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ON BOUNDARY RELATION FOR SOME DISSIPATIVE SYSTEMS

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Abstract

We consider the initial-boundary value problem for dissipative systems. The main goal is to obtain explicit relation for the boundary values and their normal differentiations. We calculate out the relation for three basic examples in continuum physics. Following the standard approach, we apply the Fourier-Laplace transforms. Our main effort is to explicitly invert these transforms.

1. Introduction

Consider linear system of conservation laws with dissipations and sources

$$\mathbf{u}_t+\sum_{j=1}^{m}\mathbb{A}_j\mathbf{u}_{x_j}=\sum_{j,k=1,\ j\le k}^{m}\mathbb{B}_{jk}\partial^2_{x_jx_k}\mathbf{u}+\mathbf{S}(\mathbf{x},t), \tag{1.1}$$

where the basic dependent variables $\mathbf{u}=\mathbf{u}(\mathbf{x},t)$ and the space variables $\mathbf{x}=(x_1,x_2,\cdots,x_m)$ are in $\mathbb{R}^m$. $\mathbb{A}_j,\ j=1,2,\cdots,m$, are the convective matrices and $\mathbb{B}_{jk},\ j,k=1,2,\cdots,m$, the viscosity matrices. We take these to be constant matrices. We are interested in the *initial-boundary* value problem, $x_1>0,\ \bar{\mathbf{x}}\equiv(x_2,\cdots,x_m)\in\mathbb{R}^{m-1}$ with suitable boundary condition at

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$x_1 = 0$. Let $\mathbb{G}(\mathbf{x},\mathbf{y},t,s) = \mathbb{G}(\mathbf{x}-\mathbf{y},t-s)$ be the Green's function for the *initial value problem*:

$$\begin{cases} \mathbb{G}_t + \sum_{j=1}^m \mathbb{A}_j \mathbb{G}_{x_j} = \sum_{j,k=1}^m \mathbb{B}_{jk} \partial^2_{x_j x_k} \mathbb{G}, \ \mathbf{x} \in \mathbb{R}^m, \ t > 0, \\ \mathbb{G}(\mathbf{x},0) = \delta(\mathbf{x})\mathbb{I}, \ \mathbf{x} \in \mathbb{R}^m, \end{cases} \tag{1.2}$$

where $\mathbb{I}$ is the identity matrix. For many physical systems, the Green's function for the initial value problem has been explicitly constructed for viscous conservation laws of the above form (1.2), see, for instance, [8], [5], and also for the dissipative equation in the kinetic theory, the Boltzmann equation, [3], [4]. When the Green's function is constructed, we can multiply the system (1.2) with the Green's function and integrate over $x_1 > 0$ to obtain

$$\begin{aligned} \mathbf{u}(\mathbf{x},t) \ = \ & \int_{y_1>0} \mathbb{G}(\mathbf{x}-\mathbf{y},t)\mathbf{u}(\mathbf{y},0)d\mathbf{y} + \int_0^t \int_{y_1>0} \mathbb{G}(\mathbf{x}-\mathbf{y},t-s)\mathbf{S}(\mathbf{y},s)d\mathbf{y}ds \\ & + \int_0^t \int_{\mathbb{R}^{m-1}} \mathbb{G}(x_1,\bar{\mathbf{x}}-\bar{\mathbf{y}},t-s)\mathbb{A}_1 \mathbf{u}(0,\bar{\mathbf{y}},s)d\bar{\mathbf{y}}ds \\ & - \int_0^t \int_{\mathbb{R}^{m-1}} \mathbb{G}(x_1,\bar{\mathbf{x}}-\bar{\mathbf{y}},t-s)\sum_{k=1}^m \mathbb{B}_{1k}\mathbf{u}_{x_k}(0,\bar{\mathbf{y}},s)d\bar{\mathbf{y}}ds \\ & + \int_0^t \int_{\mathbb{R}^{m-1}} \mathbb{G}_{x_1}(x_1,\bar{\mathbf{x}}-\bar{\mathbf{y}},t-s)\mathbb{B}_{11}\mathbf{u}(0,\bar{\mathbf{y}},s)d\bar{\mathbf{y}}ds. \end{aligned} \tag{1.3}$$

This formula for the solution demands both the values of the differentials $\sum_{k=1}^m \mathbb{B}_{1k}\mathbf{u}_{x_k}$ and the function $\mathbb{B}_{11}\mathbf{u}$ on the boundary $x_1 = 0$. On the other hand, for the well-posedness of the initial-boundary value problem, only part of these values are given on the boundary. The main goal of the present paper is to initiate a new method for deriving the relation between the function and its differentials at the boundary. With such a *boundary relation*, a well-posed boundary problem, with partial information of the $\mathbf{u}$ and $\nabla_{\mathbf{x}}\mathbf{u}$ given, would have a solution formula. In other words, the Green's function $\mathbb{G}$, together with the boundary relation, yield the explicit expression for the Green's function for the initial-bounday value problem. The explicit expression of the Green's function is essential for many quantitative and qualitative understanding of physical phenomena that these problems aim to model.

Our analysis starts with the Fourier transform with respect to the tangential variables $\bar{\mathbf{x}}$ and the Laplace transform for the normal spatial variable x_1 and the time variable t. This reduces the system (1.2) to a system of algebraic equations. The boundary relation in the transformed variables is given by the standard requirement that there is no unstable modes as $x_1 \to \infty$. We aim at *explicitly inverting* the Laplace-Fourier transform of the boundary relation. Our pointwise approach is useful for the global, in time and space, understanding of the solution under the boundary effects. We present three examples in continuum physics to illustrate our specially designed method for this inversion. We hope to generalize this method to more general systems in continuum physics.

There have been much studies on the well-posedness of hyperbolic systems. For hyperbolic systems, the boundary condition depends on the characteristic directions, corresponding to the convection matrices $\mathbb{A}_j,\ j = 1, 2, \cdots, m$, in (1.2). For the dissipative systems, there is the additional factor of the viscosity matrices $\mathbb{B}_{jk},\ j, k = 1, 2, \cdots, m$. For the approach using the energy methods, see, for instance, [2], [6], [7].

2. Heat equation

Consider the heat equation in the quarter plane with zero initial value

$$\begin{cases} u_t = u_{xx},\ x, t \geq 0, \\ u(x, 0) = 0,\ x \geq 0. \end{cases} \tag{2.1}$$

The boundary is $x = 0,\ t \geq 0$ and we denote the boundary values by

$$\begin{cases} u^0(t) \equiv u(0, t),\ \text{Dirichlet boundary value}, \\ u_x^0(t) \equiv u_x(0, t),\ \text{Neumann boundary value}. \end{cases} \tag{2.2}$$

Take the Laplace transform of the heat equation with respect to time and make use of the zero initial value to obtain

$$\begin{cases} sU = U_{xx}, \\ U(x, s) \equiv \int_0^\infty e^{-st} u(x, t) dt. \end{cases} \tag{2.3}$$

Next take the Laplace transform with respect to the space variable:

$$\begin{cases}(s-\xi^2)\hat{U}=-U_x^0-\xi U^0,\\ \hat{U}(\xi,s)\equiv\int_0^\infty e^{-\xi x}U(x,s)dx,\end{cases} \tag{2.4}$$

where

$$U^0=U^0(s)\equiv U(0,s),\ U_x^0=U_x^0(s)\equiv U_x(0,s)$$

are the boundary values of the transformed function. The function

$$p_H(\xi,s)\equiv\xi^2-s=(\xi-\lambda_1(s))(\xi-\lambda_2(s)) \tag{2.5}$$

as a polynomial of ξ is called the *characteristic polynomial* and has two roots

$$\lambda_1(s)=-\sqrt{s}<0<\lambda_2(s)=\sqrt{s}. \tag{2.6}$$

We have

$$\hat{U}=\frac{U_x^0+\xi U^0}{p_H(\xi,s)}=\frac{U_x^0+\xi U^0}{(\xi-\lambda_1(s))(\xi-\lambda_2(s))}. \tag{2.7}$$

We now invert the Laplace transform, first the one with respect to the space variable x, using the standard formula:

$$U(x,s)=\frac{1}{2\pi i}\int_{-i\infty+D}^{i\infty+D}e^{\xi x}\hat{U}(\xi,s)d\xi=\frac{1}{2\pi i}\int_{-i\infty+D}^{i\infty+D}e^{\xi x}\frac{U_x^0+\xi U^0}{(\xi-\lambda_1(s))(\xi-\lambda_2(s))}d\xi \tag{2.8}$$

where D is chosen to be greater than $\sqrt{s}$ so that the integrant is analytic to the right of the line of integration $\xi=D+iy,\ -\infty<y<\infty$. It is clear that the integral can be expressed in terms of the residues $Res(\xi=\pm\sqrt{s})$:

$$U(x,s)=e^{\lambda_1(s)x}\frac{U_x^0+\lambda_1(s)U^0}{\lambda_1(s)-\lambda_2(s)}+e^{\lambda_2(s)x}\frac{U_x^0+\lambda_2(s)U^0}{\lambda_2(s)-\lambda_1(s)}. \tag{2.9}$$

As $\lambda_2(s)=\sqrt{s}>0$, the factor $e^{\lambda_2(s)x}$ represents the unstable mode as $x\to\infty$. Thus we require its coefficient to be zero:

$$U_x^0+\lambda_2(s)U^0=U_x^0+\sqrt{s}U^0=0. \tag{2.10}$$

This is the *Dirichlet-Neumann relation* in the transformed variables. It remains to invert the Laplace transform of this relation. Instead of inverting

$\sqrt{s}$, we will invert

$$\frac{\sqrt{s}}{s}=\frac{1}{\sqrt{s}} \tag{2.11}$$

by the usual inversion formula of the Laplace transform:

$$L(t)\equiv\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}e^{st}\frac{1}{\sqrt{s}}ds=\frac{1}{2\pi\sqrt{t}}\int_{-\infty}^{\infty}e^{is}\frac{1}{\sqrt{is}}ds=\frac{1}{\sqrt{\pi t}}. \tag{2.12}$$

Note that the inversion of the Laplace transform has the property that division in s corresponds to differentiation in t. Thus the Dirichlet-Neumann relation (2.10) becomes

$$u_x(0,t)=\frac{\partial}{\partial t}\int_0^t\frac{1}{\sqrt{\pi(t-s)}}u(0,s)ds. \tag{2.13}$$

Remark 2.1. The general initial-boundary value problem for the heat equation with source

$$\begin{cases}u_t=u_{xx}+h(x,t),\ \ x,t\geq 0,\\ u(x,0)=u_0(x),\\ u(0,t)=u_b(t),\ \ t>0,\end{cases} \tag{2.14}$$

can be reduced to the case with zero initial data for

$$\bar{u}(x,t)\equiv u(x,t)-u_I(x,t),\ \ \bar{u}(x,0)=0, \tag{2.15}$$

where $u_I(x,t)$ is the convolution of the initial data with the heat kernel:

$$u_I(x,t)\equiv\int_0^\infty H(x-y,t)u_0(y)dy,\ \ H(x,t)\equiv\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}. \tag{2.16}$$

This way the above analysis yields the Neumann data and the solution formula for $\bar{u}$:

$$\begin{aligned}\bar{u}(x,t)\ &=\ \int_0^t\int_0^\infty H(x-y,t-s)h(y,s)dyds\\ &\quad-\int_0^t[H(x,t-s)\bar{u}_x(0,s)-H_x(x,t-s)\bar{u}(0,s)]ds\end{aligned} \tag{2.17}$$

with the boundary Dirichlet value $\bar{u}(0,s)=u_b(s)-u_I(0,s)$ and the Neumann value $\bar{u}_x(0,s)$ given in terms of the Dirichlet value through the Dirichlet-

Neumann relation (2.13).

Remark 2.2. The general initial-boundary value problem (2.14) can also be solved by the standard reflection method as follows, [1]: First reduce the Dirichlet value to zero, then extend the initial data and the solution by an odd function

$$u(x,t) \equiv -u(-x,t), \; x < 0,$$

and use the heat kernel to solve the resulting initial value problem. This way one can also easily derive the Dirichlet-Neumann relation (2.13). On the other hand, the present method can be applied to other problems for which no reflection method is available.

3. Navier-Stokes Equations

Consider the one-dimensional, isentropic compressible Navier-Stokes equations

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ (\rho v)_t + (\rho v^2 + p(\rho))_x = (\mu v_x)_x. \end{cases} \tag{3.1}$$

Here $\rho, \; v, p(\rho)$ are the density, velocity, and pressure, respectively. The first equation, the continuity equation, has no dissipation. The second, the momentum equation has the dissipation term $(\mu v_x)_x$, with $\mu > 0$ the viscosity coefficient. Thus this system is not uniformly parabolic and is hyperbolic-parabolic. It turns out that for the initial-boundary value problem $x, t \geq 0$, the boundary value cannot be posed for both ρ and v, but only for v.

The sound speed c is given as

$$c^2 = p'(\rho).$$

Linearize the Navier-Stokes equations around a constant state (ρ_0, v_0). With the viscosity μ and the sound speed $c_0 = \sqrt{p'(\rho_0)}$ normalized to be unity and the base state taken to be $(\rho_0, v_0) = (1, 0)$, the linearized system becomes

$$\begin{cases} \rho_t + v_x = 0, \\ v_t + \rho_x = v_{xx}. \end{cases} \tag{3.2}$$

Again, we consider zero initial values

$$(\rho, v)(x,0) = 0,\ \ x > 0,$$

and the Laplace transforms

$$\begin{cases} \begin{pmatrix} P \\ V \end{pmatrix}(x,s) \equiv \int_0^\infty e^{-st} \begin{pmatrix} \rho \\ v \end{pmatrix}(x,t)dt, \\ \begin{pmatrix} \hat{P} \\ \hat{V} \end{pmatrix}(\xi,s) \equiv \int_0^\infty e^{-\xi x} \begin{pmatrix} P \\ V \end{pmatrix}(x,s)dx. \end{cases} \tag{3.3}$$

The linearized Navier-Stokes equations (3.2) with zero initial values become, in the transformed functions,

$$\begin{cases} s\hat{P} + \xi\hat{V} = V^0, \\ s\hat{V} + \xi\hat{P} = \xi^2\hat{V} + P^0 - V_x^0 - \xi V^0. \end{cases} \tag{3.4}$$

Here, as in the above, the boundary values of the transformed functions are:

$$V^0 = V^0(s) \equiv V(0,s),\ \ P^0 = P^0(s) \equiv P(0,s),\ \ V_x^0 = V_x^0(s) \equiv V_x(0,s).$$

Note that, due to the hyperbolic-parabolic, and not uniform parabolic, nature of the system (3.2), the function P_x^0 does not appear in the transformed system (3.4). In fact, the boundary function P^0 in the system can also be elliminatted as follows: Take the Laplace transform in time of the first equation in the linearized Navier-Stokes equations (3.2) to obtain

$$sP + V_x = 0$$

and, evaluating it on the boundary $x = 0$,

$$sP^0 + V_x^0 = 0. \tag{3.5}$$

Replace P^0 in the second equation in (3.4) by $-V_x^0/s$ to obtain

$$\begin{cases} s\hat{P} + \xi\hat{V} = V^0, \\ s\hat{V} + \xi\hat{P} = \xi^2\hat{V} - (\frac{1}{s} + 1)V_x^0 - \xi V^0. \end{cases} \tag{3.6}$$

From this we can elliminate $\hat{P}$ to obtain a single equation for the transformed variables V, $\hat{V}$ and their boundary values:

$$(\xi^2 - s^2 + s\xi^2)\hat{V} = (s+1)[\xi V^0 + V_x^0]. \tag{3.7}$$

The characteristic polynomial is

$$p_{NS}(\xi, s) \equiv (s+1)\xi^2 - s^2 \tag{3.8}$$

with two roots $\lambda_j,\ j = 1, 2$:

$$\begin{aligned} p_{NS}(\xi, s) &= (s+1)(\xi - \lambda_1(s))(\xi - \lambda_2(s)), \\ \lambda_1(s) &= -\frac{s}{\sqrt{s+1}} < 0 < \lambda_2(s) = \frac{s}{\sqrt{s+1}}, \end{aligned} \tag{3.9}$$

$$\hat{V} = \frac{\xi V^0 + V_x^0}{(\xi - \lambda_1(s))(\xi - \lambda_2(s))}. \tag{3.10}$$

The inversion of the Laplace transform in x as in (2.9) yields

$$V(x, s) = e^{\lambda_1(s)x} \frac{\lambda_1(s)V^0 + V_x^0}{(\lambda_1(s) - \lambda_2(s))} + e^{\lambda_2(s)x} \frac{\lambda_2(s)V^0 + V_x^0}{(\lambda_2(s) - \lambda_1(s))}. \tag{3.11}$$

The stability condition, the vanishing of the coefficient of the growth term $e^{\lambda_2(s)x}$, yields the Dirichlet-Neumann relation in the transformed variables:

$$\lambda_2(s)V^0 + V_x^0 = \frac{s}{\sqrt{s+1}} V^0 + V_x^0 = 0. \tag{3.12}$$

As in the first example, we consider first the inverse Laplace transform:

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{st}}{\sqrt{s+1}} ds = \frac{1}{2\pi\sqrt{t}} e^{-t} \int_{-\infty}^{\infty} \frac{e^{iz}}{\sqrt{iz}} dz = \frac{e^{-t}}{\sqrt{\pi t}} \tag{3.13}$$

and the Dirichlet-Neumann relation becomes

$$v_x(0, t) = \frac{d}{dt}[\frac{e^{-t}}{\sqrt{\pi t}} \star v(0, t)] = \int_0^t \frac{e^{-(t-\tau)}}{\sqrt{\pi(t-\tau)}} v(0, \tau) d\tau. \tag{3.14}$$

From (3.5) we can construct the boundary value for the density in terms of the boundary gradient of the velocity; or it follows directly from the first equation of the system (3.2) that

$$\rho(0,t) = -\int_0^t v_x(0,y)dy = -\int_0^t \int_0^y \frac{e^{-(y-\tau)}}{\sqrt{\pi(y-\tau)}} v(0,\tau)d\tau dy. \tag{3.15}$$

Remark 3.1. Unlike the first example, and third example to be presented in the following section, the Navier-Stokes equation (3.2) cannot easily solved by simple reflection around the t-axis.

We now consider the general initial-boundary value problem for the Navier-Stokes equations (3.2) with sources:

$$\begin{cases} \rho_t + v_x = S_1(x,t), \\ v_t + \rho_x = v_{xx} + S_2(x,t), \ \ x, t \geq 0, \\ \begin{pmatrix} \rho \\ v \end{pmatrix} (x,0) = \begin{pmatrix} \rho_0 \\ v_0 \end{pmatrix} (x), \ \ x \geq 0, \\ v(0,t) = v_b(t), \ \ t \geq 0, \end{cases} \tag{3.16}$$

where the initial values $(\rho_0(x), v_0(x)$, the boundary value $v_b(0,t)$, and the sources $S_1(x,t),\ S_2(x,t)$ are given functions. The resolution of this problem can be used for solving the corresponding problem for the full Navier-Stokes equations (3.1), with the sources as the nonlinear terms when it is approximated by the linear Navier-Stokes equations (3.2). To solve (3.16), we need the Green's function $\mathbb{G}(x,t)$ for the initial value problem, a 2×2 matrix defined by

$$\begin{cases} \mathbb{G}_t + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbb{G}_x = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbb{G}_{xx}, \\ \mathbb{G}(x,0) = \delta(x)\mathbb{I} \end{cases} \tag{3.17}$$

where $\mathbb{I}$ is the 2×2 identity matrix. The Green's function for a system with *physical viscosity* such as (3.2) contains rich wave structure and has been done in [8], see also [5] for general systems. The Green's function contains a δ-function;

$$\mathbb{G}_1(x,t) \equiv e^{-\frac{t}{2}}\delta(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.18}$$

The system (3.2) is not uniformly parabolic, and this has two implications, the first is that the Dirichlet boundary data in the present situation is posted only for the velocity $v_b(t)$; and the second is the presence of this delta function in the Green's function. The corresponding inviscid system for the Navier-Stokes equations (3.2) is the linear Euler equations

$$\begin{cases} \rho_t + v_x = 0, \\ v_t + \rho_x = 0. \end{cases} \tag{3.19}$$

The Euler equations can be easily diagonalized and have waves propagating with sound speed ± 1. Thus its Green's function $\mathbb{G}_E(x,t)$ is of the form

$$\mathbb{G}_E(x,t) = \delta(x+t)\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} + \delta(x-t)\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{3.20}$$

Consequently, the Navier-Stokes equations possess the heat kernels propagating with the sound speeds ± 1 :

$$\mathbb{G}_2(x,t) \equiv \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x+t)^2}{2t}} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-t)^2}{2t}} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}. \tag{3.21}$$

Due to the coupling of the Navier-Stokes equations, its Green's function contains an extra remaining terms decaying faster than the heat kernel; precisely, for any constant $D > 2$,

$$\mathbb{G}(x,t) = \mathbb{G}_1(x,t) + \mathbb{G}_2(x,t) + O(1)(t+1)^{-\frac{1}{2}} t^{-\frac{1}{2}} [e^{-\frac{(x+t)^2}{Dt}} + e^{-\frac{(x-t)^2}{Dt}}]. \tag{3.22}$$

With the Green's function $\mathbb{G}(x,t)$ explicitly constructed, and the Dirichlet-Neumann relations given in (3.14) and (3.15), we can derive the explicit solution formula for the initial-boundary value problem (3.16) as follows: The first step is to use the Green's function to construct functions representing the effects of the initial values and the sources:

$$\begin{cases} \begin{pmatrix} \rho_I \\ v_I \end{pmatrix}(x,t) \equiv \int_0^\infty \mathbb{G}(x-y,t) \begin{pmatrix} \rho_0 \\ v_0 \end{pmatrix}(y)dy, \\ \begin{pmatrix} \rho_S \\ v_S \end{pmatrix}(x,t) \equiv \int_0^t \int_0^\infty \mathbb{G}(x-y,t-s) \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}(y,s)dyds. \end{cases} \tag{3.23}$$

Then the functions

$$\left\{\begin{pmatrix}\bar\rho\\ \bar v\end{pmatrix}(x,t)\equiv\begin{pmatrix}\rho\\ v\end{pmatrix}(x,t)-\begin{pmatrix}\rho_I\\ v_I\end{pmatrix}(x,t)-\begin{pmatrix}\rho_S\\ v_S\end{pmatrix}(x,t)\right. \tag{3.24}$$

has zero initial values and sources, but with a new boundary values $\bar v_b(t)$:

$$\begin{cases}\bar\rho_t+\bar v_x=0,\\ \bar v_t+\bar\rho_x=\bar v_{xx},\ \ x,t\geq 0,\\ \begin{pmatrix}\bar\rho\\ \bar v\end{pmatrix}(x,0)=\begin{pmatrix}0\\ 0\end{pmatrix},\ \ x\geq 0,\\ \bar v(0,t)=\bar v_b(t)\equiv v_b(t)-v_I(0,t)-v_S(0,t).\end{cases} \tag{3.25}$$

From the above Dirichlet-Neumann relations (3.14) and (3.15),

$$\bar v_x(0,t) \ = \ \int_0^t \frac{e^{-(t-\tau)}}{\sqrt{\pi(t-\tau)}}\bar v_b(\tau)d\tau, \tag{3.26}$$

$$\bar\rho(0,t) \ = \ \int_0^t\int_0^y \frac{e^{-(y-\tau)}}{\sqrt{\pi(y-\tau)}}\bar v_b(\tau)d\tau dy. \tag{3.27}$$

With the full boundary values thus given, we integrate the Navier-Stokes equations times the Green's function to yield the solution formula for the initial-boundary value problem (3.25):

$$\begin{aligned}\begin{pmatrix}\bar\rho\\ \bar v\end{pmatrix}(x,t) \ &= \ \int_0^t \mathbb{G}(x,t-s)\begin{pmatrix}\bar v\\ \bar\rho\end{pmatrix}(0,s)ds-\int_0^t \mathbb{G}(x,t-s)\begin{pmatrix}0\\ \bar v_x\end{pmatrix}(0,s)ds\\ &\quad+\int_0^t \mathbb{G}_x(x,t-s)\begin{pmatrix}0\\ \bar v\end{pmatrix}(0,s)ds.\end{aligned} \tag{3.28}$$

4. Dissipative Wave Equations

The two-dimensional wave equation

$$u_{tt}=u_{xx}+u_{yy} \tag{4.1}$$

can be written as a system of two first order equations

$$\begin{cases} u_t + u_x + v_y = 0, \\ v_t - v_x + u_y = 0, \end{cases} \quad v \equiv \int^y -(u_t + u_x)dy. \tag{4.2}$$

Note that both components (u, v) satisfy the wave equation.

We consider the system with *artificial viscosity* and its initial-boundary problem with zero initial values and given boundary data $u_b(y,t)$ and $v_b(y,t)$ at $x = 0$:

$$\begin{cases} u_t + u_x + v_y = \Delta u \\ v_t - v_x + u_y = \Delta v \end{cases}$$
$$u(0,y,t) = u_b(y,t), v(0,y,t) = v_b(y,t) \text{ for } y \in \mathbb{R}, t > 0,$$
$$(u(x,y,0), v(x,y,0)) = (0,0). \tag{4.3}$$

4.1. Dirichlet-Neumann relation in the Fourier-Laplace variables

Consider the transformed functions:

$$\begin{cases} U(x,\eta,s) \equiv \int_0^\infty \int_{-\infty}^\infty e^{-st-i\eta y} u(x,y,t)dydt, \\ V(x,\eta,s) \equiv \int_0^\infty \int_{-\infty}^\infty e^{-st-i\eta y} v(x,y,t)dydt, \end{cases}$$
$$\text{(Fourier-Laplace transformation)}$$
$$\begin{cases} \hat{U}(\xi,\eta,s) = \int_0^\infty e^{-\xi x} U(x,\eta,s)dx, \\ \hat{V}(\xi,\eta,s) = \int_0^\infty e^{-\xi x} V(x,\eta,s)dx. \end{cases}$$
$$\text{(Laplace-Fourier-Laplace transformation)} \tag{4.4}$$

In the transformed variables, the system (4.3) is turned into an algebraic system:

$$\begin{pmatrix} s + \xi - \xi^2 + \eta^2 & i\eta \\ i\eta & s - \xi - \xi^2 + \eta^2 \end{pmatrix} \begin{pmatrix} \hat{U} \\ \hat{V} \end{pmatrix} = \begin{pmatrix} (1-\xi)U^0 - U_x^0 \\ (-1-\xi)V^0 - V_x^0 \end{pmatrix}, \tag{4.5}$$

where

$$\begin{aligned} U^0 &= U^0(\eta, s) = U(0, \eta, s),\ V^0 = V^0(\eta, s) = V(0, \eta, s), \\ U^0_x &= U^0_x(\eta, s) = U_x(0, \eta, s),\ V^0_x = V^0_x(\eta, s) = V_x(0, \eta, s), \end{aligned} \tag{4.6}$$

are the boundary data for the Fourier-Laplace transform of the unknown functions. Thus $(\hat{U}, \hat{V})$ can be expressed as a rational function in ξ-variable of the boundary values as follows:

$$\begin{aligned} \begin{pmatrix} \hat{U} \\ \hat{V} \end{pmatrix} &= \frac{1}{p(\xi, \eta, s)} \begin{pmatrix} s - \xi - \xi^2 + \eta^2 & -i\eta \\ -i\eta & s + \xi - \xi^2 + \eta^2 \end{pmatrix} \begin{pmatrix} (1-\xi)U^0 - U^0_x \\ (-1-\xi)V^0 - V^0_x \end{pmatrix} \\ &\equiv \frac{\mathbb{K}(\xi, \eta, s; U^0, U^0_x, V^0, V^0_x)}{p(\xi, \eta, s)}. \end{aligned} \tag{4.7}$$

The characteristic polynomial $p(\xi, \eta, s)$, in ξ variable,

$$\begin{aligned} p(\xi, \eta, s) &= det \begin{pmatrix} s + \xi - \xi^2 + \eta^2 & i\eta \\ i\eta & s - \xi - \xi^2 + \eta^2 \end{pmatrix} \\ &= s^2 + \eta^2 + 2s\eta^2 + \eta^4 - (1 + 2s + 2\eta^2)\xi^2 + \xi^4 \end{aligned} \tag{4.8}$$

has four roots:

$$\begin{aligned} \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \equiv \Bigg\{ &\sqrt{\frac{1}{2} + s + \frac{1}{2}\sqrt{1+4s} + \eta^2},\ \sqrt{\frac{1}{2} + s - \frac{1}{2}\sqrt{1+4s} + \eta^2}, \\ &-\sqrt{\frac{1}{2} + s - \frac{1}{2}\sqrt{1+4s} + \eta^2},\ -\sqrt{\frac{1}{2} + s + \frac{1}{2}\sqrt{1+4s} + \eta^2} \Bigg\}. \end{aligned} \tag{4.9}$$

The inverse Laplace transform yields the solution in terms of $Res_{\lambda_j}, j = 1, 2, 3, 4$, the residues at the roots:

$$\begin{pmatrix} U \\ V \end{pmatrix} (\eta, s) = \sum_{j=1}^{4} e^{\lambda_j x} Res_{\lambda_j} = \sum_{j=1}^{4} e^{\lambda_j x} \frac{\mathbb{K}(\lambda_j, \eta, s; U^0, U^0_x, V^0, V^0_x)}{p'_\xi(\lambda_j, \eta, s)}. \tag{4.10}$$

For $s > 0$ and $\eta \in \mathbb{R}$, the roots are real:

$$\lambda_1 > \lambda_2 > 0 > \lambda_3 > \lambda_4.$$

The well-posedness of (4.3) imposes that the solution (U, V) does not contain exponential growth components

$$e^{\lambda_j x},\ j=1,2,$$

and so from (4.10)

$$\mathbb{K}(\lambda_j,\eta,s;U^0,U^0_x,V^0,V^0_x)=0,\ j=1,2.$$

Both of these two relations are degenerate. This is consistent with the fact that, for well-posedness, only one condition, say Dirichlet or Neumann, should be given on the boundary. This yields the following relation:

$$\begin{pmatrix} 1+\sqrt{1+4s}+\sqrt{2+4s+2\sqrt{1+4s}+4\eta^2} & 2i\eta \\ 1-\sqrt{1+4s}+\sqrt{2+4s-2\sqrt{1+4s}+4\eta^2} & 2i\eta \end{pmatrix}\begin{pmatrix} U^0_x \\ V^0_x \end{pmatrix}+$$
$$\begin{pmatrix} 2s+2\eta^2-\sqrt{\frac{1}{2}+s+\sqrt{\frac{1}{4}+s}+\eta^2}+\sqrt{\frac{1}{2}+2s}\sqrt{1+2s+\sqrt{1+4s}+2\eta^2} & 2i\eta+i\eta\sqrt{2+4s+2\sqrt{1+4s}+4\eta^2} \\ 2s+2\eta^2-\sqrt{\frac{1}{2}+s-\sqrt{\frac{1}{4}+s}+\eta^2}-\sqrt{\frac{1}{2}+2s}\sqrt{1+2s-\sqrt{1+4s}+2\eta^2} & 2i\eta+i\eta\sqrt{2+4s-2\sqrt{1+4s}+4\eta^2} \end{pmatrix}\begin{pmatrix} U^0 \\ V^0 \end{pmatrix}$$
$$=\begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.11}$$

This is simplified to the following Dirichlet-Neumann relation in the transformed variables:

$$U^0_x = \frac{1}{2}(1-\alpha-\beta)U^0+\frac{i(1+\alpha-\beta)\left(1-2\alpha+\alpha^2-\beta^2\right)}{4\eta}V^0 \tag{4.12}$$

and

$$\begin{aligned} V^0_x = &\frac{1}{8(1+4s)\eta^3}i(\alpha-\beta)\left(-\alpha+\alpha^2-\beta(1+\beta)\right)\\ &\cdot\left(1+4s+\beta^2-2\beta^3+\alpha^2(-1+2\beta)\right)\\ &\cdot(-1+\beta^2-2s(2+\alpha+\beta)+\alpha^2(1+2\beta)-2\eta^2-2\beta\eta^2+2\alpha\left(\beta+\beta^2-\eta^2\right))U^0\\ &+\frac{1}{2}(-1-\alpha-\beta)V^0, \end{aligned} \tag{4.13}$$

where

$$\begin{cases} \alpha=\lambda_2\equiv\sqrt{(\frac{1}{2}-\sqrt{s+1/4})^2+\eta^2}, \\ \beta=\lambda_1\equiv\sqrt{(\frac{1}{2}+\sqrt{s+1/4})^2+\eta^2}. \end{cases} \tag{4.14}$$

4.2. Inverse transformation in time variable, I

The Dirichlet-Neumann relations, (4.12), (4.13), in the transformed variables contain expressions which are polynomials in α and β. Thus their inverse Laplace transforms are convolution of those of the inverse transform of α and β. We will use the contour integral in the complex domain for the calculations. However, such a calculation is hindered as the functions α and β do not decay to zero for $s \in i\mathbb{R}$ as $|s| \to \infty$. On the other hand, their differentiations

$$\frac{\partial \alpha}{\partial s} = \frac{\sqrt{\frac{1}{4}+s}-\frac{1}{2}}{2\sqrt{\frac{1}{4}+s}\sqrt{\left(\frac{1}{2}-\sqrt{\frac{1}{4}+s}\right)^2+\eta^2}},$$

$$\frac{\partial \beta}{\partial s} = \frac{\sqrt{\frac{1}{4}+s}+\frac{1}{2}}{2\sqrt{\frac{1}{4}+s}\sqrt{\left(\frac{1}{2}+\sqrt{\frac{1}{4}+s}\right)^2+\eta^2}},$$

decays as $|s| \to \infty$. Thus, instead of finding the inverse Laplace transformation of α and β, we consider the inverse of the operator α_s and β_s. The inverse Laplace transform of α and β is recovered by the relations:

$$\begin{cases} \displaystyle\int_{Re(s)=0} \alpha e^{st} ds = -\frac{1}{t}\int_{Re(s)=0} \frac{\partial \alpha}{\partial s} e^{st} ds, \\ \displaystyle\int_{Re(s)=0} \beta e^{st} ds = -\frac{1}{t}\int_{Re(s)=0} \frac{\partial \beta}{\partial s} e^{st} ds. \end{cases} \tag{4.15}$$

Remark 4.1. The choice of the partial differentiations α_s and β_s turns out to be crucial in the following calculations. Recall that, for the first example, instead of studying the inverse Laplace transform of $\sqrt{s}$, we consider $\sqrt{s}/s$, (2.11). The reason for that has been explained, (2.13). Similarly, in the second example, instead of considering $s/(\sqrt{s+1})$, we consider $1/\sqrt{s+1}$, (3.13), for the same reason. On the other hand, for these two examples we could also consider the differentiation of the original functions. We had, however, followed the more intuitive way in that two examples. For this third example, though, the choice of differentiation is the only right choice.

Consider first the Bromwich's integral of β_s for the inverse Laplace transform:

$$\frac{1}{2\pi i}\int_{Re(s)=0}\left(\frac{\partial\beta}{\partial s}\right)e^{st}ds. \tag{4.16}$$

We consider the contour integral over the path illustrated in Figure 1:

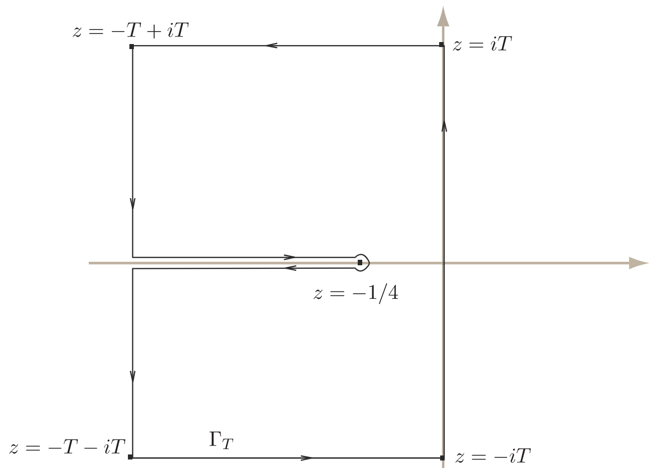

Figure 1: The contour of the path integral of Γ_T.

The function β_s is analytic in the domain $\mathbb{C}\setminus(-\infty,-1/4]$ and so

$$\frac{1}{2\pi i}\int_{\Gamma_T}\frac{\frac{1}{2}+\sqrt{\frac{1}{4}+s}}{2\sqrt{\frac{1}{4}+s}\sqrt{\left(\frac{1}{2}+\sqrt{\frac{1}{4}+s}\right)^2+\eta^2}}e^{st}ds=0. \tag{4.17}$$

Letting $T\to\infty$, the vertical integrals along the real part of $z=-T$ and also horizontal integrals along the imaginary part of $z=\pm T$ tend to zero, and we obtain:

$$\begin{aligned}&\left|\frac{1}{2\pi i}\int_{Re(s)=0}\left(\frac{\partial\beta}{\partial s}\right)e^{st}ds\right|\\ &\quad=\left|\frac{1}{2\pi i}\int_{Re(s)=0}\frac{\frac{1}{2}+\sqrt{\frac{1}{4}+s}}{2\sqrt{\frac{1}{4}+s}\sqrt{\left(\frac{1}{2}+\sqrt{\frac{1}{4}+s}\right)^2+\eta^2}}e^{st}ds\right|\end{aligned}$$

$$= \left| \frac{1}{2\pi i} \int_{\infty}^{0} \frac{2e^{-\frac{1}{4}t\left(1+4x^2\right)} x \left(\frac{1-ix}{\sqrt{1-4ix-4x^2+4\eta^2}} - \frac{1+ix}{\sqrt{1+4ix-4x^2+4\eta^2}} \right)}{\sqrt{-x^2}} dx \right|$$

$$\leq O(1) \frac{e^{-t/4}}{\sqrt{t}(1+|\eta|)}. \tag{4.18}$$

4.3. Inverse transformation in time variable, II

Note in (4.18) the exponential decay in t of the inverse Laplace transform of β_s. The reason is that the real part of β_s is bounded above by $-1/4$, the spectral gap. This is not so for α_s, which can be arbitrarily close to zero as η varies. An entirely different, more sophisticated thinking is needed for computing the Bromwich integral,

$$\frac{1}{2\pi i} \int_{Re(s)=0} \frac{\partial \alpha}{\partial s} e^{st} ds = \frac{1}{2\pi i} \int_{Re(s)=0} \frac{1 - \frac{1}{2\sqrt{\frac{1}{4}+s}}}{2\sqrt{\frac{1}{2}+s-\sqrt{\frac{1}{4}+s}+\eta^2}} e^{st} ds. \tag{4.19}$$

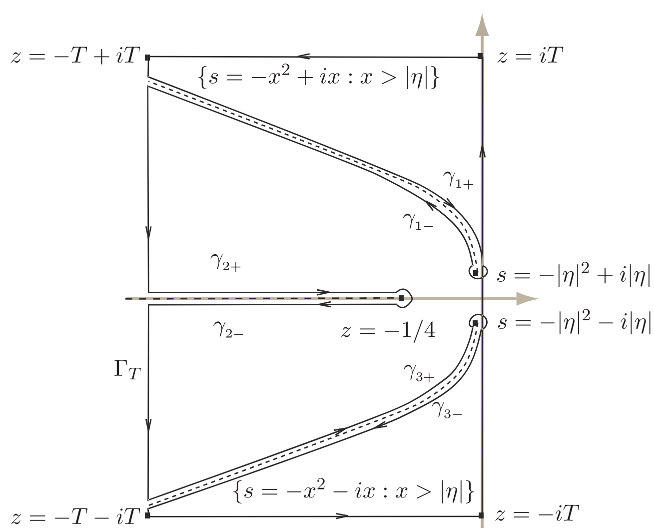

Figure 2: The contour of the path integral of Γ_T.

The function α_s is analytic in the cut domain

$$\mathbb{C}\backslash\left(\{z=-x^2\pm ix : x\geq|\eta|\}\cup(-\infty,-1/4]\right).$$

To compute the Bromwich integral (4.19) we integrate α_s along the boundary of this domain, the contour Γ_T as illustrated in Figure 2:

$$\int_{\Gamma_T}\alpha_s e^{st}ds=0. \tag{4.20}$$

By taking the limit $T\to\infty$, one has

$$\frac{1}{2\pi i}\int_{\{Re(s)=0\}+\gamma_{1+}+\gamma_{1-}+\gamma_{2+}+\gamma_{2-}+\gamma_{3+}+\gamma_{3-}}\alpha_s e^{st}ds=0, \tag{4.21}$$

where these paths are given as follows, Figure 2:

$$\begin{cases}\gamma_{1+}=\{s=-(\tau^2+|\eta|^2)+i\sqrt{\tau^2+|\eta|^2}\ | \qquad \tau\in(\infty,0),\\ \qquad\sqrt{1/2+s-\sqrt{1/4+s}+\eta^2}=i\tau,\sqrt{s+1/4}=1/2+i\sqrt{\tau^2+|\eta|^2}\},\\ \gamma_{1-}=\{s=-(\tau^2+|\eta|^2)+i\sqrt{\tau^2+|\eta|^2}\ | \qquad \tau\in(0,\infty),\\ \qquad\sqrt{1/2+s-\sqrt{1/4+s}+\eta^2}=-i\tau,\sqrt{s+1/4}=1/2+i\sqrt{\tau^2+|\eta|^2}\},\\ \gamma_{2+}=\{s=-\tau^2-1/4\ |\tau\in(\infty,0),\sqrt{1/4+s}=i\tau\},\\ \gamma_{2-}=\{s=-\tau^2-1/4\ |\tau\in(0,\infty),\sqrt{1/4+s}=-i\tau\},\\ \gamma_{3+}=\{s=-(\tau^2+|\eta|^2)-i\sqrt{\tau^2+|\eta|^2}\ | \qquad \tau\in(\infty,0),\\ \qquad\sqrt{1/2+s-\sqrt{1/4+s}+\eta^2}=i\tau,\sqrt{s+1/4}=1/2-i\sqrt{\tau^2+|\eta|^2}\},\\ \gamma_{3-}=\{s=-(\tau^2+|\eta|^2)-i\sqrt{\tau^2+|\eta|^2}\ | \qquad \tau\in(0,\infty),\\ \qquad\sqrt{1/2+s-\sqrt{1/4+s}+\eta^2}=-i\tau,\sqrt{s+1/4}=1/2-i\sqrt{\tau^2+|\eta|^2}\},\end{cases} \tag{4.22}$$

On paths γ_{1+}, γ_{1-}, γ_{3+}, and γ_{3-}, $s+1/2-\sqrt{s+1/4}+\eta^2=-\tau^2$ for some $\tau\in\mathbb{R}$. These four paths can thus be parametrized as

$$s=-(\tau^2+|\eta|^2)\pm i\sqrt{\tau^2+|\eta|^2}\text{ for }\tau>0,$$

and we make the change of variables of integration $s\to\tau$. After long computations, we obtain

$$\begin{aligned}
&\int_{\gamma_{1+}+\gamma_{1-}} \frac{-\frac{1}{2}+\sqrt{\frac{1}{4}+s}}{2\sqrt{\frac{1}{4}+s}\sqrt{\left(-\frac{1}{2}+\sqrt{\frac{1}{4}+s}\right)^2+\eta^2}} e^{st} ds \\
&= \int_{\infty}^{0} e^{-t\left(\eta^2+\tau^2\right)} \left(-1+\sqrt{1+4s}\right) \left(1+2i\sqrt{\eta^2+\tau^2}\right) \\
&\quad \times \left(\cos(t\sqrt{\eta^2+\tau^2})+i\sin(t\sqrt{\eta^2+\tau^2})\right) /(2\sqrt{1+4s}\sqrt{\eta^2+\tau^2}) d\tau \\
&\quad +\int_{0}^{\infty} (e^{-t\left(\eta^2+\tau^2\right)} \left(-1+\sqrt{1+4s}\right) \left(-i+2\sqrt{\eta^2+\tau^2}\right) \\
&\quad \times \left(-i\cos\left(t\sqrt{\eta^2+\tau^2}\right)+\sin\left(t\sqrt{\eta^2+\tau^2}\right)\right) /(2\sqrt{1+4s}\sqrt{\eta^2+\tau^2}) d\tau \\
&= 2i\int_{0}^{\infty} e^{-t\left(\tau^2+\eta^2\right)} \left(\cos\left(t\sqrt{\tau^2+\eta^2}\right)+i\sin\left(t\sqrt{\tau^2+\eta^2}\right)\right) d\tau; \qquad (4.23)
\end{aligned}$$

$$\begin{aligned}
&\int_{\gamma_{3+}+\gamma_{3-}} \frac{-\frac{1}{2}+\sqrt{\frac{1}{4}+s}}{2\sqrt{\frac{1}{4}+s}\sqrt{\left(-\frac{1}{2}+\sqrt{\frac{1}{4}+s}\right)^2+\eta^2}} e^{st} ds \\
&= \int_{\infty}^{0} \frac{e^{-t\left(\eta^2+\tau^2\right)} \left(-1+2i\sqrt{\eta^2+\tau^2}\right) \left(\cos\left(t\sqrt{\eta^2+\tau^2}\right)-i\sin\left(t\sqrt{\eta^2+\tau^2}\right)\right)}{i+2\sqrt{\eta^2+\tau^2}} d\tau \\
&\quad +\int_{0}^{\infty} \frac{e^{-t\left(\eta^2+\tau^2\right)} \left(i+2\sqrt{\eta^2+\tau^2}\right) \left(\cos\left(t\sqrt{\eta^2+\tau^2}\right)-i\sin\left(t\sqrt{\eta^2+\tau^2}\right)\right)}{-1+2i\sqrt{\eta^2+\tau^2}} d\tau \\
&= -2\int_{0}^{\infty} e^{-t\left(\eta^2+\tau^2\right)} \left(i\cos\left(t\sqrt{\eta^2+\tau^2}\right)+\sin\left(t\sqrt{\eta^2+\tau^2}\right)\right) d\tau. \qquad (4.24)
\end{aligned}$$

We will discuss the geometric significance of these expressions when we combine this inversion of the Laplace transform with the inversion of the Fourier transform in the tangential spatial variable η later.

Similar to the case with β_s, the path integral along $\gamma_{2+}+\gamma_{2-}$ decays exponentially in t because of the spectral gap:

$$\left|\int_{\gamma_{2+}+\gamma_{2-}} \alpha_s e^{st} ds\right|$$

$$
\begin{aligned}
&= \left| \int_0^\infty -2ie^{-\frac{1}{4}t\left(1+4\tau^2\right)} \left(\frac{1-2i\tau}{2\sqrt{1+4\eta^2-4i\tau-4\tau^2}} + \frac{-\frac{1}{2}-i\tau}{\sqrt{1+4\eta^2+4i\tau-4\tau^2}} \right) d\tau \right| \\
&\le O(1)\frac{e^{-t/4}}{\sqrt{t}(1+|\eta|)}. \qquad (4.25)
\end{aligned}
$$

4.4. Inverse transform in space variable

In the previous subsection we have established the inverse of $\alpha(s,\eta)$ and $\beta(s,\eta)$ in the time variable s in terms of (4.15) and the Bromwich's integrals (4.18), (4.23), (4.24), and (4.25). We now inverse the space variable η. The terms which decays with the exponential rate $e^{-t/4}$ have local effects and there is no need to explicitly invert them. We only need to perform the inverse Fourier transformation of $\int_{Re(s)=0} \alpha_s e^{st} ds$.
From (4.23), (4.24), and (4.25), one has that

$$
\frac{1}{2\pi i}\int_{Re(s)=0} \frac{\partial \alpha}{\partial s} e^{st} ds = -\frac{2}{\pi}\int_0^\infty e^{-t\left(\eta^2+\tau^2\right)} \cos\left(t\sqrt{\eta^2+\tau^2}\right) d\tau + O(1)\frac{e^{-t/4}}{\sqrt{t}}. \qquad (4.26)
$$

Thus we need to find the inverse Fourier transformation of

$$
-\frac{2}{\pi}\int_0^\infty e^{-t\left(\eta^2+\tau^2\right)} \cos\left(t\sqrt{\eta^2+\tau^2}\right) d\tau :
$$

$$
\mathscr{W}_2(y,t) \equiv -\frac{1}{\pi^2}\int_{\mathbb{R}} e^{iy\eta} \int_0^\infty e^{-t\left(\eta^2+\tau^2\right)} \cos\left(t\sqrt{\eta^2+\tau^2}\right) d\tau d\eta. \qquad (4.27)
$$

This function can be identified with the convolution of the 2-dimensional heat kernel with the solution u of the following initial value problem of the wave equation at time t:

$$
\begin{cases}
\left(\dfrac{\partial^2}{\partial t'^2} - \dfrac{\partial^2}{\partial y'^2} - \dfrac{\partial^2}{\partial z'^2}\right) u(y',z',t') = 0, \\
u(y',z',0) = -\dfrac{1}{\pi^2}\delta(y')\delta(z'), \\
u_{t'}(y',z',0) = 0.
\end{cases}
$$

The Green's function u for the 2-dimensional wave equation has been constructed by the Hadamard's descending method in terms of the Kirchhoff formula for 3-dimensional wave equation. The convolution with the heat kernel becomes:

$$
\begin{aligned}
&-\frac{1}{\pi^2}\int_{\mathbb{R}} e^{iy\eta}\int_0^\infty e^{-t\left(\eta^2+\tau^2\right)}\cos\left(t\sqrt{\eta^2+\tau^2}\right)d\tau d\eta \\
&= -\frac{1}{\pi^2}\,\partial'_t\left[\frac{1}{4\pi t'}\iint_{|\vec{y}-\vec{z}|=t'} k(\vec{z},t)dS_{\vec{y}}\right]\Bigg|_{t'=t}, \qquad (4.28)
\end{aligned}
$$

where

$$
\begin{cases}
\vec{y}=(y,0,0)\in\mathbb{R}^3,\\
\vec{z}=(z^1,z^2,z^3)\in\mathbb{R}^3,\\
k(\vec{\boldsymbol{z}},t)\equiv \dfrac{e^{-\frac{|z^1|^2+|z^2|^2}{4t}}}{4\pi t}.
\end{cases}
$$

This explicit expression can easily be estimated:

$$
|\mathscr{W}_2(y,t)|\le O(1)W_2(0,y,t;2), \qquad (4.29)
$$

where

$$
W_2(\vec{\boldsymbol{x}},t;D)\equiv
\begin{cases}
\dfrac{1}{t-|\vec{\boldsymbol{x}}|} & \text{for } |\vec{\boldsymbol{x}}|\le t-\sqrt{Dt},\\
\dfrac{1}{t^{3/4}D^{1/4}} & \text{for } ||\vec{\boldsymbol{x}}|-t|\le\sqrt{Dt},\\
\dfrac{e^{-\frac{(|\vec{\boldsymbol{x}}|-t)^2}{Dt}}}{t^{3/4}D^{1/4}} & \text{for } |\vec{\boldsymbol{x}}|\ge t+\sqrt{Dt},
\end{cases} \qquad (4.30)
$$

where $\vec{\boldsymbol{x}}=(x,y)$. From (4.26),

$$
\begin{aligned}
&\sup_{y\in\mathbb{R}}\left|\int_{\mathbb{R}} e^{i\eta y}\left(\int_{Re(s)=0}(\alpha_s e^{st}ds+4i\int_0^\infty e^{-t(|\eta|^2+\tau^2)}\cos(t\sqrt{|\eta|^2+\tau^2})d\tau\right)d\eta\right|\\
&\le\ O(1)e^{-t/4}/\sqrt{t}. \qquad (4.31)
\end{aligned}
$$

From (4.27), (4.29), and (4.31), one has

$$
\left|\int_{\mathbb{R}} e^{i\eta y}\int_{Re(s)=0}\frac{\partial\alpha}{\partial s}e^{st}dsd\eta\right|\le O(1)\left(W_2(0,y,t;2)+e^{-t/4}/\sqrt{t}\right). \qquad (4.32)
$$

Combining (4.32) and (4.15) to yield

$$\left|\int_{\mathbb{R}} e^{i\eta y}\int_{Re(s)=0}\alpha e^{st}dsd\eta\right| \leq O(1)\frac{1}{t}\left(W_2(0,y,t;2)+e^{-t/4}/\sqrt{t}\right). \quad (4.33)$$

The following theorem is a consequence of (4.18), (4.33), and (4.13).

Theorem 4.2. *There exist* $K_1,\ K_2 \in \mathbb{R}$, *functions* $\mathscr{K}_{ij}(y,t)$ *and* $\mathscr{L}_{ij}(y,t)$, $i,j=1,2$ *such that*

$$\begin{aligned}\int_{-\infty}^{\infty}\Bigg|u_x(0,y,t)-K_1u_b(y,t)-\left(\mathscr{K}_{11}+\mathscr{F}^{-1}\left(\frac{1}{\eta}\right)\underset{(y,t)}{*}\mathscr{K}_{12}\right)\underset{(y,t)}{*}u_b(y,t)\\ -\left(\mathscr{K}_{21}+\mathscr{F}^{-1}\left(\frac{1}{\eta}\right)\underset{(y,t)}{*}\mathscr{K}_{22}\right)\underset{(y,t)}{*}v_b(y,t)\Bigg|^2dy\\ \leq\ O(1)\int_0^t e^{-(t-\tau)/4}\int_{\mathbb{R}}\left(|u_b(y,\tau)|^2+|v_b(y,\tau)|^2\right)dyd\tau \end{aligned} \quad (4.34)$$

and

$$\begin{aligned}\int_{-\infty}^{\infty}\Bigg|v_x(0,y,t)-K_2v_b(y,t)-\left(\mathscr{L}_{11}+\mathscr{F}^{-1}\left(\frac{1}{\eta^3}\right)\underset{(y,t)}{*}\mathscr{L}_{12}\right)\underset{(y,t)}{*}u_b(y,t)\\ -\left(\mathscr{L}_{21}+\mathscr{F}^{-1}\left(\frac{1}{\eta}\right)\underset{(y,t)}{*}\mathscr{L}_{22}\right)\underset{(y,t)}{*}v_b(y,t)\Bigg|^2dy\\ \leq\ O(1)\int_0^t e^{-(t-\tau)/4}\int_{\mathbb{R}}\left(|u_b(y,\tau)|^2+|v_b(y,\tau)|^2\right)dyd\tau, \end{aligned} \quad (4.35)$$

and the functions $\mathscr{K}_{ij}$ *and* $\mathscr{L}_{ij}$ *satisfy*

$$|\mathscr{K}_{ij}(y,t)|,|\mathscr{L}_{ij}(y,t)|\leq\frac{O(1)}{t}W_2(0,y,t;8), \quad (4.36)$$

where W_2 *is given* (4.30). *Here,* $\mathscr{F}^{-1}$ *is the inverse Fourier transformation from* η *variable to* y *variable, the operator* $\underset{(y,t)}{*}$ *is the convolution operator defined by*

$$[h\underset{(y,t)}{*}g](y,t)\equiv\int_0^t\int_{\mathbb{R}}h(z,t-\tau)g(y-z,\tau)dzd\tau.$$

Remark 4.3. The functions $\mathscr{K}_{ij}$ and $\mathscr{L}_{ij}$ give the wave propagation structure within the boundary induced by the boundary data u_b and v_b; other operators with the terms $\mathscr{F}^{-1}(1/\eta)$ and $\mathscr{F}^{-1}(1/\eta^3)$ give interactions global in space but local in time. In the above estimates, the right hand sides register the minor time-asymptotic effects, exponentially decaying memory of $u_x(0,y,t)$ and $v_x(0,y,t)$ on the Dirichlet data u_b and v_b.

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