Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **5** (2010), No. 2, pp. 215-224

SOME NOTES ON A CERTAIN CLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH THE DZIOK-SRIVASTAVA OPERATOR

BY

JIN-LIN LIU

Abstract

The main object of the present paper is to investigate several further interesting properties of the class $V_{p,q,s}(\alpha_1; A, B)$ which was recently introduced and studied by Dziok and Srivastava [Appl.Math.Comput.103(1999)1-13].

1. Introduction and Definitions

For $p \in \mathbb{N} := \{1, 2, 3, \ldots\}$, we denote by $A(p)$ the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $U := \{z\colon z \in C \text{ and } |z| < 1\}$.

Let $f(z)$ and $g(z)$ be analytic in U. We say that the function $g(z)$ is subordinate to $f(z)$ if there exists an analytic function $w : U \to U$ with $w(0) = 0$ such that $g(z) = f(w(z))$ for $z \in U$. This relation is denoted by $g(z) \prec f(z)$. In case $f(z)$ is univalent in U we have that the subordination $g(z) \prec f(z)$ is equivalent to $g(0) = f(0)$ and $g(U) \subset f(U)$.

Received September 22, 2008 and in revised form June 30, 2009.

AMS Subject Classification: Primary: 30C45; 26A33; Secondary: 33C20.

Key words and phrases: Analytic function, hadamard product, Dziok-Srivastava operator, generalized hypergeometric function, δ-neighborhood.

For analytic functions

$$f(z)=\sum_{n=0}^{\infty}a_nz^n \quad \text{and} \quad g(z)=\sum_{n=0}^{\infty}b_nz^n,$$

by $f*g$ we denote the Hadamard product (or convolution) of f and g, defined by

$$(f*g)(z):=\sum_{n=0}^{\infty}a_nb_nz^n. \tag{1.2}$$

Making use of the Hadamard product (or convolution) given by (1.2), we now define the Dziok-Srivastava operator:

$$H_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s):A(p)\to A(p),$$

which was introduced and studied in a series of recent papers by Dziok and Srivastava ([2, 3, 4]; see also [5, 9, 10, 11]).

We recall that for $q,s\in N_0=\mathbb{N}\cup 0$ with $q\le s+1$ and complex parameters $\alpha_1,\ldots,\alpha_q$ and $\beta_1,\ldots,\beta_s$ with $\beta_j\ne 0,-1,-2,\ldots$ for $j=1,\ldots,s$, the generalized hypergeometric function ${}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)$ is defined by

$${}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_q)_n}{(\beta_1)_n\cdots(\beta_s)_n}\cdot\frac{z^n}{n!}$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\begin{cases}1 & ,n=0\\ \lambda(\lambda+1)\cdots(\lambda+n-1) & ,n\in\mathbb{N}.\end{cases}$$

Corresponding to a function $h_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)$ defined by

$$h_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)=z^p{}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z),$$

Dziok and Srivastava [2] considered a linear operator defined on $A(p)$ by the following Hadamard product

$$H_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f(z)=h_p(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)*f(z),z\in U.$$

For convenience, we write

$$H_{p,q,s}(\alpha_1) = H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s).$$

Thus, after some calculations, we have

$$z(H_{p,q,s}(\alpha_1)f(z))' = \alpha_1 H_{p,q,s}(\alpha_1+1)f(z) - (\alpha_1 - p)H_{p,q,s}(\alpha_1)f(z). \quad (1.3)$$

We observe that, for a function f of the form (1.1), we have

$$H_{p,q,s}(\alpha_1)f(z) = z^p + \sum_{n=p+1}^{\infty} \Gamma_n(\alpha_1)a_n z^n, \quad (1.4)$$

where

$$\Gamma_n(\alpha_1) := \frac{(\alpha_1)_{n-p} \cdots (\alpha_q)_{n-p}}{(\beta_1)_{n-p} \cdots (\beta_s)_{n-p}(n-p)!}. \quad (1.5)$$

The Dziok-Srivastava operator $H_{p,q,s}(\alpha_1)$ includes various other linear operators which were considered in earlier works. In particular, for $p = s = 1$ and $q = 2$, we obtain the liner operator

$$\mathscr{F}(\alpha_1, \alpha_2, \beta_1)f(z) = H_1(\alpha_1, \alpha_2; \beta_1)f(z),$$

which was defined by Hohlov [7]. Setting moreover $\alpha_2 = 1$,we obtain the Carlson-Shaffer operator

$$\mathscr{L}(\alpha_1, \beta_1)f(z) = H_1(\alpha_1, 1; \beta_1)f(z),$$

which was introduced by Carlson and Shaffer [1].

Many interesting subclasses of analytic functions, associated with the Dziok-Srivastava operator $H_{p,q,s}(\alpha_1)$ and its many special cases, were investigated recently by (for example) Dziok and Srivastava [2, 3, 4], Gangadharan et al. [5], Liu [9], Liu and Srivastava [10, 11] and others (see also [8, 13, 15, 16, 17]).

Let $p, q, s \in N$ and suppose that the parameters $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ are positive real numbers. Also let

$$0 < B < 1 \quad \text{and} \quad -B \leq A < B.$$

We denote by $V_{p,q,s}(\alpha_1; A, B)$ the class of functions $f \in A(p)$ of the form

$$f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n \quad , z \in U, \tag{1.6}$$

where $a_n \geq 0, n \in \mathbb{N}$,which also satisfy the following condition

$$\alpha_1 \frac{H_{p,q,s}(\alpha_1 + 1)f(z)}{H_{p,q,s}(\alpha_1)f(z)} + p - \alpha_1 \prec p\frac{1 + Az}{1 + Bz}. \tag{1.7}$$

The class $V_{p,q,s}(\alpha_1; A, B)$ was first introduced and studied by Dziok and Srivastava [2]. Many interesting properties such as coefficients estimates, distortion theorems, extreme points, and the radii of convexity and starlikeness for the class $V_{p,q,s}(\alpha_1; A, B)$ were given by Dziok and Srivastava [2]. In the present sequel to these earlier works, we shall derive some interesting characteristics of the δ -neighborhood associated with the class $V_{p,q,s}(\alpha_1; A, B)$.

2. Main results

We begin by recalling each of the following lemmas which will be required in our present investigation.

Lemma 1. (Dziok and Srivastava [2]). *A function f of the form* (1.6) *belongs to $V_{p,q,s}(\alpha_1; A, B)$ if and only if*

$$\sum_{n=p+1}^{\infty} ((B + 1)n - (A + 1)p)\Gamma_n(\alpha_1)a_n \leq p(B - A), \tag{2.1}$$

where $\Gamma_n(\alpha_1)$ is defined by Eq.(1.5).

Making use of the same method by Liu and Srivastava [10, 12], we immediately have:

Lemma 2. *Let $\lambda > -p$. If $f \in V_{p,q,s}(\alpha_1; A, B)$, then the function $F(z)$ defined by*

$$F(z) = \frac{\lambda + p}{z^\lambda} \int_0^z t^{\lambda - 1} f(t)dt \tag{2.2}$$

also belongs to $V_{p,q,s}(\alpha_1; A, B)$.

Following the earlier works (see [6, 14]; see also [10, 12]), we now define the δ-neighborhood of a function $f \in A(p)$ for $\delta \geq 0, 0 < B < 1$ and $-B \leq A < B$ by

$$N_{\delta,A,B}(f) = \left\{ g \in A(p) : g(z) = z^p - \sum_{n=p+1}^{\infty} b_n z^n \quad (b_n \geq 0) \text{ and} \right.$$

$$\left. \sum_{n=p+1}^{\infty} \frac{((B+1)n - (A+1)p)\Gamma_n(\alpha_1)}{p(B-A)} |b_n - a_n| \leq \delta \right\}. \tag{2.3}$$

Theorem 1. *If $f \in V_{p,q,s}(\alpha_1 + 1; A, B)$,then*

$$N_{\delta,A,B}(f) \subset V_{p,q,s}(\alpha_1; A, B), \tag{2.4}$$

where $\delta := \frac{1}{\alpha_1+1}$.The result is sharp in the sense that the number δ cannot be increased.

Proof. It is easily seen from (1.7) that a function $g \in A(p)$ belongs to the class $V_{p,q,s}(\alpha_1; A, B)$ if and only of for $z \in U$ we have:

$$\frac{\alpha_1(H_{p,q,s}(\alpha_1+1)g(z) - H_{p,q,s}(\alpha_1)g(z))}{p(A-B)H_{p,q,s}(\alpha_1)g(z) - B\alpha_1(H_{p,q,s}(\alpha_1+1)g(z) - H_{p,q,s}(\alpha_1)g(z))} \neq \sigma \tag{2.5}$$

for any $\sigma \in \mathbb{C}$ with $|\sigma| = 1$,which is equivalent to:

$$\frac{(g * h)(z)}{z^p} \neq 0 \quad , z \in U, \tag{2.6}$$

where we denoted by $h : U \to \mathbb{C}$ the function defined by

$$\begin{aligned} h(z) &= z^p + \sum_{n=p+1}^{\infty} c_n z^n \\ &= z^p + \sum_{n=p+1}^{\infty} \frac{(n-p)(1+B\sigma) + p\sigma(B-A)}{p\sigma(A-B)} \Gamma_n(\alpha_1) z^n. \end{aligned} \tag{2.7}$$

We easily find from (2.7) that

$$\begin{aligned}|c_n| &= \left|\frac{(n-p)(1+B\sigma)+p\sigma(B-A)}{p\sigma(A-B)}\Gamma_n(\alpha_1)\right| \\ &\leq \frac{n(1+B)-p(1+A)}{p(B-A))}\Gamma_n(\alpha_1)\end{aligned}$$

for any $n > p$. If $f \in V_{p,q,s}(\alpha_1+1; A, B)$ is given by (1.6), we obtain that

$$\begin{aligned}\left|\frac{(f*h)(z)}{z^p}\right| &= \left|1-\sum_{n=p+1}^{\infty} a_n c_n z^{n-p}\right| \\ &\geq 1-\sum_{n=p+1}^{\infty}\frac{n(1+B)-p(1+A)}{p(B-A)}\Gamma_n(\alpha_1)a_n|z|^{n-p} \\ &> 1-\frac{\alpha_1}{\alpha_1+1}\sum_{n=p+1}^{\infty}\frac{n(1+B)-p(1+A)}{p(B-A)}\Gamma_n(\alpha_1+1)a_n \\ &\geq 1-\frac{\alpha_1}{\alpha_1+1} = \frac{1}{\alpha_1+1} := \delta\end{aligned}$$

by appealing to Lemma 1.

Now, if we let

$$\varphi(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n \in N_\delta(f),$$

where $\delta = \frac{1}{\alpha_1+1}$, then

$$\begin{aligned}\left|\frac{(f(z)-\varphi(z))*h(z)}{z^p}\right| &= \left|\sum_{n=p+1}^{\infty}(a_n-b_n)c_n z^{n-p}\right| \\ &\leq \sum_{n=p+1}^{\infty}\frac{n(1+B)-p(1+A)}{p(B-A)}\Gamma_n(\alpha_1)|a_n-b_n|\cdot|z|^{n-p} \\ &< \delta\end{aligned}$$

for any $z \in U$. Thus, for any complex number σ with $|\sigma| = 1$, we have

$$\frac{(\varphi*h)(z)}{z^p} \neq 0 \quad , z \in U,$$

which implies that $\varphi \in V_{q,s}(\alpha_1; A, B)$.

In order to see the sharpness of the assertion of the theorem, we consider the functions $f, g : U \to \mathbb{C}$ defined by

$$f(z) = z^p - \frac{p(B-A)}{((B+1)(p+1)-(A+1)p)\Gamma_{p+1}(\alpha_1+1)} z^{p+1}$$

and

$$g(z) = z^p - \Big(\frac{p(B-A)}{((B+1)(p+1)-(A+1)p)\Gamma_{p+1}(\alpha_1+1)}$$

$$+\frac{p(B-A)\delta'}{((B+1)(p+1)-(A+1)p)\Gamma_{p+1}(\alpha_1)}\Big) z^{p+1},$$

where $\delta' > \delta = \frac{1}{\alpha_1+1}$ are arbitrarily fixed . It is easy to see that we have $f \in V_{p,q,s}(\alpha_1+1; A, B)$ and $g \in N_{\delta',A,B}(f)$. Thus, by observing from Lemma 1 that the function g is not in the class $V_{p,q,s}(\alpha_1; A, B)$, it follows that the constant δ in the statement of the theorem cannot be increased, completing the proof of the theorem. □

Theorem 2. *If the function $f(z)$ is in the class $V_{p,q,s}(\alpha_1; A, B)$, then for any $\lambda > -p$ the function $F(z)$ defined by* (2.2) *belongs to $N_1(f)$. The result is sharp in the sense that the constant* 1 *can not be decreased.*

Proof. Suppose that the function $f(z)$ is in the class $V_{p,q,s}(\alpha_1; A, B)$. Then it follows from (2.2) and Lemma 2 that

$$\begin{aligned} F(z) &:= z^p - \sum_{n=p+1}^{\infty} b_n z^n \\ &= z^p - \sum_{n=p+1}^{\infty} \frac{\lambda+p}{\lambda+n} a_n z^n \in V_{p,q,s}(\alpha_1; A, B). \end{aligned} \tag{2.8}$$

Since by hypothesis $f \in V_{p,q,s}(\alpha_1; A, B)$, we have

$$\begin{aligned} &\sum_{n=p+1}^{\infty} \frac{((B+1)n-(A+1)p)\Gamma_n(\alpha_1)}{p(B-A)} |b_n - a_n| \\ &= \sum_{n=p+1}^{\infty} \frac{((B+1)n-(A+1)p)\Gamma_n(\alpha_1)}{p(B-A)} \cdot \frac{n-p}{\lambda+n} a_n \end{aligned}$$

$$\leq \sum_{n=p+1}^{\infty} \frac{((B+1)n-(A+1)p)\Gamma_n(\alpha_1)}{p(B-A)} a_n$$
$$\leq 1,$$

which shows that $F(z) \in N_1(f)$.

In order to see the sharpness for an arbitrarily fixed $n \geq p+1$, we consider the function $f : U \to \mathbb{C}$ defined by

$$f(z) = z^p - \frac{p(B-A)}{((B+1)n-(A+1)p)\Gamma_n(\alpha_1)} z^n, \quad z \in U.$$

It is easy to see that $f \in V_{p,q,s}(\alpha_1; A, B)$, and from (2.2) we have

$$\begin{aligned} F(z) &= \frac{\lambda+p}{z^\lambda} \int_0^z t^{\lambda-1} f(t) dt \\ &= z^p - \frac{p(B-A)}{((B+1)n-(A+1)p)\Gamma_n(\alpha_1)} \cdot \frac{\lambda+p}{\lambda+n} z^n, \quad z \in U. \end{aligned}$$

We have

$$\frac{((B+1)n-(A+1)p)\Gamma_n(\alpha_1)}{p(B-A)} |b_n - a_n| = \frac{n-p}{\lambda+n},$$

which shows that the function F belongs to the class $N_{\delta,A,B}(f)$ for any $\delta \leq \frac{n-p}{n+\lambda}$. Since $\frac{n-p}{n+\lambda} \nearrow 1$ as $n \to \infty$, this shows that the statement of the theorem cannot be decreased, concluding the proof. □

Acknowledgments

The present investigation is partly supported by Jiangsu Gaoxiao Natural Science Foundation (04KJB110154).

References

1. B. C. Carlson and D. B. Shaffer, Starlike and prestarlike hypergeometric functions, *SIAM J. Math. Anal.*, **15**(1984), 737-745.

2. J. Dziok and H. M. Srivastava, Classes of analytic functions associated with the generalized hypergeometric function, *Appl. Math. Comput.*, **103**(1999), 1-13.

3. J. Dziok and H. M. Srivastava, Some subclasses of analytic functions with fixed

argument of coefficients associated with the generalized hypergeometric function, *Adv. Stud. Contemp. Math.*, **5**(2002), 115-125.

4. J. Dziok and H. M. Srivastava, Certain subclasses of analytic functions associated with the generalized hypergeometric function, *Integral Transform. Spec. Funct.*, **14**(2003), 7-18.

5. A. Gangadharan, T. N. Shanmugam and H. M. Srivastava, Generalized hypergeometric functions associated with k-uniformly convex functions, *Comput. Math. Appl.* **44**(2002), 1515-1526.

6. A. W. Goodman, Univalent functions and nonanalytic curves, *Proc. Amer. Math. Soc.*, **8**(1957), 598-601.

7. Yu. E. Hohlov, Operators and operations in the class of univalent functions, *Izv. Vysš. Učebn. Zaved. Matematika*, **10**(1978), 83-89.

8. Y. C. Kim and H. M. Srivastava, Fractional integral and other linear operators associated with the Gaussian hypergeometric function, *Complex Variables Theory Appl.*, **34**(1997), 293-312.

9. J.-L. Liu, Strongly starlike functions associated with the Dziok-Srivastava operator, *Tamkang J.Math.*, **35**(2004), 37-42.

10. J.-L. Liu and H. M. Srivastava, Classes of meromorphically multivalent functions associated with the generalized hypergeometric function, *Math. Comput. Modelling*, **38**(2004), 21-34.

11. J.-L. Liu and H. M. Srivastava, Certain properties of the Dziok-Srivastava operator, *Appl. Math. Comput.*, **159**(2004), 485-493.

12. J.-L. Liu and H. M. Srivastava, A linear operator and associated families of meromorphically multivalent functions, *J. Math. Anal. Appl.*, **259**(2001), 566-581.

13. S. Owa and H. M. Srivastava, Univalent and starlike generalized hypergeometric functions, *Canad. J. Math.*, **39**(1987), 1057-1077.

14. S. Ruscheweyh, Neighborhoods of univalent functions, *Proc. Amer. Math. Soc.*, **81**(1981), 521-527.

15. H. M. Srivastava and S. Owa, Some characterization and distortion theorems involving fractional calculus, generalized hypergeometric functions, Hadamard products, linear operators, and certain subclasses of analytic functions, *Nagoya Math. J.*, **106**(1987), 1-28.

16. H. M. Srivastava and S. Owa(Eds.), *Univalent Functions, Fractional calculus, and Their Applications*, Halsted Press (Ellis Horwood Limited, Chichester), John wiley and Sons, New York, Chichester, Brisbane, and Toronto, 1989.

17. H. M. Srivastava, S. Owa(Eds.), *Current Topics in Analytic Function Theory, World Scientific Publishing Company*, Singapore, New Jersey, London and Hong Kong, 1992.

Department of Mathematics, Yangzhou University, Yangzhou 225002, Jiangsu, People's Republic of China.

E-mail: jlliu@yzu.edu.cn