{k-q}{p-q})^m-\gamma\right]} \quad (k \ge j+p; p,j \in N). \tag{5.9}$$

Hence , in light of the inequality (5.7), it is sufficient to prove that

$$\frac{(p-q-\alpha)\delta(p,q)}{(\frac{k-q}{p-q})^n\left[(p-q)(\frac{k-q}{p-q})^m-\alpha\right]\delta(k,q)} \le \frac{(p-q-\gamma)\left[(p-q)(\frac{k-q}{p-q})^m-\alpha\right]}{(p-q-\alpha)\left[(p-q)(\frac{k-q}{p-q})^m-\gamma\right]} \quad (k \ge j+p; p,j \in N). \tag{5.10}$$

It follows from (5.10) that

$$\gamma \le (p-q) \times\left\{1-\frac{(p-q-\alpha)^2\left[(\frac{k-q}{p-q})^m-1\right]\delta(p,q)}{(\frac{k-q}{p-q})^n\left[(p-q)(\frac{k-q}{p-q})^m-\alpha\right]^2\delta(k,q)-(p-q-\alpha)^2\delta(p,q)}\right\} \quad (k \ge j+p; p,j \in N). \tag{5.11}$$

Now, defining the function $G(k)$ by

$$G(k) = (p-q) \times\left\{1-\frac{(p-q-\alpha)^2\left[(\frac{k-q}{p-q})^m-1\right]\delta(p,q)}{(\frac{k-q}{p-q})^n\left[(p-q)(\frac{k-q}{p-q})^m-\alpha\right]^2\delta(k,q)-(p-q-\alpha)^2\delta(p,q)}\right\} \quad (k \ge j+p; p,j \in N), \tag{5.12}$$

we see that $G(k)$ is an increasing function of k. Therefore, we concldue that

$$\gamma \le G(j+p) = (p-q) \times\left\{1-\frac{(p-q-\alpha)^2\left[(\frac{j+p-q}{p-q})^m-1\right]\delta(p,q)}{(\frac{j+p-q}{p-q})^n\left[(p-q)(\frac{j+p-q}{p-q})^m-\alpha\right]^2\delta(j+p,q)-(p-q-\alpha)^2\delta(p,q)}\right\} \tag{5.13}$$

which evidently completes the proof of Theorem 4.

Putting (i) $n = 0$ and $m = 1$ (ii) $n = m = 1$ in Theorem 4, we obtain

Corollary 2. *Let the functions $f_\nu(z)(\nu = 1, 2)$ defined by (5.1) be in the class $S_j(p, q, \alpha)$. Then $(f_1 \circledast f_2)(z) \in S_j(p, q, \gamma)$, where*

$$\gamma = (p-q) - \frac{j(p-q-\alpha)^2\delta(p,q)}{(j+p-q-\alpha)^2\delta(j+p,q) - (p-q-\alpha)^2\delta(p,q)}. \tag{5.14}$$

The result is sharp.

Remark 3. .We note that the result obtained by Chen et al. [4, Theorem 5] is not correct. The correct result is given by (5.14).

Corollary 3. *Let the functions $f_\nu(z)(\nu = 1, 2))$ defined by (5.1) be in the class $C_j(p, q, \alpha)$. Then $(f_1 \circledast f_2)(z) \in C_j(p, q, \gamma)$, where*

$$\gamma = (p-q) - \frac{j(p-q-\alpha)^2\delta(p,q+1)}{(j+p-q-\alpha)^2\delta(j+p,q+1) - (p-q-\alpha)^2\delta(p,q+1)}. \tag{5.15}$$

The result is sharp.

Remark 4. We note that the result obtained by Chen et al. [4, Theorem 6] is not correct. The correct result is given by (5.15).

Using arguments similar to those in the proof of Theorem 4, we obtain the following result.

Theorem 5. *Let the functions $f_\nu(z)(\nu = 1, 2)$ defined by (5.1) be in the class $T_j(n, m, p, q, \alpha)$. Then the function*

$$h(z) = z^p - \sum_{k=j+p}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k \tag{5.16}$$

belongs to the class $T_j(n, m, p, q, \xi)$, where

$$\xi = (p-q) \times \left\{ 1 - \frac{2(p-q-\alpha)^2 \left[(\frac{j+p-q}{p-q})^m - 1 \right] \delta(p,q)}{(\frac{j+p-q}{p-q})^n \left[(p-q)(\frac{j+p-q}{p-q})^m - \alpha \right]^2 \delta(j+p,q) - 2(p-q-\alpha)^2\delta(p,q)} \right\}. \tag{5.17}$$

The result is the sharp for the functions $f_\nu(z)(\nu = 1, 2)$ defined by (5.4).

Proof. Noting that

$$\sum_{k=j+p}^{\infty}\left\{\frac{(\frac{k-q}{p-q})^n\left[(p-q)(\frac{k-q}{p-q})^m-\alpha\right]\delta(k,q)}{(p-q-\alpha)\delta(p,q)}\right\}^2 a_{k,\nu}^2$$
$$\leq \left\{\sum_{k=j+p}^{\infty}\frac{(\frac{k-q}{p-q})^n\left[(p-q)(\frac{k-q}{p-q})^m-\alpha\right]\delta(k,q)}{(p-q-\alpha)\delta(p,q)}a_{k,\nu}^2\right\}^2 \leq 1$$
$$(f_\nu(z)\in T_j(n,m,p,q,\alpha) \quad (\nu=1,2)), \tag{5.18}$$

we have

$$\sum_{k=j+p}^{\infty}\frac{1}{2}\left\{\frac{(\frac{k-q}{p-q})^n\left[(p-q)(\frac{k-q}{p-q})^m-\alpha\right]\delta(k,q)}{(p-q-\alpha)\delta(p,q)}\right\}^2 (a_{k,1}^2+a_{k,2}^2)\leq 1. \tag{5.19}$$

Therefore, we have to find the largest ξ such that

$$\frac{\left[(p-q)(\frac{k-q}{p-q})^m-\xi\right]}{(p-q-\xi)} \leq \frac{(\frac{k-q}{p-q})^n\left[(p-q)(\frac{k-q}{p-q})^m-\alpha\right]\delta(k,q)}{2(p-q-\alpha)\delta(p,q)}$$
$$(k\geq j+p; p,j\in N), \tag{5.20}$$

that is, that

$$\xi=(p-q)$$
$$\times\left\{1-\frac{2(p-q-\alpha)^2\left[(\frac{k-q}{p-q})^m-1\right]\delta(p,q)}{(\frac{k-q}{p-q})^n\left[(p-q)(\frac{k-q}{p-q})^m-\alpha\right]^2\delta(k,q)-2(p-q-\alpha)^2\delta(p,q)}\right\}$$
$$(k\geq j+p; p,j\in N)\ . \tag{5.21}$$

Now, defined the function $\Psi(k)$ by

$$\Psi(k)=(p-q)$$
$$\times\left\{1-\frac{2(p-q-\alpha)^2\left[(\frac{k-q}{p-q})^m-1\right]\delta(p,q)}{(\frac{k-q}{p-q})^n\left[(p-q)(\frac{k-q}{p-q})^m-\alpha\right]^2\delta(k,q)-2(p-q-\alpha)^2\delta(p,q)}\right\}$$
$$(k\geq j+p; p,j\in N)\,, \tag{5.22}$$

we observe that $\Psi(k)$ is an increasing function of k. We thus conclude that

$$\xi \leq \Psi(j+p) = (p-q)$$
$$\times\left\{1-\frac{2(p-q-\alpha)^2\left[(\frac{j+p-q}{p-q})^m-1\right]\delta(p,q)}{(\frac{j+k-q}{p-q})^n\left[(p-q)(\frac{j+p-q}{p-q})^m-\alpha\right]^2\delta(j+p,q)-2(p-q-\alpha)^2\delta(p,q)}\right\}, \quad (5.23)$$

which completes the proof of Theorem 5. □

6. Applications of Fractional Calculus

Various operators of fractional calculus (that is, fractional integral and fractional derivatives) have been studied in the literature rather extensively (cf., e.g., [5], [11], [19] and [20]; see also the various references cited therein). For our present investigation, we recall the following definitions.

Definition 1. The fractional integral of order μ is defined, for a function $f(z)$, by

$$D_z^{-\mu}f(z) = \frac{1}{\Gamma(\mu)}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\mu}}d\zeta \quad (\mu > 0), \tag{6.1}$$

where the function $f(z)$ is analytic in a simply- connected domain of the complex z - plane containing the origin and the multiplicity of $(z-\zeta)^{\mu-1}$ is removed by requiring $log(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 2. The fractional derivative of order μ is defined, for a function $f(z)$, by

$$D_z^{\mu}f(z) = \frac{1}{\Gamma(1-\mu)}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{\mu}}d\zeta \quad (0 \leq \mu < 1), \tag{6.2}$$

where the function $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\mu}$ is removed, as in Definition 1.

Definition 3. Under the hypothses of Definition 2, the fractional derivative of order $n+\mu$ is defined, for a function $f(z)$, by

$$D_z^{n+\mu}f(z) = \frac{d^n}{dz^n}\{D_z^{\mu}f(z)\} \quad (0 \leq \mu < 1; n \in N_0). \tag{6.3}$$

In this section, we investigate the growth and distortion properties of functions in the class $T_j(n,m,p,q,\alpha)$, involving the operators $J_{c,p}$ and D_z^{μ}. In order to derive our results, the following lemma given by Chen et al. [5] are used.

Lemma 1. (see Chen et al. [5]). *Let the function $f(z)$ defined by* (1.1). *Then*

$$D_z^{\mu}\{(J_{c,p}f)(z)\} = \frac{\Gamma(p+1)}{\Gamma(p+1-\mu)}z^{p-\mu} - \sum_{k=j+p}^{\infty}\frac{(c+p)\Gamma(k+1)}{(c+k)\Gamma(k+1-\mu)}a_k z^{k-\mu}$$

$$(\mu\in R; c>-p; p,j\in N) \tag{6.4}$$

and

$$J_{c,p}(D_z^{\mu}\{f(z)\}) = \frac{(c+p)\Gamma(p+1)}{(c+p-\mu)\Gamma(p+1-\mu)}z^{p-\mu} - \sum_{k=j+p}^{\infty}\frac{(c+p)\Gamma(k+1)}{(c+k-\mu)\Gamma(k+1-\mu)}a_k z^{k-\mu}$$

$$(\mu\in R; c>-p; p,j\in N), \tag{6.5}$$

provided that no zeros appear in the denominators in (6.4) *and* (6.5).

Theorem 6. *Let the function $f(z)$ defined by* (1.1) *be in the class $T_j(n,m,p,q,\alpha)$. Then*

$$\left|D_z^{-\mu}\{(J_{c,p}f)(z)\}\right| \geq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1+\mu)} - \frac{(c+p)\Gamma(j+p+1)(p-q-\alpha)\delta(p,q)}{(c+j+p)\Gamma(j+p+1+\mu)(\frac{j+p-q}{p-q})^n\left[(p-q)(\frac{j+p-q}{p-q})^m-\alpha\right]\delta(j+p,q)}|z|^j\right\}|z|^{p+\mu}$$

$$(z\in U; 0\leq\alpha<p-q; \mu>0; c>-p; p,j,m\in N; q,n\in N_0; p>q) \tag{6.6}$$

and

$$\left|D_z^{-\mu}\{(J_{c,p}f)(z)\}\right| \leq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1+\mu)} + \frac{(c+p)\Gamma(j+p+1)(p-q-\alpha)\delta(p,q)}{(c+j+p)\Gamma(j+p+1+\mu)(\frac{j+p-q}{p-q})^n\left[(p-q)(\frac{j+p-q}{p-q})^m-\alpha\right]\delta(j+p,q)}|z|^j\right\}|z|^{p+\mu}$$

$$(z\in U; 0\leq\alpha<p-q; \mu>0; c>-p; p,j,m\in N; q,n\in N_0; p>q). \tag{6.7}$$

Each of the assertions (6.6) *and* (6.7) *is sharp.*

Proof. In view of Theorem 1, we have

$$\frac{(\frac{j+p-q}{p-q})^n\left[(p-q)(\frac{j+p-q}{p-q})^m-\alpha\right]\delta(j+p,q)}{(p-q-\alpha)\delta(p,q)}\sum_{k=j+p}^{\infty}a_k$$
$$\leq \sum_{k=j+p}^{\infty}\frac{(\frac{k-q}{p-q})^n\left[(p-q)(\frac{k-q}{p-q})^m-\alpha\right]\delta(k,q)}{(p-q-\alpha)\delta(p,q)}a_k\leq 1, \tag{6.8}$$

which readily yields

$$\sum_{k=j+p}^{\infty}a_k\leq\frac{(p-q-\alpha)\delta(p,q)}{(\frac{j+p-q}{p-q})^n\left[(p-q)(\frac{j+p-q}{p-q})^m-\alpha\right]\delta(j+p,q)}. \tag{6.9}$$

Consider the function $F(z)$ defined in U by

$$\begin{aligned}F(z) &= \frac{\Gamma(p+1+\mu)}{\Gamma(p+1)}z^{-\mu}D_z^{-\mu}\{(J_{c,p}f)(z)\}\\ &= z^p-\sum_{k=j+p}^{\infty}\frac{(c+p)\Gamma(k+1)\Gamma(p+1+\mu)}{(c+k)\Gamma(k+1+\mu)\Gamma(p+1)}a_kz^k\\ &= z^p-\sum_{k=j+p}^{\infty}\Phi(k)a_kz^k\quad(z\in U),\end{aligned}$$

where

$$\Phi(k)=\frac{(c+p)\Gamma(k+1)\Gamma(p+1+\mu)}{(c+k)\Gamma(k+1+\mu)\Gamma(p+1)}\quad(k\geq j+p;p,j\in N;\mu>0)\ . \tag{6.10}$$

Since $\Phi(k)$ is a decreasing function of k when $\mu>0$, we get

$$0<\Phi(k)\leq\Phi(j+p)\;=\;\frac{(c+p)\Gamma(j+p+1)\Gamma(p+1+\mu)}{(c+j+p)\Gamma(j+p+1+\mu)\Gamma(p+1)}$$
$$(c>-p;p,j\in N;\mu>0). \tag{6.11}$$

Thus, by using (6.9) and (6.11), we deduce that

$$|F(z)|\;\geq\;|z|^p-\Phi(j+p)|z|^{j+p}\sum_{k=j+p}^{\infty}a_k$$

$$\geq |z|^p - \frac{(c+p)\Gamma(j+p+1)\Gamma(p+1+\mu)(p-q-\alpha)\delta(p,q)}{(c+j+p)\Gamma(j+p+1+\mu)\Gamma(p+1)(\frac{j+p-q}{p-q})^n\left[(p-q)(\frac{j+p-q}{p-q})^m-\alpha\right]\delta(j+p,q)}|z|^{j+p} \quad (z \in U)$$

and

$$|F(z)| \leq |z|^p + \Phi(j+p)|z|^{j+p}\sum_{k=j+p}^{\infty} a_k$$
$$\leq |z|^p + \frac{(c+p)\Gamma(j+p+1)\Gamma(p+1+\mu)(p-q-\alpha)\delta(p,q)}{(c+j+p)\Gamma(j+p+1+\mu)\Gamma(p+1)(\frac{j+p-q}{p-q})^n\left[(p-q)(\frac{j+p-q}{p-q})^m-\alpha\right]\delta(j+p,q)}|z|^{j+p} \quad (z \in U),$$

which yield the inequalities (6.6) and (6.7) of Theorem 6. The equalities in (6.6) and (6.7) are attained for the function $f(z)$ given by

$$D_z^{-\mu}\{(J_{c,p}f)(z)\} = \left\{\frac{\Gamma(p+1)}{\Gamma(p+1+\mu)} - \frac{(c+p)\Gamma(j+p+1)(p-q-\alpha)\delta(p,q)}{(c+j+p)\Gamma(j+p+1+\mu)(\frac{j+p-q}{p-q})^n\left[(p-q)(\frac{j+p-q}{p-q})^m-\alpha\right]\delta(j+p,q)}z^j\right\}z^{p+\mu} \quad (6.12)$$

or, equivalently, by

$$(J_{c,p}f)(z) = z^p - \frac{(c+p)(p-q-\alpha)\delta(p,q)}{(c+j+p)(\frac{j+p-q}{p-q})^n\left[(p-q)(\frac{j+p-q}{p-q})^m-\alpha\right]\delta(j+p,q)}z^{j+p}. \quad (6.13)$$

Thus we complete the proof of Theorem 6. □

Theorem 7. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $T_j(n,m,p,q,\alpha)$. *Then*

$$|D_z^{\mu}\{(J_{c,p}f)(z)\}| \geq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\mu)} - \frac{(c+p)\Gamma(j+p+1)(p-q-\alpha)\delta(p,q)}{(c+j+p)\Gamma(j+p+1-\mu)(\frac{j+p-q}{p-q})^n\left[(p-q)(\frac{j+p-q}{p-q})^m-\alpha\right]\delta(j+p,q)}|z|^j\right\}|z|^{p-\mu}$$

$$(z\in U; 0\leq\alpha<p-q; 0\leq\mu<1; c>-p; p,j,m\in N; q,n\in N_0; p>q) \quad (6.14)$$

and

$$|D_z^{\mu}\{(J_{c,p}f)(z)\}| \leq \Bigg\{\frac{\Gamma(p+1)}{\Gamma(p+1-\mu)}$$
$$+\frac{(c+p)\Gamma(j+p+1)(p-q-\alpha)\delta(p,q)}{(c+j+p)\Gamma(j+p+1-\mu)(\frac{j+p-q}{p-q})^n(\frac{j+p-q}{p-q})^n\Big[(p-q)(\frac{j+p-q}{p-q})^m-\alpha\Big]\delta(j+p,q)}|z|^j\Bigg\}|z|^{p-\mu}$$
$$(z\in U; 0\leq\alpha<p-q; 0\leq\mu<1; c>-p; p,j,m\in N; q,n\in N_0; p>q). \qquad (6.15)$$

Each of the assertions (6.14) *and* (6.15) *is sharp.*

Proof. It follows from Theorem 1, that

$$\sum_{k=j+p}^{\infty} ka_k \leq \frac{(j+p)(p-q-\alpha)\delta(p,q)}{(\frac{j+p-q}{p-q})^n\left[(p-q)(\frac{j+p-q}{p-q})^m-\alpha\right]\delta(j+p,q)}\ . \qquad (6.16)$$

We consider the function $H(z)$ defined in U by

$$\begin{aligned} H(z) &= \frac{\Gamma(p+1-\mu)}{\Gamma(p+1)}z^{\mu}D_z^{\mu}\{(J_{c,p}f)(z)\} \\ &= z^p-\sum_{k=j+p}^{\infty}\frac{(c+p)\Gamma(k)\Gamma(p+1-\mu)}{(c+k)\Gamma(k+1-\mu)\Gamma(p+1)}ka_kz^k \\ &= z^p-\sum_{k=j+p}^{\infty}\Psi(k)ka_kz^k \quad (z\in U) \end{aligned}$$

where, for convenience,

$$\Psi(k)=\frac{(c+p)\Gamma(k)\Gamma(p+1-\mu)}{(c+k)\Gamma(k+1-\mu)\Gamma(p+1)} \quad (k\geq j+p; p,j\in N; 0\leq\mu<1).$$

Since $\Psi(k)$ is a decreasing function of k when $\mu<1$, we find that

$$0<\Psi(k)\leq\Psi(j+p)=$$

$$\frac{(c+p)\Gamma(j+p)\Gamma(p+1-\mu)}{(c+j+p)\Gamma(j+p+1-\mu)\Gamma(p+1)} \quad (c>-p; p,j\in N; 0\leq\mu<1). \quad (6.17)$$

Consequently, with the aid of (6.16) and (6.17), we find that

$$|H(z)| \geq |z|^p - \Psi(j+p)|z|^{j+p} \sum_{k=j+p}^{\infty} ka_k$$
$$\geq |z|^p - \frac{(c+p)\Gamma(j+p+1)\Gamma(p+1-\mu)(p-q-\alpha)\delta(p,q)}{(c+j+p)\Gamma(j+p+1-\mu)\Gamma(p+1)(\frac{j+p-q}{p-q})^n\left[(p-q)(\frac{j+p-q}{p-q})^m-\alpha\right]\delta(j+p,q)}|z|^{j+p} \quad (z \in U)$$

and

$$|H(z)| \leq |z|^p + \Psi(j+p)|z|^{j+p} \sum_{k=j+p}^{\infty} ka_k$$
$$\leq |z|^p + \frac{(c+p)\Gamma(j+p+1)\Gamma(p+1-\mu)(p-q-\alpha)\delta(p,q)}{(c+j+p)\Gamma(j+p+1-\mu)\Gamma(p+1)(\frac{j+p-q}{p-q})^n\left[(p-q)(\frac{j+p-q}{p-q})^m-\alpha\right]\delta(j+p,q)}|z|^{j+p} \quad (z \in U)$$

which yield the inequalities (6.14) and (6.15) of Theorem 7. The equalities in (6.14) and (6.15) are attained for the function $f(z)$ given by

$$D_z^{\mu}\{(J_{c,p}f)(z)\} = \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\mu)} - \frac{(c+p)\Gamma(j+p+1)(p-q-\alpha)\delta(p,q)}{(c+j+p)\Gamma(j+p+1-\mu)(\frac{j+p-q}{p-q})^n\left[(p-q)(\frac{j+p-q}{p-q})^m-\alpha\right]\delta(j+p,q)}z^j\right\}z^{p-\mu} \quad (6.18)$$

or for the function $(J_{c,p}f)(z)$ given by (6.13). The proof of Theorem 7 is thus completed.

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