Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **5** (2010), No. 2, pp. 225-232

POISSON'S EQUATION AND GENERALIZED FUNCTIONS

BY

DENNIS NEMZER

Abstract

Solutions of Poisson's equation in the space of Boehmians are investigated. In particular, given that the forcing function is a Boehmian with compact support, necessary and sufficient conditions are established for Poisson's equation to have a solution with compact support.

1. Introduction

In this note, we will be concerned with a space of generalized functions known as Boehmians (see [3]). The space of Schwartz distributions [7] can be identified with a proper subspace of Boehmians.

Consider Poisson's equation

$$\Delta u = f, \tag{1.1}$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_d^2}$ and $f \in \mathcal{E}'(\mathbb{R}^d)$, the space of distributions on $\mathbb{R}^d$ with compact supports.

Notice that the Dirac delta measure $\delta \in \mathcal{E}'(\mathbb{R}^d)$ and $\Delta E = \delta$, where E is the fundamental solution of (1.1). However, E does not have compact support. This raises the question; when does (1.1) have a solution in $\mathcal{E}'(\mathbb{R}^d)$?

The following, which is a special case of Theorem 8.4 in [6], gives a complete answer to the above question for the space of distributions.

Received November 25, 2008 and in revised form June 02, 2009.

AMS Subject Classification: 35D99, 44A40, 46F99.

Key words and phrases: Boehmian, compact support, distribution, Poisson equation.

Theorem 1.1. *Let $f \in \mathcal{E}'(\mathbb{R}^d)$. Then* (1.1) *has a solution in $\mathcal{E}'(\mathbb{R}^d)$ if and only if $\frac{\widehat{f}(z_1,\ldots,z_d)}{z_1^2+\ldots+z_d^2}$ is an entire function, where $\widehat{f}$ is the Fourier transform of f. When this condition is satisfied,* (1.1) *has a unique solution u in $\mathcal{E}'(\mathbb{R}^d)$; the support of this u lies in the convex hull of the support of f.*

The space of Boehmians contains objects which have compact supports but are not distributions. Thus, it is natural to ask; is there a similar theorem for Boehmians?

In [5], a conjecture for the space of Boehmians similar to Theorem 1.1 was proposed by the author for $d \geq 3$. In this note, we will establish the validity of this conjecture.

The proof of Theorem 1.1 relies on the Paley-Wiener theorem for distributions [6]. The proof of the corresponding result, Theorem 3.1, for Boehmians would be simplified if a Paley-Wiener theorem for Boehmians were available. Moreover, the case for $d = 2$ would most likely be included. J. Burzyk [2] did prove a Paley-Wiener theorem for Boehmians for $d = 1$.

2. Some Preliminaries

Let $C(\mathbb{R}^d)$ denote the space of all continuous functions on $\mathbb{R}^d$, and let $\mathcal{D}(\mathbb{R}^d)$ denote the subspace of $C(\mathbb{R}^d)$ of all infinitely differentiable functions with compact support. If $x, y \in \mathbb{R}^d$, then $x = (x_1, x_2, \ldots, x_d)$, $y = (y_1, y_2, \ldots, y_d)$, $x \cdot y = x_1y_1 + x_2y_2 + \ldots + x_dy_d$, and $||x|| = \sqrt{x \cdot x}$.

A sequence $\varphi_n \in \mathcal{D}(\mathbb{R}^d)$ is called a *delta sequence* provided:

(i) $\int \varphi_n = 1$ for all $n \in \mathbb{N}$,

(ii) $\int |\varphi_n| \leq M$ for some constant M and all $n \in \mathbb{N}$,

(iii) For every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $\varphi_n(x) = 0$ for $||x|| > \varepsilon$ and $n > n_\varepsilon$.

A pair of sequences (f_n, φ_n) is called a *quotient of sequences* if $f_n \in C(\mathbb{R}^d)$ for $n \in \mathbb{N}$, $\{\varphi_n\}$ is a delta sequence, and $f_k * \varphi_m = f_m * \varphi_k$ for all $k, m \in \mathbb{N}$, where $*$ denotes convolution:

$$(f * \varphi)(x) = \int_{\mathbb{R}^d} f(x-u)\varphi(u)du, \qquad (2.1)$$

Two quotients of sequences (f_n, φ_n) and (g_n, ψ_n) are said to be equivalent if $f_k * \psi_m = g_m * \varphi_k$ for all $k, m \in \mathbb{N}$. A straightforward calculation shows that this is an equivalence relation. The equivalence classes are called Boehmians. The space of all Boehmians will be denoted by $\beta(\mathbb{R}^d)$ and a typical element of $\beta(\mathbb{R}^d)$ will be written as $F = \left[\frac{f_n}{\varphi_n}\right]$. A function $f \in C(\mathbb{R}^d)$ can be identified with the Boehmian $\left[\frac{f*\varphi_n}{\varphi_n}\right]$, where $\{\varphi_n\}$ is any delta sequence. It is convenient to view $C(\mathbb{R}^d)$ as a subspace of $\beta(\mathbb{R}^d)$.

For $\psi \in \mathcal{D}(\mathbb{R}^d)$ and $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta(\mathbb{R}^d)$, $F * \psi$ is defined as

$$F * \psi = \left[\frac{f_n * \psi}{\varphi_n}\right]. \tag{2.2}$$

Definition 2.1. A sequence $\{F_n\} \in \beta(\mathbb{R}^d)$ is said to be δ-*convergent* to $F \in \beta(\mathbb{R}^d)$, denoted $\delta\text{-}\lim_{n\to\infty} F_n = F$, if there exists a delta sequence $\{\varphi_n\}$ such that for for all $k, n \in \mathbb{N}$, $F_n * \varphi_k$, $F * \varphi_k \in C(\mathbb{R}^d)$ and, for each $k \in \mathbb{N}$, $F_n * \varphi_k \to F * \varphi_k$ uniformly on compact sets as $n \to \infty$.

Let Ω be an open subset of $\mathbb{R}^d$. A Boehmian F is said to vanish on Ω, provided that there exists a delta sequence $\{\varphi_n\}$ such that $F * \varphi_n \in C(\mathbb{R}^d)$ for all $n \in \mathbb{N}$, and $F * \varphi_n \to 0$ uniformly on compact subsets of Ω as $n \to \infty$.

The support of a Boehmian F is the complement of the largest open set on which F vanishes. The space of all Boehmians with compact support will be denoted by $\beta_c(\mathbb{R}^d)$.

A Boehmian $F = \left[\frac{f_n}{\varphi_n}\right]$ has compact support if and only if there exists $r > 0$ such that $\operatorname{supp} f_n \subset B(0, r)$ for all $n \in \mathbb{N}$, where $B(0, r)$ is the open ball centered at the origin with radius r.

Convolution can be extended to $\beta(\mathbb{R}^d) \times \beta_c(\mathbb{R}^d)$. Let $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta(\mathbb{R}^d)$ and $G = \left[\frac{g_n}{\psi_n}\right] \in \beta_c(\mathbb{R}^d)$. Then $F * G = \left[\frac{f_n * g_n}{\varphi_n * \psi_n}\right] \in \beta(\mathbb{R}^d)$.

Let $F, G \in \beta_c(\mathbb{R}^d)$. T. K. Boehme [1] proved that

$$\langle \operatorname{supp} F * G \rangle = \langle \operatorname{supp} F \rangle + \langle \operatorname{supp} G \rangle, \tag{2.3}$$

where $\langle \cdot \rangle$ denotes the convex hull.

The Fourier transform of the Boehmian $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta_c(\mathbb{R}^d)$, denoted $\widehat{F}$, is the entire function given by

$$\widehat{F}(z) = \lim_{n\to\infty} \widehat{f}_n(z), \quad z \in \mathbb{C}^d \tag{2.4}$$

where $\widehat{f}_n(z) = \int_{\mathbb{R}^d} f_n(x) e^{-ix\cdot z} dx$.

Remarks.

(i) The limit in (2.4) exists and is independent of the representative.

(ii) The convergence in (2.4) is uniform on compact subsets of $\mathbb{C}^d$.

Lemma 2.2. *Let $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta_c(\mathbb{R}^d)$ and $G \in \beta(\mathbb{R}^d)$. Then, $\delta\text{-}\lim_{n\to\infty}(G * f_n) = G * F$.*

Proof. Let $G = \left[\frac{g_n}{\psi_n}\right]$. Since $\delta\text{-}\lim_{n\to\infty} f_n = F$ (see [3]), there exists a delta sequence $\{\gamma_n\}$ such that for all $k, n \in \mathbb{N}, f_n * \gamma_k$, $F * \gamma_k \in C(\mathbb{R}^d)$ and, for each $k \in \mathbb{N}$, $f_n * \gamma_k \to F * \gamma_k$ uniformly on compact sets as $n \to \infty$.

Now, let $\sigma_n = \gamma_n * \psi_n, n \in \mathbb{N}$. Then $\{\sigma_n\}$ is a delta sequence and $G * \sigma_k = g_k * \gamma_k \in C(\mathbb{R}^d)$, $k \in \mathbb{N}$. And, $(G * f_n) * \sigma_k = g_k * (f_n * \gamma_k) \in C(\mathbb{R}^d)$, $k, n \in \mathbb{N}$.

Moreover, for each k, $(G * f_n) * \sigma_k = g_k * (f_n * \gamma_k) \to g_k * (F * \gamma_k) = (G * F) * \sigma_k$ where the convergence is uniform on compact sets as $n \to \infty$.

That is, $\delta\text{-}\lim_{n\to\infty}(G * f_n) = G * F$. □

3. The Main Result

Theorem 3.1. *Let $F \in \beta_c(\mathbb{R}^d)$, where $d \geq 3$. Then there exists $U \in \beta_c(\mathbb{R}^d)$ such that $\Delta U = F$ if and only if $\frac{\widehat{F}(z_1,\ldots,z_d)}{z_1^2+\ldots+z_d^2}$ is an entire function. When this condition is satisfied, $\langle supp\, U\rangle = \langle supp\, F\rangle$ and $U \in \beta_c(\mathbb{R}^d)$ is unique.*

Proof. Let $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta_c(\mathbb{R}^d)$.

Suppose $\frac{\widehat{F}(z_1,\ldots,z_d)}{z_1^2+\ldots+z_d^2}$ is entire. Then, $\frac{\widehat{f}_n(z_1,\ldots,z_d)}{z_1^2+\ldots+z_d^2} = \frac{\widehat{F}(z_1,\ldots,z_d)\widehat{\varphi}_n(z_1,\ldots,z_d)}{z_1^2+\ldots+z_d^2}$ is entire for all $n \in \mathbb{N}$.

Now, $\Delta(E * f_n) = f_n$, $n \in \mathbb{N}$, where E is the fundamental solution for the Laplacian. By Theorem 4.2 in [5] and Theorem 1.1, $\operatorname{supp}(E * f_n)$ is compact for all $n \in \mathbb{N}$. Also,

$$\operatorname{supp}(E * f_n) \subset \langle \operatorname{supp} f_n \rangle \subset clB(0, r), \tag{3.1}$$

for some $r > 0$ and all $n \in \mathbb{N}$. $clB(0, r)$ denotes the closed ball centered at the origin with radius r.

By Lemma 2.2

$$\delta\text{-}\lim_{n \to \infty} (E * f_n) = E * F. \tag{3.2}$$

(3.1) and (3.2) give

$$\operatorname{supp}(E * F) \subset clB(0, r).$$

Thus, $\operatorname{supp}(E * F)$ is compact.
Since $\Delta(E*F) = (\Delta E)*F = \delta * F = F$, $U = E*F$ is the desired Boehmian.

For the other direction, let $U \in \beta_c(\mathbb{R}^d)$ such that $\Delta U = F$. Then, $U = E * F$ (see [5]). Now,

$$E * f_n = E * (F * \varphi_n) = (E * F) * \varphi_n = U * \varphi_n \in \beta_c(\mathbb{R}^d), n \in \mathbb{N}.$$

By above and the fact that for each $n \in \mathbb{N}, E * f_n \in D'(\mathbb{R}^d)$, we obtain

$$E * f_n \in \mathcal{E}'(\mathbb{R}^d), \text{ for all } n \in \mathbb{N}.$$

Also,

$$\Delta(E * f_n) = f_n, \, n \in \mathbb{N}.$$

Since $f_n \in \mathcal{E}'(\mathbb{R}^d)$, Theorem 1.1 yields

$$\frac{\widehat{f}_n(z_1, \ldots, z_d)}{z_1^2 + \ldots + z_d^2} \text{ is entire }, n \in \mathbb{N}.$$

Now, $\frac{\widehat{F}(z_1,\ldots,z_d)}{z_1^2+\ldots+z_d^2} = \frac{\widehat{F}(z_1,\ldots,z_d)\widehat{\varphi}_n(z_1,\ldots,z_d)}{z_1^2+\ldots+z_d^2} \frac{1}{\widehat{\varphi}_n(z_1,\ldots,z_d)} = \frac{\widehat{f}_n(z_1,\ldots,z_d)}{z_1^2+\ldots+z_d^2} \frac{1}{\widehat{\varphi}_n(z_1,\ldots,z_d)}$,

for all $n \in \mathbb{N}$, provided $\widehat{\varphi}_n(z_1, \ldots, z_d) \neq 0$.

Also,

$$\widehat{f}_n\widehat{\varphi}_k = \widehat{f}_k\widehat{\varphi}_n, \text{ for all } k, n \in \mathbb{N}.$$

This follows from

$$f_n * \varphi_k = f_k * \varphi_n, \text{ for all } k, n \in \mathbb{N}.$$

By the above and the fact that

$$\widehat{\varphi}_n \to 1 \text{ uniformly on compact sets as } n \to \infty,$$

we see that

$$\frac{\widehat{F}(z_1, \ldots, z_d)}{z_1^2 + \ldots + z_d^2} \text{ is entire.}$$

Now assume that the condition is satisfied. We see from the previous part of the proof that $E * F$ is the unique solution in $\beta_c(\mathbb{R}^d)$ and $\operatorname{supp}(E * f_n)$ is compact for all $n \in \mathbb{N}$. By (2.3) and Theorem 1.1, we obtain

$$\operatorname{supp}(E * f_n) \subset \langle \operatorname{supp} f_n \rangle = \langle \operatorname{supp}(F * \varphi_n) \rangle = \langle \operatorname{supp} F \rangle + \langle \operatorname{supp} \varphi_n \rangle,$$

for all $n \in \mathbb{N}$.

So, given $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\operatorname{supp}(E * f_n) \subset \langle \operatorname{supp} F \rangle + B(0, \varepsilon), \tag{3.3}$$

for all $n \geq n_\varepsilon$.

By Lemma 2.2,

$$\delta\text{-}\lim_{n\to\infty}(E * f_n) = E * F. \tag{3.4}$$

(3.3) and (3.4) yield

$$\operatorname{supp}(E * F) \subset \langle \operatorname{supp} F \rangle.$$

Therefore,

$$\langle \operatorname{supp}(E * F) \rangle \subset \langle \operatorname{supp} F \rangle. \tag{3.5}$$

Since

$$\operatorname{supp} F = \operatorname{supp} \Delta(E * F) \subset \operatorname{supp}(E * F),$$

we have

$$\langle \operatorname{supp} F \rangle \subset \langle \operatorname{supp} (E * F) \rangle. \tag{3.6}$$

So by (3.5) and (3.6),

$$\langle \operatorname{supp} (E * F) \rangle = \langle \operatorname{supp} F \rangle,$$

and the proof is complete. □

From the theorem we obtain a useful corollary which gives necessary conditions for Poisson's equation to have a solution in $\beta_c(\mathbb{R}^d)$.

Corollary 3.2. *Let $F \in \beta_c(\mathbb{R}^d)$, $d \geq 3$. If $\Delta U = F$ for some $U \in \beta_c(\mathbb{R}^d)$, then*

$$\widehat{F}(0) = \frac{\partial \widehat{F}}{\partial z_i}(0) = \frac{\partial^2 \widehat{F}}{\partial z_i \partial z_j}(0) = \frac{\partial^3 \widehat{F}}{\partial z_i \partial z_j \partial z_k}(0) = \ldots = \frac{\partial^d \widehat{F}}{\partial z_1 \partial z_2 \ldots \partial z_d}(0) = 0,$$

for $i, j, k, \ldots \in \{1, 2, \ldots, d\}$ and $i, j, k, \ldots$ distinct.

Remarks. Let $f \in \mathcal{E}'(\mathbb{R}^d)$. Since there exist harmonic Boehmians which are not distributions [4], there exist new solutions to Poisson's equation $\Delta u = f$. That is, there are solutions which are not distributions. However, Theorem 3.1 together with Theorem 1.1 show there are no new solutions in $\beta_c(\mathbb{R}^d)$.

Recently, Theorem 3.1 has been proven for the case $d = 2$ (Poisson's Equation and Generalized Functions in the Plane, *Bull. Pure Appl. Math.*).

References

1. T. K. Boehme, The Support of Mikusiński Operators, *Trans. A.M.S.*, **176**(1973), 319-334.

2. J. Burzyk, A Paley-Wiener Type Theorem for Regular Operators of Bounded Support, *Studia Math.*, **93** (1989), 187-200.

3. P. Mikusiński, Convergence of Boehmians, *Japan. J. Math.* (*N.S.*), **9**(1983), 159-179.

4. P. Mikusiński, On Harmonic Boehmians, *Proc. A.M.S.*, **106**(1989), 447-449.

5. D. Nemzer, Some Results on Harmonic Boehmians, *Bull. Inst. Math. Acad. Sinica.(N.S.)*, **4** (2009), 25-34.

6. W. Rudin, *Functional Analysis*, McGraw-Hill Inc., New York, 1973.

7. L. Schwartz, *Theorie des Distributions*, Herman, Paris, 1966.

Department of Mathematics, California State University, Stanislaus, One University Circle, Turlock, CA 95382, USA.

E-mail: jclarke@csustan.edu