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OSCILLATORY AND ASYMPTOTIC BEHAVIOR OF $\frac{dx}{dt} + Q(t)G(x(t-\sigma(t))) = f(t)$

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Abstract

Consider the equation

$$\frac{dx}{dt} + Q(t)G(x(t-\sigma(t))) = f(t) \tag{$*$}$$

where $f, \sigma, Q \in C([0,\infty),[0,\infty)), G \in C(R,R), G(-x) = -G(x), xG(x) > o$ for $x \neq 0$, G is non-decreasing, $t > \sigma(t)$, $\sigma(t)$ is decreasing and $t - \sigma(t) \to \infty$ as $t \to \infty$. When$f(t) \equiv 0$, a sufficient condition in terms of the constants

$$k \ = \ \liminf_{t\to\infty} \int_{t-\sigma(t)}^{t} Q(s)ds$$

and

$$L \ = \ \limsup_{t\to\infty} \int_{t-\sigma(t)}^{t} Q(s)ds$$

is established for all solutions of $(*)$ to be oscillatory.The present results improve the earlier results of the literature by both weakening the conditions and considering a general non linear and non-homogeneous differential equation.

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1. Introduction

In this paper the first order forced delay differential equation of the form

$$\frac{dx}{dt} + Q(t)G(x(t-\sigma(t))) = f(t) \tag{1.1}$$

is considered, where Q, f, σ and G satisfy

$$Q, f, \sigma \in C([0,\infty),[0,\infty)) \tag{1.2}$$

$$G \in C(R,R), G \text{ is non-decreasing} \tag{1.3}$$

$$xG(x) > 0 \text{ for } x \neq 0, \sigma(t) < t, t-\sigma(t) \to \infty \text{ as } t \to \infty \text{ and } \sigma'(t) \leq 0. \tag{1.4}$$

The literature on oscillatory/non oscillatory and asymptotic behaviour of solutions of associated homogeneous equation.

$$\frac{dy}{dt} + Q(t)G(y(t-\sigma(t))) = 0, \tag{1.5}$$

is relatively rich. There are thousands of papers on Eq.(1.5) establishing sufficient conditions for oscillation of all its solutions. The conditions are generally formulated in terms of the parameters Q, G and σ. Separate cases of G are dealt where it is *linear*, *sublinear* or *superlinear*.

The work on Eq (1.1) is not satisfactory. Obviously,there are many papers dealing with oscillatory/nonoscillatory and asymptotic behaviour of solutions of (1.1) among which some papers establish the existence of a non-oscillatory solutions.

The following definitions of *linear*, *sub-linear* and *super-linear* are used throughout the paper.

Following Shreve [28], equations (1.1) or (1.5) is called *sub-linear* or *super-linear* accordingly respectively as G satisfies

$$\lim_{\theta\to 0} \frac{G(\theta)}{\theta} = \infty$$

or

$$\lim_{\theta\to 0} \frac{G(\theta)}{\theta} = M$$

where M is some positive real constant. When $M = 1$ it is said to be *linear*.

The above definitions of *sub-linear* or *super-linear* or *linear* include the commonly used sub-linear,super-linear and linear cases where $G(\theta) = \theta^\alpha, \alpha$ is a ratio of odd positive integers satisfying $0 < \alpha < 1$ and $1 < \alpha < \infty$ and $\alpha = 1$ respectively.

By a solution of (1.1) we mean a continuously differentiable function $x(t)$ defined on $[T_0 - \sigma(T_0), \infty)$ for some $T_0 \geq 0$ and such that (1.1) is satisfied for $t \geq T_0$ As usual, a non-trivial solution $x(t)$ of (1.1) is called oscillatory if the set of zeros of x in $(0, \infty)$ is unbounded. Otherwise, $x(t)$ is called non-oscillatory.

The motivation of this work may be viewed as follows:

From the literature it appears that the authors in [10], [11] and [27] have surveyed the results for (1.5) $G(x) = x$ and according to them the development in chronological order is as follows : Myshkis [25] in 1950 proved that every solution of

$$\frac{dy}{dt} + Q(t)y(t - \sigma(t)) = 0 \tag{1.6}$$

oscillates if

$$\limsup_{t\to\infty} \sigma(t) < \infty$$

$$\liminf_{t\to\infty} \sigma(t) \liminf_{t\to\infty} Q(t) > \frac{1}{e}$$

In 1972, Ladas et al. [21] established the same conclusion replacing the above by

$$L = \limsup_{t\to\infty} \int_{t-\sigma(t)}^{t} Q(s)ds > 1 \tag{1.7}$$

In 1979, Ladas [16] and in 1982 Koplatadze and Chanturija [13] established it under the criterion

$$k = \liminf_{t\to\infty} \int_{t-\sigma(t)}^{t} Q(s)ds > \frac{1}{e} \tag{1.8}$$

which was a weaker condition than that of Myshkis [25] and independent of (1.7) Concerning the lower bound $1/e$ in (1.8), it needs to be pointed out that if

$$\int_{t-\sigma(t)}^{t} Q(s)ds \leq \frac{1}{e} \tag{1.9}$$

for large t, then according to a result of [13], (1.6) admits a non-oscillatory solution. The obvious gap between (1.7), (1.8) and (1.9) is transparent when

$$\lim_{t\to\infty} \int_{t-\sigma(t)}^{t} Q(s)ds$$

does not exist. What happens when

$$0 < k \leq \frac{1}{e} \text{ and } \frac{1}{e} < L \leq 1$$

is a question which needs to be settled.

In 1988, Erbe and Zhang [8] established a new oscillation criterion in terms of the constants k and L showing that

$$L > 1 - \frac{k^2}{4} \tag{1.10}$$

implies that every solution of Eq.(1.6) oscillates.In 1991, Jian Chao [2] improved (1.10) to

$$L > 1 - \frac{k^2}{2(1-k)} \tag{1.11}$$

and in 1992, Yu and Wang [30], Yu, Wang , Zhang and Qian [31] established the same conclusion replacing (1.11) by

$$L > 1 - \frac{(1-k) - \sqrt{1-2k-k^2}}{2} \tag{1.12}$$

In 1992 ,Elbert and Stavroulakis [6],using different techniques, improved (1.7) to

$$L > 1 - \left(1 - \frac{1}{\sqrt{\lambda_1}}\right)^2, \tag{1.13}$$

where λ_1 is the smaller real root of the equation

$$F(\lambda) = e^{k\lambda} - \lambda = 0 \tag{1.14}$$

In 1991 Kwong [15] improved (1.7) to

$$L > \frac{\ln \lambda_1 + 1}{\lambda_1} \tag{1.15}$$

The improvement follows immediately from the fact that

$$\max_{x>0} \frac{\ln x + 1}{x} = 1 \tag{1.16}$$

In 1994, Koplatadze and Kvinikadze[14] improved (1.12) while in 1998 Philos and Sficas[26],in 1999 Jaros and Stavroulakis [10] and in 2000 Kon,Sficas and Stavroulakis [11] derived the following conditions

$$\begin{aligned} L &> 1 - \frac{k^2}{2(1-k)} - \frac{k^2}{2}\lambda_1 &(1.17)\\ L &> \frac{\ln \lambda_1 + 1}{\lambda_1} - \frac{(1-k) - \sqrt{1-2k-k^2}}{2} &(1.18)\end{aligned}$$

and

$$L > 2k + \frac{2}{\lambda_1} - 1$$

respectively.

In 2003, Sficas and Stavroulakis[27] established the oscillation condition

$$L > \frac{ln\lambda_1 - 1 + \sqrt{5 - 2\lambda_1 + 2k\lambda_1}}{\lambda_1} \tag{1.19}$$

which might have given the lowest lower bound for L when $k = 1/e$. In 2003, Das et al. [3] proved the oscillation of all solutions of (1.6) if

$$\limsup_{t\to\infty} \left[\int_{t-\sigma(t)}^{t} Q(s)ds + \left(\int_{t-\frac{1}{2}\sigma(t)}^{t} Q(s)ds \right) \left(\int_{t}^{t+\frac{1}{2}\sigma(t)} Q(s)ds \right) \right] > \frac{1 + \ln \lambda_1}{\lambda_1} \tag{1.20}$$

whose methods of proof were different from the earlier ones.The purpose of this paper is to improve (1.19) for nonlinear equations. In particular, for linear equations the present condition reduces to

$$\begin{aligned} \limsup_{t\to\infty} \Bigg\{ \int_{t-\sigma(t)}^{t} Q(s)ds + \left[\int_{t-\frac{1}{2}\sigma(t)}^{t} Q(s)ds \right] \left[\int_{t}^{t+\frac{1}{2}\sigma(t)} Q(s)ds \right] \\ \times \left[1 - \int_{t}^{t+\frac{1}{2}\sigma(t)} Q(s)ds \right]^{-1} \Bigg\} > \frac{1 + \ln \lambda_1}{\lambda_1} \end{aligned} \tag{1.21}$$

where λ_1 is as discussed earlier. The present result improves (1.7), (1.15) and (1.20). The merit of this result is that the condition (1.21) is easily verifiable. Further, the conditions (1.10), (1.11), (1.12), (1.13), (1.15), (1.17), (1.18) reduce to (1.7) as $k \to 0$ except (1.19) , (1.20) and (1.21).

2. Main Results

Theorem 2.1. *Suppose* (1.2) *to* (1.4) *hold* ,$0 < k \leq \frac{1}{Me} < L \leq \frac{1}{M}$ *and*

$$\lim_{y \to 0} \frac{G(y)}{y} = M \tag{2.1}$$

for some $M \in (0, \infty)$.*Set*

$$p(t) = \int_t^{t+\frac{1}{2}\sigma(t)} Q(s)ds \tag{2.2}$$

and

$$q(t) = \int_{t-\frac{1}{2}\sigma(t)}^{t} Q(s)ds \tag{2.3}$$

for large t *and*

$$\limsup_{t \to \infty} \left\{ \int_{t-\sigma(t)}^{t} Q(s)ds + \frac{Mp(t)q(t)}{1 - Mp(t)} \right\} > \frac{M^2 + \ln \lambda_1}{M\lambda_1}, \tag{2.4}$$

where λ_1 *is the smaller root of*

$$F(\lambda) = e^{Mk\lambda} - \lambda = 0 \tag{2.5}$$

then every solution of (1.5) *oscillates.*

Proof. It is clear that $F(1) > 0$ and $F(e) < 0$ and hence $1 < \lambda_1 < e$ and consequently,

$$\frac{M^2 + \ln \lambda_1}{M\lambda_1} > \frac{M}{\lambda_1} > \frac{M}{e} \tag{2.6}$$

From (2.4) and (2.6) it follows that

$$\int_0^{\infty} Q(s)ds = \infty \tag{2.7}$$

Indeed, otherwise, the functions $\int_{t-\sigma(t)}^{t} Q(s)ds$, $p(t)$ and $q(t)$ tend to zero as $t \to \infty$. Thus the term on the left hand side of (2.4) approaches zero as $t \to \infty$, is a contradiction to (2.4) and (2.6). If possible, suppose $y(t)$is a non-oscillatory solution of (1.5) .Without loss of generality, assume that $y(t) > 0, y(t-\sigma(t)) > 0$ for $t \geq t_0$.From (1.5) , it follows that $y'(t) < 0, t \geq t_0$. It can be shown that

$$\lim_{t\to\infty} y(t) = 0 \tag{2.8}$$

Indeed, otherwise integrating (1.5) from t_0 to t and using (2.7) in the resultant integral, it leads to $y(t) \to -\infty$ as $t \to \infty$, is a contradiction. Now set

$$w(t) = \frac{y(t-\sigma(t))}{y(t))} \tag{2.9}$$

Dividing Eq.(1.5) throughout by $y(t)$ then integrating it from $t - \sigma(t)$ to t, it yields

$$w(t) = exp\left(\int_{t-\sigma(t)}^{t} Q(s)\frac{G(y(s-\sigma(s)))}{y(s-\sigma(s))} w(s)ds \right) \tag{2.10}$$

Denoting

$$\lim_{t\to\infty} w(t) = \alpha, \tag{2.11}$$

from (2.10) it follows that α satisfies

$$\alpha \geq \exp(Mk\alpha), \tag{2.12}$$

where k is given in (1.8).That is 'α' lies between the real roots of (2.5).If λ_1 and λ_2 ($\lambda_1 < \lambda_2$) are the real roots of (2.5) then

$$\lambda_1 < \alpha < \lambda_2 \tag{2.13}$$

The first inequality in (2.13) becomes an equality if $Mk = 1/e$ and in that case $\alpha = e$. From (2.13) it follows that for sufficiently small ϵ ($0 < \epsilon < \lambda_1 - 1$) there exists $T_0 \geq t_o$ such that

$$\frac{y(t-\sigma(t))}{y(t)} > \lambda_1 - \epsilon \tag{2.14}$$

and

$$M-\epsilon<\frac{G(y(t-\sigma(t)))}{y(t-\sigma(t))}<M+\epsilon \tag{2.15}$$

Obviously, the function

$$g(s,t)=\frac{y(t-\sigma(t))}{y(s)}$$

satisfies $g(t-\sigma(t),t)=1<\lambda_1-\epsilon$, $g(t,t)>\lambda_1-\epsilon$ and hence by continuity of g, there exists $\sigma^*(t)$ such that $t-\sigma(t)\le\sigma^*(t)\le t$ and $g(\sigma^*(t),t)=\lambda_1-\epsilon$.

That is,

$$\frac{y(t-\sigma(t))}{y(\sigma^*(t))}=\lambda_1-\epsilon \tag{2.16}$$

Dividing (1.5) throughout by $y(t)$, integrating it from $t-\sigma(t)$ to $\sigma^*(t)$ and using (2.14) in it we obtain for some $t\ge T_1\ge t_0$,

$$\int_{t-\sigma(t)}^{\sigma^*(t)} Q(s)\frac{G(y(s-\sigma(s)))}{y(s-\sigma(s))}.\frac{y(s-\sigma(s))}{y(s)}ds=-\int_{t-\sigma(t)}^{\sigma^*(t)}\frac{y'(s)}{y(s)}ds$$

That is,

$$(M-\epsilon)(\lambda_1-\epsilon)\int_{t-\sigma(t)}^{\sigma^*(t)}Q(s)ds\le\ln\left(\frac{y(t-\sigma(t))}{y(\sigma^*(t))}\right)=\ln(\lambda_1-\epsilon)$$

This inequality gives

$$\int_{t-\sigma(t)}^{\sigma^*(t)}Q(s)ds\le\frac{ln(\lambda_1-\epsilon)}{(M-\epsilon)(\lambda_1-\epsilon)} \tag{2.17}$$

Integrating (1.5) from $\sigma^*(t)$ to t and using (2.14) and (2.15) we get

$$\begin{aligned}\int_{\sigma^*(t)}^{t}Q(s)ds &\le \frac{y(\sigma^*(t))}{G(y(t-\sigma(t)))}-\frac{y(t)}{G(y(t-\sigma(t)))}\\ &\le \frac{1}{(M-\epsilon)(\lambda_1-\epsilon)}-\frac{y(t)}{G(y(t-\sigma(t)))} \qquad (2.18)\end{aligned}$$

Adding (2.17) and (2.18), we have

$$\int_{t-\sigma(t)}^{t} Q(s)ds \leq \frac{\ln(\lambda_1-\epsilon)}{(M-\epsilon)(\lambda_1-\epsilon)} + \frac{1}{(M-\epsilon)(\lambda_1-\epsilon)} - \frac{y(t)}{G(y(t-\sigma(t)))} \tag{2.19}$$

Now, integrating (1.5) from t to $t+\sigma(t)/2$ for $t \geq T_1$ we obtain

$$y(t+\sigma(t)/2) - y(t) + G(y(t-\sigma(t)/2))p(t) \leq 0$$

That is,

$$y(t) \geq (M-\epsilon)y(t-\sigma(t)/2)p(t) \tag{2.20}$$

Similarly, integrating (1.5) from $t-\sigma(t)/2$ to t ,we get

$$y(t) - y(t-\sigma(t)/2) + G(y(t-\sigma(t)))q(t) \leq 0.$$

That is,

$$y(t-\sigma(t)/2) \geq y(t) + G(y(t-\sigma(t)))q(t) \tag{2.21}$$

Using (2.21) in (2.20) we get

$$y(t) \geq (M-\epsilon)p(t)\Big\{y(t) + G(y(t-\sigma(t)))q(t)\Big\}$$

This gives

$$\frac{y(t)}{G(y(t-\sigma(t)))} \geq \frac{(M-\epsilon)p(t)q(t)}{\Big[1-(M-\epsilon)p(t)\Big]}$$

Since ϵ is arbitrary, using the above inequality in (2.19) and taking limit superior of both sides we obtain.

$$\limsup_{t\to\infty}\left\{\int_{t-\sigma(t)}^{t} Q(s)ds + \frac{Mp(t)q(t)}{1-Mp(t)}\right\} \leq \frac{\ln\lambda_1}{M\lambda_1} + \frac{1}{M\lambda_1},$$

a contradiction to our assumption. This completes the proof. □

Note: Condition (1.21) follows from (2.4) for the linear equations where $G(y) = y$. □

Theorem 2.2. *Suppose that* $Q,f \in C([0,\infty), R^+)$,$\sigma(t) = \sigma \in (0,\infty)$,$G \in$

(R,R) such that $xG(x)>0$ for $x\neq 0$ and G is nondecreasing. Further, if

$$\int_0^\infty Q(s)ds \;=\; \infty \tag{2.22}$$

$$\lim_{x\to 0}\frac{x}{G(x)} \;=\; M_0\in(0,\infty), G(-x)=-G(x) \tag{2.23}$$

and for $\mu>0$

$$\liminf_{t\to\infty}\int_{t-\sigma}^t\left\{\mu f(s)+\frac{Q(s)}{M_0}\right\}ds>\frac{1}{e}, \tag{2.24}$$

then a nontrival solution $x(t)$ of (1.1) is either oscillatory or satisfies

$$0\le x(t)\le\int_0^t f(s)ds \tag{2.25}$$

for large t.

Proof. Let $x(t)$ be a solution of (1.1). If $x(t)$ is oscillatory, then there is nothing to prove. Assume that $x(t)$ is non-oscillatory. By definition, there exists a $t_0>0$ such that either

$$x(t)\;>\;0,\qquad t\ge t_0 \tag{2.26}$$

or

$$x(t)\;<\;0,\qquad t\ge t_0 \tag{2.27}$$

Suppose that $x(t)>0$ for $t\ge t_0$. Set

$$w(t)\;=\;\int_0^t f(s)ds \tag{2.28}$$

and

$$z(t)\;=\;x(t)-w(t) \tag{2.29}$$

From (1.1) and (2.29) it follows that

$$z'(t)+Q(t)G(x(t-\sigma))=0 \tag{2.30}$$

The above gives $z'(t)<0$, $t\ge t_0+\sigma$. Thus, there exists $t_1\ge t_0+\sigma$ such that either

$$z(t)\;>\;0,\qquad t\ge t_1 \tag{2.31}$$

or

$$z(t) \ < \ 0, \qquad t \geq t_1 \tag{2.32}$$

Suppose that $z(t) > 0$, for $t \geq t_1$. From (2.29) it follows that

$$x(t) > w(t), \qquad t \geq t_1 \tag{2.33}$$

From (2.30) and (2.33) we obtain

$$z'(t) + Q(t)G(w(t-\sigma)) \leq 0, \qquad t \geq t_1. \tag{2.34}$$

Integrating (2.34) from t_1 to t , using (2.33) and the fact that G and w are non-decreasing in $(0, \infty)$,we obtain

$$z(t) - z(t_1) \leq -\int_{t_1}^{t} Q(s)G(w(s-\sigma))ds \leq -G(w(t_1-\sigma)) \int_{t_1}^{t} Q(s)ds \tag{2.35}$$

Taking limit $t \to \infty$ in (2.35) with the use of (2.22) we see that $z(t) \to -\infty$ as $t \to \infty$. This is a contradiction to our assumption. Hence $z(t) < 0$, $t \geq t_1$. That is $x(t) < w(t)$,$t \geq t_1$ This gives finally

$$0 < x(t) < \int_0^t f(s)ds \ , \qquad t \geq t_1$$

Next, suppose that

$$x(t) < 0, \qquad t \geq t_0 \tag{2.36}$$

From (1.1) it follows that

$$x'(t) \geq 0, \qquad t \geq t_0 + \sigma.$$

and

$$\lim_{t\to\infty} x(t) = \lambda \in (-\infty, 0] \tag{2.37}$$

Here we claim that $\lambda = 0$. If possible, let $\lambda \in (-\infty, 0)$. By definition there exists $t_1 \geq t_0 + \sigma$ such that $G(x(s-\sigma)) > G(\lambda/2)$. Integrating (1.1) from t_1 to t we get

$$x(t) - x(t_1) \ = \ \int_{t_1}^{t} f(s)ds - \int_{t_1}^{t} Q(s)G(x(s-\sigma))ds$$

$$\geq -G(\lambda/2)\int_{t_1}^{t} Q(s)ds \tag{2.38}$$

Taking limit as $t \to \infty$ in (2.38) and using (2.22) we see that $x(t) \to \infty$ as $t \to \infty$. This contradicts to (2.36). Hence

$$\lim_{t\to\infty} x(t) = 0 \tag{2.39}$$

From (2.24), it follows that there exists $t^* \in (t-\sigma, t)$ such that

$$\int_{t-\sigma}^{t^*} \Big\{\mu f(s) + \frac{Q(s)}{M_0}\Big\} ds \;>\; \frac{1}{2e} \tag{2.40}$$

and

$$\int_{t^*}^{t} \Big\{\mu f(s) + \frac{Q(s)}{M_0}\Big\} ds \;>\; \frac{1}{2e} \tag{2.41}$$

From (2.23) and (2.39) it follows that for every $\epsilon > 0$, there exists $T_\epsilon > 0$ such that

$$M_0 - \epsilon \leq \frac{x(t)}{G(x(t))} \leq M_0 + \epsilon, \qquad t \geq T_\epsilon \tag{2.42}$$

Now, integrating (1.1) from t^* to t $(t > T_\epsilon + \sigma)$ we obtain

$$\begin{aligned} x(t) - x(t^*) &= \int_{t^*}^{t} \Big\{ f(s) - Q(s)G(x(s-\sigma))\Big\} ds \\ &= \int_{t^*}^{t} -x(t-\sigma)\Big\{\frac{f(s)}{-x(s-\sigma)} - Q(s)\frac{G(x(s-\sigma))}{-x(s-\sigma)}\Big\} ds \\ &\geq -x(t-\sigma)\int_{t^*}^{t}\Big\{\frac{f(s)}{-x(s-\sigma)} + \frac{Q(s)}{M_0+\epsilon}\Big\} ds \end{aligned} \tag{2.43}$$

Integrating, similarly, from $t-\sigma$ to t^* we get

$$\begin{aligned} x(t^*) - x(t-\sigma) &= \int_{t-\sigma}^{t^*} \Big\{ f(s) - Q(s)G(x(s-\sigma))\Big\} ds \\ &\geq \int_{t-\sigma}^{t^*} -x(s-\sigma)\Big\{\frac{f(s)}{-x(s-\sigma)} - Q(s)\frac{G(x(s-\sigma))}{-x(s-\sigma)}\Big\} ds \\ &\geq -x(t^*-\sigma)\int_{t-\sigma}^{t^*}\Big\{\frac{f(s)}{-x(s-\sigma)} - Q(s)\frac{G(x(s-\sigma))}{-x(s-\sigma)}\Big\} ds \\ &\geq -x(t^*-\sigma)\int_{t-\sigma}^{t^*}\Big\{\frac{f(s)}{-x(s-\sigma)} + \frac{Q(s)}{M_0+\epsilon}\Big\} ds \end{aligned} \tag{2.44}$$

From (2.39), (2.43) and (2.40) it follows that for large t,

$$x(t) - x(t^*) \geq -x(t-\sigma)\frac{1}{2e} \tag{2.45}$$

and similarly from (2.39), (2.44) and (2.41),

$$x(t-\sigma) \leq x(t^*-\sigma)\frac{1}{2e} \tag{2.46}$$

Combining (2.45) and (2.46) it reduces to

$$x(t^*) \leq x(t^*-\sigma)\left(\frac{1}{2e}\right)^2 \tag{2.47}$$

since $x(t^*)$ is negative , from (2.47) we get

$$\frac{x(t^*-\sigma)}{x(t^*)} \leq 4e^2 \tag{2.48}$$

Since $x(t)$ is negative and increasing, let us set

$$w(t) = \frac{x(t-\sigma)}{x(t)} \tag{2.49}$$

and

$$\liminf_{t\to\infty} w(t) = \beta \tag{2.50}$$

From the above, it clearly follows that $w(t) > 0$ and β is finite. Dividing (1.1) throughout by $x(t)$ then integrating the resultant from$t-\sigma$ to t we get

$$\ln \frac{x(t)}{x(t-\sigma)} = \int_{t-\sigma}^{t} \left\{\frac{f(s)}{x(s)} - Q(s)\frac{G(x(s-\sigma))}{x(s)}\right\}ds$$

that is,

$$\ln w(t) = \int_{t-\sigma}^{t} \left\{\frac{f(s)}{-x(s-\sigma)} + Q(s)\frac{G(x(s-\sigma))}{x(s-\sigma)}\right\}w(s)ds$$

With the use of the mean value theorem,there exists $T_1 \geq T_\epsilon$ such that

$$\ln w(t) \geq w(\theta)\int_{t-\sigma}^{t} \left\{\mu f(s) + Q(s)\frac{G(x(s-\sigma))}{x(s-\sigma)}\right\}ds \tag{2.51}$$

for $t \geq T_1$,where $t - \sigma < \theta < t$ and for every $\mu > 0$. Taking limit infirm of both sides of (2.51) we obtain

$$ln\beta \geq \beta \liminf_{t\to\infty} \int_{t-\sigma}^{t} \left\{ \mu f(s) + \frac{Q(s)}{M_0} \right\} ds$$

That is,

$$\frac{ln\beta}{\beta} \geq \liminf_{t\to\infty} \int_{t-\sigma}^{t} \left\{ \mu f(s) + \frac{Q(s)}{M_0} \right\} ds \tag{2.52}$$

However, it is known that

$$\max_{\beta>0} \frac{ln\beta}{\beta} = \frac{1}{e} \tag{2.53}$$

Combining (2.52) and (2.53) ,we reach

$$\liminf_{t\to\infty} \int_{t-\sigma}^{t} \left\{ \mu f(s) + \frac{Q(s)}{M_0} \right\} ds \leq \frac{1}{e},$$

which is a contradiction. This completes the proof. □

Example 1. Consider the equation

$$\frac{dx}{dt} + \left(\frac{1}{e} + e^{2t} \right) x(t-1) = e^{t+1}$$

This equation satisfies the hypotheses of Theorem 2.2 and hence every solution is either oscillatory or satisfies (2.25). One of such solutions is $x(t) = e^{-t}$.

Example 2. Consider the non-linear equation

$$\frac{dx}{dt} + e^{2t} \left(x(t-1) + x^3(t-1) \right) = e^{9-7t} + e^{3-t} - 3e^{-3t}$$

This equation satisfies the hypotheses of Theorem 2.2 and hence every solution is either oscillatory or satisfies (2.25). One of such solutions is $x(t) = e^{-3t}$.

Corollary 1. *Supposethat* $Q \in C([0,\infty), R^+)$,$\sigma(t) = \sigma \in (0,\infty)$, $G \in C(R,R)$ *such that* $xG(x) > 0$ *for* $x \neq 0$, *G is nondecreasing and G(-x)=-*

G(x), further if

$$\lim_{x\to 0}\frac{x}{G(x)} = M_0 \in (0,\infty) \tag{2.54}$$

and

$$\liminf_{t\to\infty}\int_{t-\sigma}^{t} Q(s)ds > \frac{M_0}{e} \tag{2.55}$$

then every solution of (1.1) *with* $f(t)\equiv 0$*, oscillates.*

Proof. The proof follows directly from Theorem 2.2, because a non-trivial solution does not satisfy (2.25),when $f(t)\equiv 0$. □

Remark. Corollary 1 extends the main result of [16] for *super-linear* equations and Theorem 2.2 extends it for non homogeneous differential equations.

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