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DOTSON'S CONVEXITY, BANACH OPERATOR PAIR AND WEAK CONTRACTIONS

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Abstract

We extend the recent results of Hussain and Cho [Weak Contractions, Common Fixed Points, and Invariant Approximations, Journal of Inequalities and Applications, vol. 2009, Article ID 390634, 10 pages, 2009] to the Banach pair (T, f) where set of fixed points of f satisfies Dotson's convexity condition which is more general than the starshapedness.

1. Introduction and Preliminaries

We first review needed definitions. Let M be a subset of a normed space $(X, \|.\|)$. The set

$$P_M(u) = \{x \in M : \|x - u\| = dist(u, M)\}$$

is called *the set of best approximants* to $u \in X$ out of M, where

$$dist(u, M) = \inf\{\|y - u\| : y \in M\}.$$

We denote $\mathbb{N}$ and $cl(M)$ (resp., $wcl(M)$) by the set of positive integers and the closure (resp., weak closure) of a set M in X, respectively. Let $f,\ T : M \to M$ be mappings. The set of fixed points of T is denoted by $F(T)$. A point $x \in M$ is a coincidence point (resp., common fixed point) of f and T if $fx = Tx$ (resp., $x = fx = Tx$). The set of coincidence points of f and T is denoted by $C(f, T)$.

The pair $\{f, T\}$ is said to be:

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(1) *commuting* if $Tfx = fTx$ for all $x \in M$,

(2) *compatible* if $\lim_{n\to\infty} \|Tfx_n - fTx_n\| = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} fx_n = t$ for some t in M,

(3) *weakly compatible* if they commute at their coincidence points, i.e.,if $fTx = Tfx$ whenever $fx = Tx$,

(4) a *Banach operator pair* if the set $F(f)$ is T-invariant, namely, $T(F(f)) \subseteq F(f)$.

Obviously, the commuting pair (T, f) is a Banach operator pair, but converse is not true in general (see [8, 10]). If (T, f) is a Banach operator pair, then (f, T) need not be a Banach operator pair (see Example 1[8]).

The set M is said to be *q-starshaped* with $q \in M$ if the segment $[q, x] = \{(1-k)q + kx : 0 \le k \le 1\}$ joining q to x is contained in M for all $x \in M$. The mapping f defined on a q-starshaped set M is said to be *affine* if

$$f((1-k)q + kx) = (1-k)fq + kfx, \quad \forall x \in M.$$

Suppose that the set M is q-starshaped with $q \in F(f)$ and is both T- and f-invariant. Then T and f are said to be:

(5) C_q-commuting ([17, 18]) if $fTx = Tfx$ for all $x \in C_q(f, T)$, where $C_q(f, T) = \cup\{C(f, T_k) : 0 \le k \le 1\}$ where $T_k x = (1-k)q + kTx$,

(6) *pointwise R-subweakly commuting* ([2]) if, for given $x \in M$, there exists a real number $R > 0$ such that $\|fTx - Tfx\| \le R\, dist(fx, [q, Tx])$,

(7) *R-subweakly commuting* on M ([15]) if, for all $x \in M$, there exists a real number $R > 0$ such that $\|fTx - Tfx\| \le R\, dist(fx, [q, Tx])$.

Following important extension of the concept of starshapedness was defined by Dotson [7] and has been studied by many authors.

Definition 1.1.(Dotson's convexity) Let M be subset of a normed space X and $\mathbb{F} = \{h_x\}_{x \in M}$ a family of functions from $[0, 1]$ into M such that $h_x(1) = x$ for each $x \in M$. The family $\mathbb{F}$ is said to be contractive [7, 11, 22, 24] if there exists a function $\varphi : (0, 1) \to (0, 1)$ such that for all $x, y \in M$ and all $t \in (0, 1)$, we have $\|h_x(t) - h_y(t)\| \le \varphi(t)\|x - y\|$. The family $\mathbb{F}$ is said to be jointly (weakly) continuous if $t \to t_0$ in [0,1] and $x \to x_o$ ($x \rightharpoonup x_0$) in M, then

$h_x(t) \to h_{x_0}(t_0)$ $(h_x(t) \rightharpoonup h_{x_0}(t_o))$ in M(here $\rightharpoonup$ denotes weak convergence). We observe that if $M \subset X$ is q-starshaped and $h_x(t) = (1-t)q + tx$, $(x \in M; t \in [0,1])$, then $\mathbb{F} = \{h_x\}_{x \in M}$ is a contractive jointly continuous and jointly weakly continuous family with $\varphi(t) = t$. Thus the class of subsets of X with the property of contractiveness and joint continuity contains the class of starshaped sets which in turn contains the class of convex sets (see [7, 12, 22, 24]).

Suppose that $\mathbb{H} = \{h_x\}_{x \in M}$ is a family of functions from $[0,1]$ into M having the property that for each sequence $\{k_n\}$ in $(0,1]$, with $k_n \to 1$, we have

$$h_x(k_n) = k_n x. \tag{$*$}$$

We observe that $\mathbb{H} \subseteq \mathbb{F}$ and it has the additional property that it is contractive and jointly continuous.

Example 1.2.([4, 16, 23]) Any subspace, a convex set with 0, a starshaped subset with center 0 and a cone of a normed linear space have a family of functions associated with them which satisfy condition $(*)$.

Definition 1.3.([12, 22]) Let T be a selfmap of the set M having a family of functions $\mathbb{F} = \{h_x\}_{x \in M}$ as defined above. Then T is said to satisfy the property (KHT), if $T(h_x(t)) = h_{Tx}(t)$ for all $x \in M$ and $t \in [0,1]$.

Example 1.4. An affine map T defined on q-starshaped set with $Tq = q$ satisfies the property (KHT). For this note that each q-starshaped set M has a contractive jointly continuous family of functions $\mathbb{F} = \{h_x\}_{x \in M}$ defined by $h_x(t) = tx + (1-t)q$, for each $x \in M$ and $t \in [0,1]$. Thus $h_x(1) = x$ for all $x \in M$. Also, if the selfmap T of M is affine and $Tq = q$, we have $T(h_x(t)) = T(tx + (1-t)q) = tTx + (1-t)q = h_{Tx}(t)$ for all $x \in M$ and all $t \in [0,1]$. Thus T satisfies the property (KHT).

Recently, Chen and Li [8] introduced the class of Banach operator pairs, as a new class of noncommuting maps and it has been further studied by Akbar and Khan [1], Hussain [10], Hussain and Cho [14], Khan and Akbar [20, 21] and Pathak and Hussain [26]. In this paper, we improve and extend the recent common fixed point and invariant best approximation results of

Hussain and Cho [14] and Chen and Li [8], to the class of $(f, \theta, L)-$ weak contractions where $F(f)$ need not be starshaped.

2. Main Results

Let (X, d) be a metric space. A mapping $T : X \to X$ is called a *weak contraction* if there exists two constants $\theta \in (0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \theta d(x, y) + Ld(y, Tx), \quad \forall x,\ y \in X. \tag{2.1}$$

Due to the symmetry of the distance, the weak contraction condition (2.1) includes the following:

$$d(Tx, Ty) \leq \theta d(x, y) + Ld(x, Ty), \quad \forall x, y \in X, \tag{2.2}$$

which is obtained from (2.1) by formally replacing $d(Tx, Ty)$, $d(x, y)$ by $d(Ty, Tx)$, $d(y, x)$, respectively, and then interchanging x and y.

Consequently, in order to check the weak contraction of T, it is necessary to check both (2.1) and (2.2). Obviously, a Banach contraction satisfies (2.1) and hence is a weak contraction. Some examples of weak contractions are given in [5, 6].

Let f be a self-mapping on X. A mapping $T : X \longrightarrow X$ is said to be *f-weak contraction* or *(f, θ, L)-weak contraction* if there exists two constants $\theta \in (0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \theta d(fx, fy) + Ld(fy, Tx), \quad \forall x, y \in X. \tag{2.3}$$

Berinde [5] introduced the notion of a (θ, L)-weak contraction and proved that a lot of the well-known contractive conditions do imply the (θ, L)-weak contraction. The concept of (θ, L)-weak contraction does not ask $\theta + L$ to be less than 1 as happens in many kinds of fixed point theorems for the contractive conditions (see [9]) that involve one or more of the displacements $d(x, y)$, $d(x, Tx)$, $d(y, Ty)$, $d(x, Ty)$, $d(y, Tx)$. For more details, we refer to [5, 6, 14] and references cited in these papers.

We shall need the following recent result;

Lemma 2.1.(Theorem 2.4[14]). *Let M be a nonempty subset of a metric space (X, d) and let T, f be self-mappings of M. Assume that $F(f)$ is nonempty, $clT(F(f)) \subseteq F(f)$, $cl(T(M))$ is complete and T is a (f, θ, L)-weak contraction. Then $M \cap F(T) \cap F(f) \neq \emptyset$.*

We shall denote by $Y_q^{Tx} = \{h_{Tx}(k) : 0 \leq k \leq 1\}$ where $q = h_{Tx}(0)$.

The following result properly contains Theorems 3.2−3.3 of [8], Theorem 2.11 in [10] and Theorem 2.2 in [26] and improves Theorem 2.2 of [3] and Theorem 6 of [19].

Theorem 2.2. *Let M be a nonempty subset of a normed [resp. Banach] space X and T, f be self-maps of M. Suppose that $F(f)$ is nonempty and has a contractive, jointly continuous [resp. jointly weakly continuous] family of functions $\mathbb{F} = \{h_x\}_{x \in F(f)}$, $clT(F(f)) \subseteq F(f)$[resp. $wclT(F(f)) \subseteq F(f)$], $cl(T(M))$ is compact [resp. $wcl(T(M))$ is weakly compact], T is continuous [resp. weakly continuous] on M and there exists a constant $L \geq 0$ such that*

$$\|Tx - Ty\| \leq \|fx - fy\| + L.dist(fy, Y_q^{Tx}), \qquad for \quad all x, y \in M \qquad (2.4)$$

Then $M \cap F(T) \cap F(f) \neq \emptyset$.

Proof. For $n \in N$, let $k_n = \frac{n}{n+1}$. Define $T_n : F(f) \to F(f)$ by $T_n x = h_{Tx}(k_n)$ for all $x \in F(f)$. Since $F(f)$ has a contractive family and $clT(F(f)) \subseteq F(f)$ [resp. $wclT(F(f)) \subseteq F(f)$], so $clT_n(F(f)) \subseteq F(f)$] [resp. $wclT_n(F(f)) \subseteq F(f)$] for each $n \geq 1$. By (2.4), and the contractiveness of the family $\mathbb{F} = \{h_x\}_{x \in F(f)}$, we have

$$\begin{aligned} \|T_n x - T_n y\| = \|h_{Tx}(k_n) - h_{Ty}(k_n)\| &= \phi(k_n)\|Tx - Ty\| \\ &\leq \phi(k_n)(\|fx - fy\| + L.dist(fy, Y_q^{Tx})) \\ &\leq \phi(k_n)\|fx - fy\| + \phi(k_n)L.\|fy - T_n x\|, \end{aligned}$$

for each $x, y \in F(f))$ where $\phi(k_n) \in (0, 1)$.

If $cl(T(M))$ is compact, for each $n \in \mathbb{N}$, $cl(T_n(M))$ is compact and hence complete. By Lemma 2.1, for each $n \geq 1$, there exists $x_n \in F(f)$ such that $x_n = fx_n = T_n x_n$. The compactness of $cl(T(M))$ implies that there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ such that $Tx_m \to w \in cl(T(M))$ as $m \to \infty$. Since $\{Tx_m\}$ is a sequence in $T(F(f))$ and $clT(F(f)) \subseteq F(f)$, therefore $w \in F(f)$. Further, the joint continuity of $\mathbb{F}$ implies that

$$x_m = T_m x_m = h_{Tx_m}(k_m) \to h_w(1) = w$$

as $m \to \infty$. By the continuity of T, we obtain $Tw = w$. Thus, $M \cap F(T) \cap F(f) \neq \emptyset$ proves the first case.

The weak compactness of $wcl(T(M))$ implies that $wcl(T_n(M))$ is weakly compact and hence complete due to completeness of X. From Lemma 2.1, for each $n \geq 1$, there exists $x_n \in F(f)$ such that $x_n = fx_n = T_n x_n$. The weak compactness of $wcl(T(M))$ implies that there is a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ converging weakly to $y \in wcl(T(M))$ as $m \to \infty$. Since $\{Tx_m\}$ is a sequence in $T(F(f))$, therefore $y \in wcl(T(F(f))) \subseteq F(f) \cap F(g)$. By the joint weak continuity of $\mathbb{F}$ we obtain,

$$x_m = T_m x_m = h_{Tx_m}(k_m) \rightharpoonup h_y(1) = y$$

as $m \to \infty$. By the weak continuity of T, we get $Ty = y$. Thus $M \cap F(T) \cap F(f) \neq \emptyset$. □

Example 2.3. Let $X = \mathbb{R}$, the set of real numbers be endowed with the usual norm and $M = [0, 1]$. Define $fx = x$ if x is rational in M and $fx = 0$ otherwise, and $Tx = 1$ for all $x \in M$. Then $F(f) = \{x : \ 0 \leq x \leq 1, \ x \in Q, \ \textit{the set of rationals}\}$. Clearly, $F(f)$ is not starshaped. Suppose that $\mathbb{F} = \{h_x\}_{x \in F(f)}$ is a family of functions from $[0, 1]$ into $F(f)$, defined by

$$h_x(t) = \begin{cases} tx & \text{for all } x, \ t \in F(f) \\ 0 & \text{for all } x \in F(f) \text{ and } t \in M \setminus F(f). \end{cases}$$

Clearly $|h_x(t) - h_y(t)| \leq t|x - y|$ for all $x, \ y \in F(f)$ and $t \in [0, 1]$. Thus the family $\mathbb{F} = \{h_x\}_{x \in F(f)}$ is a contractive family with $\phi(t) = t$, $t \in (0, 1)$. Further, $clT(F(f)) = \{1\} \subseteq F(f)$ and $clT(F(f)) = \{1\}$ is compact. All the conditions of Theorem 2.2 are satisfied and consequently T and f have a common fixed point $x = 1$. However, it is interesting to note that the results of Al- Thagafi [3], Chen and Li [8], Hussain and Cho [14] and Pathak and Hussain [26] cannot be applied, since $F(f)$ is not starshaped.

Corollary 2.4. *Let M be a nonempty subset of a normed [resp. Banach] space X and T, f be self-maps of M. Suppose that $F(f)$ is q-starshaped, $clT(F(f)) \subseteq F(f)$[resp. $wclT(F(f)) \subseteq F(f)$], $cl(T(M))$ is compact [resp. $wcl(T(M))$ is weakly compact], T is continuous [resp. weakly continuous] on M and*

$$(2.5) \quad \|Tx - Ty\| \leq \|fx - fy\| + L.dist(fy, [q, Tx]), \qquad for \quad all x, y \in M$$

Then $M \cap F(T) \cap F(f) \neq \emptyset$.

Remark 2.5. By comparing the results in [2, 11, 12, 22, 24] with the Theorem 2.2 we notice that the property (KHT) is key assumption in the results of [2, 11, 12, 22, 24].

Corollary 2.6. *Let M be a nonempty subset of a normed [resp. Banach] space X,M has a contractive, jointly continuous [resp. jointly weakly continuous] family of functions $\mathbb{F} = \{h_x\}_{x \in M}$ and T, f be self-maps of M. such that f satisfies property (KHT). Suppose that $F(f)$ is nonempty, closed [resp. weakly closed], $cl(T(M))$ is compact [resp. $wcl(T(M))$ is weakly compact], T is continuous [resp. weakly continuous] on M. If (T, f) is Banach operator pair and satisfies (2.4) for all $x, y \in M$, then $M \cap F(T) \cap F(f) \neq \emptyset$.*

Proof. Define $T_n : M \to M$ by $T_n x = h_{Tx}(k_n)$ for all $x \in M$. Since (T, f) is a Banach operator pair, we have $Tx \in F(f)$ for $x \in F(f)$. Note that f satisfies property (KHT), so we have

$$f(T_n x) = f(h_{Tx}(k_n)) = h_{fTx}(k_n) = h_{Tx}(k_n) = T_n x.$$

This implies that $T_n x \in F(f)$ for each $x \in F(f)$. Thus for each n, (T_n, f) is a Banach operator pair on M. Now the result follows from Theorem 2.2. ☐

Let $C = P_M(u) \cap C_M^f(u)$, where $C_M^f(u) = \{x \in M : \; fx \in P_M(u)\}$.

Corollary 2.7. *Let X be a normed [resp. Banach] space X and T, f be self-maps of X. If $u \in X$, $D \subseteq C$, $D_0 := D \cap F(f)$ is nonempty, has a contractive, jointly continuous [resp. jointly weakly continuous] family of functions $\mathbb{F} = \{h_x\}_{x \in D_0}$, $cl(T(D_0)) \subseteq D_0$ [resp. $wcl(T(D_0)) \subseteq D_0$], $cl(T(D))$ is compact [resp. $wcl(T(D))$ is weakly compact], T is continuous [resp. weakly continuous] on D and (2.4) holds for all $x, y \in D$, then $P_M(u) \cap F(T) \cap F(f) \neq \emptyset$.*

Corollary 2.8. *Let X be a normed [resp. Banach] space X and T, f be self-maps of X. If $u \in X$, $D \subseteq P_M(u)$, $D_0 := D \cap F(f)$ is nonempty, has a contractive, jointly continuous [resp. jointly weakly continuous] family of functions $\mathbb{F} = \{h_x\}_{x \in D_0}$, $cl(T(D_0)) \subseteq D_0$ [resp. $wcl(T(D_0)) \subseteq D_0$], $cl(T(D))$ is compact [resp. $wcl(T(D))$ is weakly compact], T is continuous [resp. weakly continuous] on D and (2.4) holds for all $x, y \in D$, then $P_M(u) \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$.*

Remark 2.9. Theorems 4.1 and 4.2 of Chen and Li [8] are particular cases of Corollaries 2.7 and 2.8.

Remark 2.10. Locally bounded topological vector spaces provide active area of research and have the following nice characterization.

A topological vector space X is Hausdorff locally bounded if and only its topology is defined by some p-norm, $0 < p \leq 1$.

A p-norm on a linear space X is a real valued function $\|\cdot\|_p$ on X with $0 < p \leq 1$, satisfying the following conditions:

(i) $\|x\|_p \geq 0$ and $\|x\|_p = 0 \Leftrightarrow x = 0$

(ii) $\|\lambda x\|_p = |\lambda|^p \|x\|_p$

(iii) $\|x + y\|_p \leq \|x\|_p + \|y\|_p$

for all $x, y \in X$ and all scalars λ. The pair $(X, \|\cdot\|_p)$ is called a p-normed space. It is a metric space with a translation invariant metric d_p defined by $d_p\ (x, y) = \|x - y\|_p$ for all $x, y \in X$. If $p = 1$, we obtain the concept of a normed space.

All of the results (Theorem 2.2–Corollary 2.8) and those proved by Hussain and Cho [14] for normed(Banach) spaces hold for the p-normed(complete p-normed) spaces. Thus from Corollary 2.13 in [14] we obtain following recent result.

Corollary 2.11.([25], Theorem 2.2) *Let T and f be selfmaps on an Opial p-normed space X and M a subset of X such that $T(\partial M) \subset M$, $x_0 \in F(f) \cap F(T)$. Suppose that $D = P_M(x_0)$ is nonempty, weakly compact and q-starshaped. Assume that f is affine, continuous, $D = fD$, $fq = q$ and T is f-nonexpansive on $D \cup \{x_0\}$. If T and f are commuting maps, then $P_M(x_0) \cap F(T) \cap F(f) \neq \emptyset$.*

Definition 2.12. A subset C of a linear space X is said to have the property (N) with respect to T [13, 16, 17] if,

(i) $T : C \to C$,

(ii) $(1 - k_n)q + k_nTx \in C$, for some $q \in C$ and a fixed sequence of real numbers $k_n(0 < k_n < 1)$ converging to 1 and for each $x \in C$.

Hussain et al. [16] noted that each q-starshaped set C has the property (N) but converse does not hold, in general.

Remark 2.13. All of the results of Hussain and Cho [14] (Th. 2.7–Cor. 2.13) remain valid, provided the q-starshapedness of the sets $F(f)$ and D_0, is replaced by the property (N). Consequently, recent results due to Hussain and Cho [14], Hussain, O'Regan and Agarwal [16] and Khan et al. [22] are improved. Here we sketch the proof of Theorem 2.7 of Hussain and Cho [14] under property (N), others follow similarly.

Theorem 2.14. Let M be a nonempty subset of a normed (resp., Banach) space X and let $T,\ f$ be self-mappings of M. Suppose that $clT(F(f)) \subseteq F(f)$ (resp., $wclT(F(f)) \subseteq F(f)$), $cl(T(M))$ is compact (resp., $wcl(T(M))$ is weakly compact and either $I - T$ is demi-closed at 0 or X satisfies Opial's condition, where I stands for the identity mapping) and there exists a constant $L \geq 0$ such that

$$\|Tx - Ty\| \leq \|fx - fy\| + L.dist(fy, [q, Tx]), \quad \forall x, y \in M. \tag{2.5}$$

If $F(f)$ has the property (N) w.r.t. T, then $M \cap F(T) \cap F(f) \neq \emptyset$.

Proof. As $T(F(f)) \subseteq F(f)$ and $F(f)$ has the property (N) w.r.t. T, for each $n \in \mathbb{N}$, we can define $T_n : F(f) \to F(f)$ by $T_nx = (1-k_n)q+k_nTx$ for all $x \in F(f)$ and a fixed sequence $\{k_n\}$ of real numbers $(0 < k_n < 1)$ converging to 1. Since $F(f)$ has the property (N) w.r.t. T and $clT(F(f)) \subseteq F(f)$ (resp., $wclT(F(f)) \subseteq F(f)$), we have $clT_n(F(f)) \subseteq F(f)$ (resp., $wclT_n(F(f)) \subseteq F(f)$) for each $n \in \mathbb{N}$. Also, by the inequality (2.6),

$$\begin{aligned}
\|T_nx - T_ny\| &= k_n\|Tx - Ty\| \\
&\leq k_n\|fx - fy\| + k_nL.dist(fy, [q, Tx]) \\
&\leq k_n\|fx - fy\| + L_n.\|fy - T_nx\|
\end{aligned}$$

for all $x, y \in F(f)$, $L_n := k_n L$ and $0 < k_n < 1$. Thus, for $n \in \mathbb{N}$, T_n is a (f, k_n, L_n)-weak contraction, where $L_n \geq 0$.

If $cl(T(M))$ is compact, then, for each $n \in \mathbb{N}$, $cl(T_n(M))$ is compact and hence complete. By Lemma 2.1, for each $n \in \mathbb{N}$, there exists $x_n \in F(f)$ such that $x_n = fx_n = T_n x_n$. Rest of the proof is similar to that of Theorem 2.7 of [14] and so is omitted. □

Example 2.15. Let $X = \mathbb{R}$ and $M = \{0, 1, 1 - \frac{1}{n+1} : n \in \mathbb{N}\}$ be endowed with the usual norm. Define $T(x) = 0$ for each $x \in M$. Clearly, M is not starshaped but has property (N) w.r.t T [16], for $q = 0$ and $k_n = 1 - \frac{1}{n+1}$. Let f be defined by $f1 = 0 = f0$ and $f(1 - \frac{1}{n+1}) = 1$ for all $n \in \mathbb{N}$. Clearly, (T, f) is a Banach operator pair and 0 is their common fixed point.

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