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MULTI-DIMENSIONAL GAS FLOW: SOME HISTORICAL PERSPECTIVES

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Abstract

Multi-dimensional gas flow with shock waves can be highly complex and the mathematical theory for such flows is still far from being complete. This is so because such a theory should resolve some of the most difficult issues in the theory of partial differential equations; among them are the free boundaries and the nonlinear equations of mixed types. These difficulties were recognized early on. We illustrate the richness of the subject with certain historical perspectives on the basic question of shock reflections.

1. Introduction

The most basic equations for the shock wave theory are the *Euler equations in gas dynamics*. We consider the isentropic system:

$$\begin{cases} \rho_t + \nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{u}) = 0, \text{ continuity equation,} \\ (\rho \boldsymbol{u})_t + \nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u} + p(\rho)\mathbb{I}) = 0, \text{ momemtum equations.} \end{cases} \tag{1.1}$$

To focus on the study of shock waves, one often considers irrotational flows by introducing the potential ϕ:

$$\boldsymbol{u} = \nabla_{\boldsymbol{x}} \phi. \tag{1.2}$$

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From (1.1) and (1.2), it follows the *Bernoulli equation*:

$$\phi_t + \frac{1}{2}|\nabla_{\boldsymbol{x}}\phi|^2 + \Pi(\rho) = A \text{ constant}, \tag{1.3}$$

where

$$\Pi'(\rho) \equiv \frac{p'(\rho)}{\rho}, \ \sqrt{p'(\rho)} = c \text{ sound speed}. \tag{1.4}$$

The *potential flow equation* is a combination of the Bernoulli equation and the continuity equation:

$$\phi_{tt} + 2\nabla_{\boldsymbol{x}}\phi \cdot \nabla_{\boldsymbol{x}}(\phi_t) + (\nabla_{\boldsymbol{x}}\phi)^t \nabla^2_{\boldsymbol{x}}\phi \nabla_{\boldsymbol{x}}\phi - c^2 \Delta\phi = 0. \tag{1.5}$$

The Euler and potential flow equations are the most basic systems for the study of multi-dimensional gas flow. For the derivation of these equations, the readers are referred to the classical treatise of Courant-Friedrichs, [4]. This book was published in 1948 and summarizes the achievements of mathematical scientists in the classical period. There has been very substantial progress on the shock wave theory since then, thanks to the pioneering works of Lax (1956), [9] and Glimm (1965), [10]. However, much of the deep analysis is on one-dimensional conservation laws, see, for instance, Dafermos [5]. Thus the achievements of the classical period are still the most important for the study of multi-dimensional gas flow. In an effort to initiate a new era, following the classical period, von Neumann called the celebrated meeting of the panel discussions on Wednesday morning, August 17, 1949:

DISCUSSION ON THE EXISTENCE AND UNIQUENESS OR MULTIPLICITY OF SOLUTIONS OF THE AERODYNAMICAL EQUATIONS

The list of participants included some of the most important mathematical scientists of the time, among them von Neumann, Burgers, Heisenberg, Liepmann, and von Karman. The text of the discussions has been reproduced in [17] and it contains vital exchanges on physical and analytical aspects of the shock wave theory by these leading scientists. The main purpose of the present article is to revisit some of these exchanges in view of the recent developments, and to offer new perspectives.

2. Preliminaries

Write the Euler equations (1.1) as a general system of hyperbolic conservation laws:

$$\begin{cases} \mathbf{U}_t + \nabla_{\boldsymbol{x}} \cdot \mathbb{F}(\mathbf{U}) = 0, \\ \mathbf{U} \equiv \begin{pmatrix} \rho \\ \rho \boldsymbol{u} \end{pmatrix}, \\ \mathbb{F}(\mathbf{U}) \equiv \rho \boldsymbol{u} \otimes \boldsymbol{u} + p(\rho)\mathbb{I}. \end{cases} \tag{2.1}$$

A discontinuity along the front $\zeta(\boldsymbol{x}, t) = 0$, ζ a scalar function, satisfies the *Rankine-Hugoniot condition*:

$$\zeta_t[\mathbf{U}_+ - \mathbf{U}_-] = \nabla_{\boldsymbol{x}}\zeta \cdot [\mathbb{F}(\mathbf{U}_+) - \mathbb{F}(\mathbf{U}_-)], \tag{2.2}$$

where $\mathbf{U}_\pm$ are the states on either side of the discontinuity. For definiteness, we make the normalization

$$|\boldsymbol{n}| = 1, \ \boldsymbol{n} \equiv \nabla_{\boldsymbol{x}}\zeta. \tag{2.3}$$

Take a Galilean transformation to make the discontinuity *stationary*, $\zeta_t = 0$ so that the Rankine-Hugoniot condition becomes

$$\boldsymbol{n} \cdot [\mathbb{F}(\mathbf{U}_+) - \mathbb{F}(\mathbf{U}_-)] = 0,$$

or

$$\boldsymbol{n} \cdot \Big(\rho_+ \boldsymbol{u}_+ \otimes \boldsymbol{u}_+ + p(\rho_+)\mathbb{I}\Big) = \boldsymbol{n} \cdot \Big(\rho_- \boldsymbol{u}_- \otimes \boldsymbol{u}_- + p(\rho_-)\mathbb{I}\Big). \tag{2.4}$$

Decompose the gas velocity $\boldsymbol{u}$ into velocities normal and tangential to the discontinuity front:

$$\boldsymbol{u}^n_\pm \equiv u^n_\pm \boldsymbol{n}, \ \boldsymbol{u}^t_\pm \equiv \boldsymbol{u}_\pm - \boldsymbol{u}^n_\pm, \ u^n_\pm \equiv \boldsymbol{u}_\pm \cdot \boldsymbol{n}. \tag{2.5}$$

There are two possibilities:

Case 1: The velocity normal to the front does not jump across the discontinuity: $\boldsymbol{u}^n_+ = \boldsymbol{u}^n_-$. Thus only the tangential velocity jumps across the front.

This is a *vortex sheet,* or, *contact discontinuity,* Figure 1. Such a rotational wave exists also for the incompressible Euler equations.

Case 2: The normal velocity jumps across the front. In this case, it follows easily from (2.4) that the tangential velocity remains continuous across the front, $\boldsymbol{u}_+^t = \boldsymbol{u}_-^t$, and we have a *shock wave*, Figure 2. Shock waves occur only in compressible fluids.

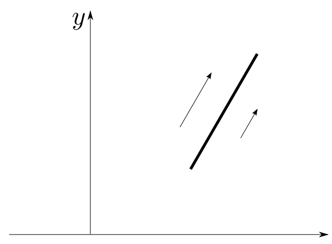

Figure 1: Contact discontinuity

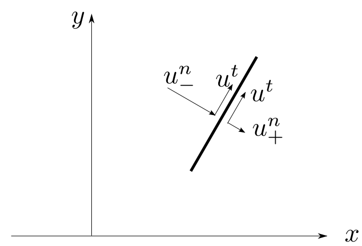

Figure 2: Shock waves

Remark 2.1. The potential flow equation (1.5) disallows vortex sheets and is often used when one wants to focus on the shock waves. When the initial value is irrotational and isentropic, and the flow is *smooth*, the flow remains irrotational and the potential flow equation is equivalent to the Euler equations. Curved shocks produce entropy variation and vorticities thus the potential flow equation is not suitable for such a shock. Unlike for incompressible flows, in a compressible flow vortex sheets can be produced at later times through the interaction of shocks and the solid boundary. These are the standard objections to the use of the potential flow equation. Mathematically, the vortex sheets are highly unstable. For the purpose of the basic understanding of the physical phenomena concerning the gas flow around a solid boundary, particularly the effects of the boundary on the shock waves, the potential flow equation is an adequate model, as it yields the qualitative behavior of flows.

Either for the Euler equations or for the potential flow equation, shock waves are produced by compression. For definiteness, we choose the sign of the front function ζ so that

$$\boldsymbol{u}_\pm \cdot \boldsymbol{n} > 0. \tag{2.6}$$

This way $\mathbf{U}_-$ is called the *upstream state* and $\mathbf{U}_+$ the *downstream state.* As shock waves are a result of the compression of gases, the downstream density is greater than the upstream density. Equivalently, the upstream normal velocity relative to the shock is greater than the downstream normal velocity. When the energy equation is included in the Euler equations, this is equivalent to the fact that the downstream entropy is greater than the upstream entropy, and so this is often formulated as the *entropy condition*, Figure 2,

$$0 < u_+^n < u_-^n. \tag{2.7}$$

For stationary flow, the shock speed is zero and the entropy condition says that the upstream normal flow velocity is supersonic and the downstream is subsonic:

$$u_-^n > c_-, \ , u_+^n < c_+. \tag{2.8}$$

In general, for a propagating shock the upstream normal velocity is supersonic *relative* to the shock. The flow speed

$$|\boldsymbol{u}| = \sqrt{(u^n)^2 + |\boldsymbol{u}^t|^2}$$

is greater than the normal velocity due to the tangential velocity component. Thus the upstream speed is always supersonic. For small tangential speed, the downstream is subsonic, and the shock is *transonic*; we call it a *weak shock.* For large tangential speed, the downstream is supersonic, and we call this shock a *supersonic, strong shock.*

Remark 2.2. Both weak and strong shock waves satisfy the entropy condition and are physically admissible. By a Galilean transformation tangential to the shock, the tangential component of the flow velocity can be made zero and the shock is then a transonic, strong shock. Similarly, through a Galilean transformation, the tangential speed can be made large to turn a weak shock into a supersonic, strong shock. Thus the notion of weak and strong shocks makes sense only when there is a particular reference frame, such as that determined by a solid flying object or a stationary obstacle. Also, the strength of the shock is determined by the jump in the normal speed and has nothing to do with whether it is supersonic or transonic.

Much attention has been paid to the study of stationary flows, and stationary potential flows, in particular, [4], [1]. It is well-known that the stationary potential flow equation

$$(\nabla_{\boldsymbol{x}}\phi)^t\nabla_{\boldsymbol{x}}^2\phi\nabla_{\boldsymbol{x}}\phi - c^2\Delta\phi = 0, \tag{2.9}$$

is

$$\begin{cases} \text{elliptic for subsonic flows, } |\nabla_{\boldsymbol{x}}\phi| = |\boldsymbol{u}| < c; \\ \text{hyperbolic for supersonic flows, } |\nabla_{\boldsymbol{x}}\phi| = |\boldsymbol{u}| > c. \end{cases} \tag{2.10}$$

Another important class of flows are the *self-similar flows*:

$$\mathbf{U}(\boldsymbol{x}, t) = \mathbf{U}(\boldsymbol{\xi}),\ \boldsymbol{\xi} \equiv \frac{\boldsymbol{x}}{t}. \tag{2.11}$$

For self similar flows, the dimension of the basic dependent variables is reduced by one, from $(\boldsymbol{x}, t)$ to the *self-similarity variables* $\boldsymbol{\xi} = \boldsymbol{x}/t$.

Remark 2.3. Both the Euler equations (1.1) and the potential flow equation (1.5) are invariant under the self-similar transformation

$$(\boldsymbol{x}, t) \to c(\boldsymbol{x}, t),\ \text{ for any positive constant } c. \tag{2.12}$$

The existence of self-similar flows requires the additional property that both the initial data $\mathbf{U}(\boldsymbol{x}, 0)$ and the geometry of the solid around which the gas flows are also invariant under the self-similar transformation (2.12). The usual boundary condition for inviscid flows

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0,\ \text{slip boundary condition,} \tag{2.13}$$

$\boldsymbol{n}$ normal to the solid boundary, is then also invariant under the self-similar transformation (2.12).

For the potential flow equation (1.5), we set

$$\phi(\boldsymbol{x}, t) = t\psi(\boldsymbol{\xi}),\ \chi(\boldsymbol{\xi}) = \psi(\boldsymbol{\xi}) - \frac{1}{2}|\boldsymbol{\xi}|^2, \tag{2.14}$$

so that we have

$$\begin{cases} \nabla_{\boldsymbol{\xi}}\psi = \nabla_{\boldsymbol{x}}\phi = \boldsymbol{u},\ \text{velocity,} \\ \nabla_{\boldsymbol{\xi}}\chi = \boldsymbol{u} - \boldsymbol{\xi},\ \text{pseudo-velocity.} \end{cases} \tag{2.15}$$

The potential flow equation (1.5) becomes the *self-similar potential flow equation*:

$$(\nabla_{\boldsymbol{\xi}}\psi - \boldsymbol{\xi})^t \nabla^2_{\boldsymbol{\xi}}\psi(\nabla_{\boldsymbol{\xi}}\psi - \boldsymbol{\xi}) - c^2\Delta\psi = 0; \tag{2.16}$$

or equivalently,

$$(\nabla_{\boldsymbol{\xi}}\chi)^t \nabla^2_{\boldsymbol{\xi}}\chi(\nabla_{\boldsymbol{\xi}}\chi) - c^2\Delta\chi = 2c^2 - |\nabla_{\boldsymbol{\xi}}\chi|^2. \tag{2.17}$$

The self-similar potential equation (2.17) is

$$\begin{cases} \text{Elliptic for pseudo-subsonic flows, } |\nabla_{\boldsymbol{\xi}}\chi| = |\boldsymbol{u} - \boldsymbol{\xi}| < c, \\ \text{Hyperbolic for pseudo-supersonic flows, } |\nabla_{\boldsymbol{\xi}}\chi| = |\boldsymbol{u} - \boldsymbol{\xi}| > c. \end{cases} \tag{2.18}$$

For *two-dimensional potential flow* there is simplification of notation, and we list the equations and their basic properties as follows:

$$\begin{cases} \boldsymbol{x} = (x, y), \ \boldsymbol{u} = (u, v) = (\phi_x, \phi_y), \\ \phi_{tt} + 2\phi_x\phi_{xt} + 2\phi_y\phi_{yt} + [(\phi_x)^2 - c^2]\phi_{xx} + 2\phi_x\phi_y\phi_{xy} + [(\phi_y)^2 - c^2]\phi_{yy} = 0, \\ \qquad \text{2d potential flow equation;} \end{cases} \tag{2.19}$$

$$\begin{cases} [(\phi_x)^2 - c^2]\phi_{xx} + 2\phi_x\phi_y\phi_{xy} + [(\phi_y)^2 - c^2]\phi_{yy} = 0, \\ \qquad \text{2d stationary potential flow equation,} \\ \begin{cases} \text{elliptic for subsonic flow } (\phi_x)^2 + (\phi_y)^2 = u^2 + v^2 < c^2, \\ \text{hyperbolic for supersonic flow } (\phi_x)^2 + (\phi_y)^2 = u^2 + v^2 > c^2; \end{cases} \end{cases} \tag{2.20}$$

$$\begin{cases} \phi(x, y, t) = t\psi(\xi, \eta), \ \chi(\xi, \eta) = \psi(\xi, \eta) - \frac{1}{2}(\xi^2 + \eta^2), \ \xi = \frac{x}{t}; \ \eta = \frac{y}{t}, \\ [(\psi_x - \xi)^2 - c^2]\psi_{xx} + 2(\psi_x - \xi)(\psi_y - \eta)\psi_{xy} + [(\psi_y - \eta)^2 - c^2]\psi_{yy} = 0, \\ \text{2d self-similar potential flow equation, I,} \\ [(\chi_\xi)^2 - c^2]\chi_{\xi\xi} + 2\chi_\xi\chi_\eta\chi_{\xi\eta} + [(\chi_\eta)^2 - c^2]\chi_{\eta\eta} = 2c^2 - |(\chi_\xi)^2 + (\chi_\eta)^2|, \\ \text{2d self-similar potential flow equation, II,} \\ \begin{cases} \text{elliptic for pseudo-subsonic flows,} \\ \qquad (\chi_\xi)^2 + (\chi_\eta)^2 = (u - \xi)^2 + (v - \eta)^2 < c^2; \\ \text{hyperbolic for pseudo-supersonic flows,} \\ \qquad (\chi_\xi)^2 + (\chi_\eta)^2 = (u - \xi)^2 + (v - \eta)^2 > c^2. \end{cases} \end{cases} \tag{2.21}$$

For the Euler equations, the stationary and self-similar flow equations are;

$$\begin{cases}\nabla_{\boldsymbol{x}}\cdot(\rho\boldsymbol{u})=0,\\ \nabla_{\boldsymbol{x}}\cdot(\rho\boldsymbol{u}\otimes\boldsymbol{u}+p(\rho)\mathbb{I})=0,\ \text{stationary Euler equations;}\end{cases}\tag{2.22}$$

$$\begin{cases}-\boldsymbol{\xi}\cdot\nabla_{\boldsymbol{\xi}}\rho+\nabla_{\boldsymbol{\xi}}\cdot(\rho\boldsymbol{u})=0,\\ -\boldsymbol{\xi}\cdot\nabla_{\boldsymbol{\xi}}(\rho\boldsymbol{u})+\nabla_{\boldsymbol{\xi}}\cdot(\rho\boldsymbol{u}\otimes\boldsymbol{u}+p(\rho)\mathbb{I})=0,\\ \qquad\qquad\qquad\qquad\text{self-similar Euler equations, I;}\end{cases}\tag{2.23}$$

or, for the n-dimensional case, $\boldsymbol{x}\in\mathbb{R}^n$,

$$\begin{cases}\nabla_{\boldsymbol{\xi}}\cdot(\rho(\boldsymbol{u}-\boldsymbol{\xi}))=-n\rho,\\ \nabla_{\boldsymbol{\xi}}\cdot(\rho(\boldsymbol{u}-\boldsymbol{\xi})\otimes(\boldsymbol{u}-\boldsymbol{\xi})+p(\rho)\mathbb{I})=-(n+1)\rho(\boldsymbol{u}-\boldsymbol{\xi}),\\ \qquad\qquad\text{self-similar Euler equations, II.}\end{cases}\tag{2.24}$$

Note from (2.22) and (2.24) that the stationary and self-similar Euler equations have the same form for their left hand sides.

From now on, we will concentrate on 2-dimensional flows.

3. Stationary and Self-Similar Flows

An interesting stationary gas flow is often produced by a constant upstream state with nonzero velocity and a flow going around a solid object. Mathematically, to construct the stationary flows one has to solve the *boundary value problem* with boundary data given at $|\boldsymbol{x}|=\infty$. In continuum physics, boundary value problems often have *multiple solutions.* This was already known to Euler for the buckling of elastic rods. For the incompressible flows, there are the well-known Taylor solutions, for example. In gas dynamics, because of the *strong nonlinearity of the shock waves,* the *nonuniqueness* of solutions to boundary value problem has been shown only in a few instances. For example, for quasi-one dimensional nozzle flows, [12], [13], [14], the multiple solutions in a nozzle with given boundary condition would contain the unstable solution with a stationary transonic shock in the portion of the nozzle contracting in the flow direction.

The equations for self-similar and stationary flows share the same property in terms of classification of partial differential equations. This is clear if we compare the two sets of equations in (2.9) and (2.17) for potential flows and (2.22) and (2.24) for Euler flows. The difference is on the lower order terms, the right hand side of the equations. We illustrate this by comparing the simplest case of the two dimensional potential flow equations, (2.20) and (2.21):

$$\begin{cases} [(\phi_x)^2 - c^2]\phi_{xx} + 2\phi_x\phi_y\phi_{xy} + [(\phi_y)^2 - c^2]\phi_{yy} = 0, \\ \text{2d stationary potential flow equation,} \\ [(\chi_\xi)^2 - c^2]\chi_{\xi\xi} + 2\chi_\xi\chi_\eta\chi_{\xi\eta} + [(\chi_\eta)^2 - c^2]\chi_{\eta\eta} = 2c^2 - |(\chi_\xi)^2 + (\chi_\eta)^2|, \\ \text{2d self-similar potential flow equation, II.} \end{cases} \tag{3.1}$$

The self-similar equation differs from the stationary equation as it contains the additional lower order term $2c^2 - |(\chi_\xi)^2 + (\chi_\eta)^2|$ on the right-hand side. The classification of partial differential equations is dictated by the higher order terms on the left-hand side of the equations. And these two sets of equations have exactly the same form for their left hand sides. Of course, there is another difference: The basic dependent variable χ for the self-similar equation has the property that its gradients are the *pseudo-velocity*:

$$\chi_\xi = u - \xi,\ \ \chi_\eta = v - \eta;$$

while the gradient of the potential ϕ is the *velocity*:

$$\phi_x = u,\ \ \phi_y = v.$$

Remark 3.1. The above are analytical considerations. Physically, the boundary value problem for the self-similar equation, with boundary value also posted for $\xi^2 + \eta^2$ at infinity, is equivalent to the initial value problem for the potential flow equation with self-similar initial data. Although it is difficult to prove, we expect the initial value problem for the potential flow equation to have a unique solution. Therefore the boundary value problem for *the self-similar equation is expected to have a unique solution*.

On the other hand, we have remarked above that the stationary solutions, also with boundary value posed at infinity, often *do not have unique solutions.* Clearly, the classification of partial differential equations alone does not suffice in clarifying the situation and the lower order term $-2c^2 + |(\chi_\xi)^2 + (\chi_\eta)^2|$ must make essential difference in this basic issue. In the following, we focus our study from this elementary point of view. A key observation is the fundamental *Ellipticity Principle* for self-similar potential flows in any space dimension.

Theorem 3.2 (Ellipticity Principle, [7])**.** *Consider the self-similar potential flow equation* (2.18)*. Suppose that there is a solution* $\chi(\boldsymbol{\xi})$*, which is elliptic, i.e. psuedo-subsonic, in some region* Ω*. Consider a continuous perturbation of the solution, either by changing the geometry of the solid or the boundary condition at infinity. Then there cannot appear a hyperbolic, i.e. psuedo-supersonic, bubble inside* Ω*.*

Remark 3.3. It is a well-documented fact that there are subsonic stationary flows around an airfoil for which a supersonic bubble can grow inside the subsonic flow. There is the question of whether a smooth, shock-less transonic flow is stable, [16]. A transonic wave pattern is experimentally obtained by increasing the Mach number M of the upstream flow. For small Mach numbers, flows globally subsonic, Figure 3. As the Mach number increases, a supersonic bubble emerges, Figure 4. This is not possible for self-similar flows as stated in the Ellipticity Principle.

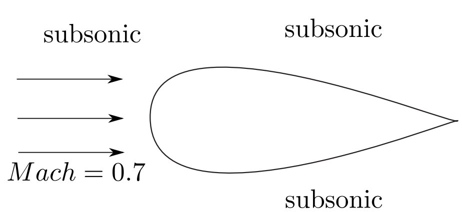

Figure 3: Subsonic stationary flow

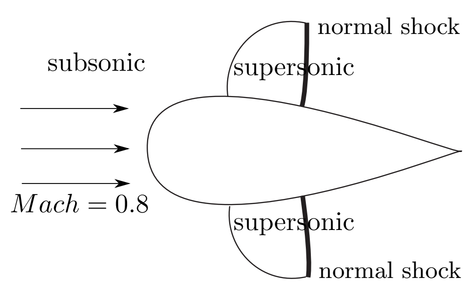

Figure 4: Transonic stationary flow

4. Historical Perspectives

In the aforementioned *DISCUSSION ON THE EXISTENCE AND UNIQUENESS OR MULTIPLICITY OF SOLUTIONS OF THE AERODYNAMICAL EQUATIONS*, [17], the chair, von Neumann, made several general comments. We quote one of his comments below that leads to substantial exchanges later:

von Neumann:

Occasionally the simplest hydrodynamical problems have several solutions, some of which are very difficult to exclude on mathematical grounds only. For instance, a very simple hydrodynamical problem is that of the supersonic flow of a gas through a concave corner, which obviously leads to the appearance of a shock wave. In general, there are two different solutions with shock waves, and it is perfectly well known from experimentation that only one of the two, the weaker shock wave, occurs in nature. But I think that all stability arguments to prove that it must be so, are of very dubious quality.

Remark 4.1. The specific example von Neumann refers to is about the shock reflection off a supersonic pointed flying object. There are two possibilities that Prandtl was aware of: One is the *weak, supersonic shock reflection*, Figure 5; and the other is the *strong transonic shock reflection*, Figure 6. Both solutions can be viewed as a supersonic flight with the *same given upstream supersonic state* I. In the case of the weak shock, the downstream state II is supersonic; while for the strong shock the downstream state III is subsonic. It has been a classical problem as to why the experiments always show the weak shock reflection. Analytically, these two solutions were obtained by Prandtl using the shock polar analysis, [4], Figure 7.

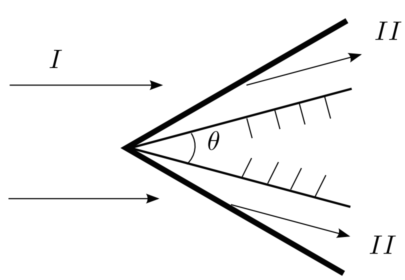

Figure 5: Weak shock reflection

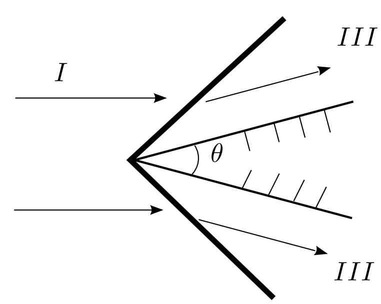

Figure 6: Strong shock reflection

Liepmann, of Liepmann-Roshko, Elements of Gas Dynamics, [11], offers the following response, which brings up interesting points in the theory of partial differential equations.

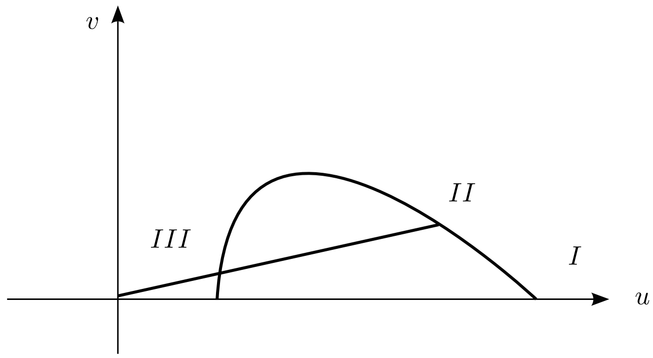

Figure 7: Shock polar

Liepmann:

I would like to add a remark about the question of the two shock waves. I think that the experiments cannot be safely cited to settle whether only the solution with the weaker shock appears in nature, because the theoretical case refers to an infinite wall (or to the flow along the two sides of an infinite wedge), which case cannot be realized in practice. With the stronger one of the two shock waves you have subsonic flow behind the shock wave, which means that behind the shock wave you have a region where the theory of the elliptic differential equation applies and where the field is influenced by the boundary conditions at a finite or an infinite distance downstream. In the case of the other shock wave the velocity remains supersonic, so that you have conditions such as those obtained with hyperbolic equations. Thus one cannot exclude a priori that conditions downstream may influence the flow and thus may lead to a predilection for one type of shock wave about the other type.

Remark 4.2. The strong, transonic shocks require geometric as well as downstream conditions to be produced and stabilized. In the nozzle flow, strong shocks are known to be stable in the portion of the nozzle expanding in the flow direction; for analytical study of this, see [13]. For supersonic flight, the engines are often designed so as to produce a strong shock inside the engine. This is to increase the gas temperature and thereby the efficiency of the combustion process. The stationary strong shocks are also used as an experimental tool for the study of the structure of gases, [20].

Next come the exchanges between von Neumann and von Karman. Von Karman makes some important observations on whether stationary solutions are physical.

von Karman:

I would like to say something about this question of uniqueness of solutions. I dont think that there is any reason that if you put a problem in a form which has no physical meaning, there shall not be two solutions. And I think the case of stationary motion as such belongs to this category, because it can occur only as a limiting case. Any physical process starts from somewhere and goes to somewhere. In the case of the two shock waves, if instead of considering a stationary motion you consider an accelerated motion, you will first get a detached shock wave ahead of the obstacle (when the Mach number has just passed through unity). Then, with increasing velocity the solution will approach the correct solution for the steady case, I should think, without any difficulty. Such a case comes near to what you can actually realize in an experiment. Is that not correct?

von Neumann:

I may not have chosen that example which fits best to your argument. It has, of course, to be admitted that to postulate stationarity is to postulate a general trait of the solution one wants, which may hold only approximately in the physical situation that can actually be realized. However, it is not necessary to take the stationary flow through a corner. The following problem also has two solutions. If you take a plane shock which hits a wall and you consider the reflection of the shock from the wall, then under a wide variety of conditions (in fact, in most cases) there are two solutions. In this case stationarity has not been postulated.

von Karman:

I only mean the following thing. I suppose we start from a certain state of rest of the gas, which must be a solution of our equations. Then we change the conditions gradually and follow the system step by step. I believe that in such a case you will always get a solution and only one solution. There is no proof that there is only one, but I believe it to be so. For, after all, a gas is a molecular system, which follows the general equations of classical mechanics. But if you take first an infinite cone, or an infinite wedge both of which are situations which can never be realized and furthermore you ask *for a stationary solution; in such a case there is no reason why there should be only one solution.*

von Karman:

Since the equations are non-linear, you can often, without violating continuity, pass from one solution to another one by following an envelope, and in such a case you can scarcely find a mathematical reason why one solution should be preferred to the other. But if you start from an actually existing (observed) state and then determine the next phase, I believe you will find only one completely determined result. Concerning Dr. von Neumann example of the reflection of waves from a wall, I do not know the answer, but I believe that no case in which infinitely extending waves or walls are involved is really defined physically.

Remark 4.3. von Karman proposes a physically realizable process: Start first with the trivial situation when the pointed flying object has zero speed and the solution is then a constant state with zero gas velocity, Figure 8. Then the flying object accelerates to a subsonic speed and a *bow shock* appears due to the compression of the gas around the flying object, Figure 9. The distance $D(t)$ of the bow shock from the flying object increases in time in general. The distance will come to a finite value if the flying object is of finite size. When the flying object is accelerated to a supersonic speed and the flying object is pointed with the wedge angle θ small, there will be an attached shock at the tip, Figure 10. von Karman then concludes that such an *attached shock would be a weak, supersonic shock.*

The basic question of *non-uniqueness* of stationary solutions is being raised here. One should not expect unique stationary solutions. On the other hand, as we have considered this question analytically in Remark 3.1, self-similar solutions should be unique. This leads to the analytical consideration in the next section.

Burgers:

Dr. von Neumann mentioned a case of nonstationary theory where you have also two solutions: a shock wave hitting a wall. But in the picture you gave the wall was infinite, so that here again one must ask: How does the situation arise, when you have an actual, finite wall? It may be that you could treat the problem for an actual situation, in which a shock wave

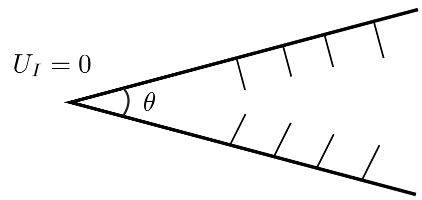

Figure 8: Zero speed, constant solution.

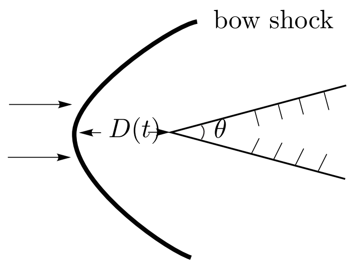

Figure 9: Subsonic upstream state.

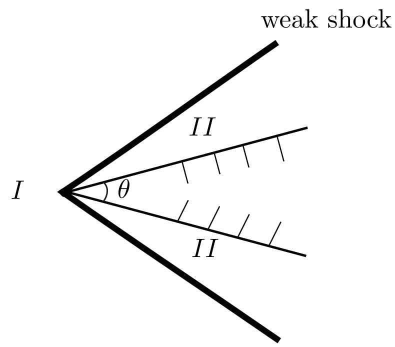

Figure 10: Supersonic upstream state.

travelling in unlimited space reaches the edge of a wall, you might obtain a definite solution.

von Neumann:

In that case you assume that the state at the time $t = 0$ *is given and you ask whether there is or is not a unique continuation of the solution at later times. The answer to this question in its full generality is not known; there seem to be a great many mathematical difficulties.*

Remark 4.4. The *great many mathematical difficulties* still exist today. However, it should be emphasized that analytical proof of uniqueness is one thing; and the plausibility of uniqueness from physical consideration is another thing. The first can be difficult. On the other hand, the latter should be raised first, both from physical and analytical point of view, and this is what these physicists have tried to do.

5. Weak Shock Reflection

Mathematically, one can expect an infinite number of shock waves occurring in the process of acceleration, making analytically difficult to construct the solutions. On the other hand, as discussed in Remark 4.3, von Karman points out that it is the *time-asymptotic* behavior that is observed and is therefore the relevant one for the study of the attached shocks. This suggests, instead of gradual acceleration, a wedge is *instantaneously accelerated* to a supersonic speed $\boldsymbol{u}_I$. This means that at the initial time, gas surrounding the flying object is set to be the supersonic upstream state, Figure 11. Such an initial value is trivially *self-similar*. Consider a 2-dimensional flying object around a wedge with angle 2α. The initial data is the upstream supersonic state

$$(\rho, \rho\boldsymbol{u})(x, y, 0) = (\rho_I, \rho_I \boldsymbol{u}_I),\ |\boldsymbol{u}_I| > c_I,\ (x, y) \in D. \tag{5.1}$$

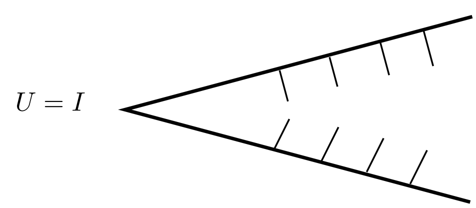

Figure 11: Initial condition

As we will see, the solution will be the same as the initial data I for $x < 0$ and, by symmetry, we will consider only the upper half plane. Thus the relevant region is

$$D \equiv \{(x, y) :\ x > 0,\ y > x \tan\alpha.\} \tag{5.2}$$

Consider the usual slip boundary condition for inviscid flows:

$$\begin{aligned} \boldsymbol{u}\cdot\boldsymbol{n} = 0,\ \text{for } y = x\tan\alpha, \quad &\text{the boundary of the wedge,} \\ \boldsymbol{n} = (\sin\alpha, \cos\alpha),\ &\text{the normal of the boundary.} \end{aligned} \tag{5.3}$$

We will consider the potential flow equation (2.19). Under the self-similar values and domain, the solution is also self-similar and satisfies (2.21). For

$x > x_0$ large, the effect of the tip of the wedge is not felt in finite time. Therefore, the solution is independent of the direction parallel to the solid boundary, and depends only on the direction normal to the solid boundary. Thus the problem is 1-dimensional and one needs only to solve the 1-dimensional Riemann problem for the time-dependent equation (2.19). The solution consists of a shock (I, IV) parallel to the solid boundary. On the other hand, the solution is self-similar and so satisfies the self-similar equation (2.21). We thus obtain the boundary value I at $x = 0$ and the boundary value of the shock (I, IV) at time $t = 1$, for sufficiently large x, for the equation (2.21). As $(x, y) = (\xi, \eta)$ for $t = 1$, the self-similar solution is the time dependent solution at time $t = 1$, and so the shock (I, IV) is located at a distance d from the solid, with d equal to its own speed. For $|\xi|^2 + |\eta|^2$ large, (2.21) is hyperbolic and so the shock (I, IV) may persists from infinity till the hyperbolicity of (2.21) terminates at the *pseudo sonic circle* pertaining to the state IV:

$$C_{IV} \equiv \{(\xi, \eta) : \; (\xi - u_{IV})^2 + (\eta - v_{IV})^2 = (c_{IV})^2.\}. \tag{5.4}$$

Around the tip $(x, y) = (\xi, \eta) = (0, 0)$, if the solution contains the weak shock reflection (I, II), as observed in the experiments, then the shock can persists till the pseudo-sonic circle C_{II} with respect to the state II:

$$C_{II} \equiv \{(\xi, \eta) : \; (\xi - u_{II})^2 + (\eta - v_{II})^2 = (c_{II})^2\}. \tag{5.5}$$

These solutions to the hyperbolic part of the self-similar equation (2.21) are unique, [15], [2]. However, these solutions may terminate before the pseudo-sonic circles. The problem of the global structure of the solution is considered in [8]. A numerical computation was performed to show that the shocks (I, II), (I, IV) persist till their respective speudo-sonic circles C_{II}, C_{IV}. These shocks are connected by a curved shock S, with the flow *pseudo-subsonic and* (2.21) *elliptic* between the pseudo-sonic circles C_{II}, C_{IV}, the curved shock S and the solid boundary. Figure 12. The following theorem in [8] says that such a solution can be analytically constructed.

Theorem 5.1. *There exists an angle α_0 such that for the wedge angle $\alpha < \alpha_0$, there exists a solution to the self-similar equation* (2.21) *with the initial data* (5.1)*, and boundary condition* (5.3) *in the region D of* (5.2)*. The solution consists of the weak shock (I, II) from the tip of the wedge till the* *pseudo-sonic circle* C_I*, the shock* (I, IV) *parallel to the solid boundary till the pseudo-sonic circle* C_{IV}*, and a curved shock* S *connecting the two shocks, and a pseudo-subsonic region between* C_{II}*,* C_{IV} *and* S*, Figure 12.*

As the solution is self-similar, the solution at later time, say $t = 2$, is the magnification of that described in the above theorem, Figure 13. Thus we have shown that the time-asymptotic solution for an accelerating wedge consists of a *weak shock reflection* attached to the tip of the wedge, Figure 5. This analytical study corresponds to the suggestions of von Karman.

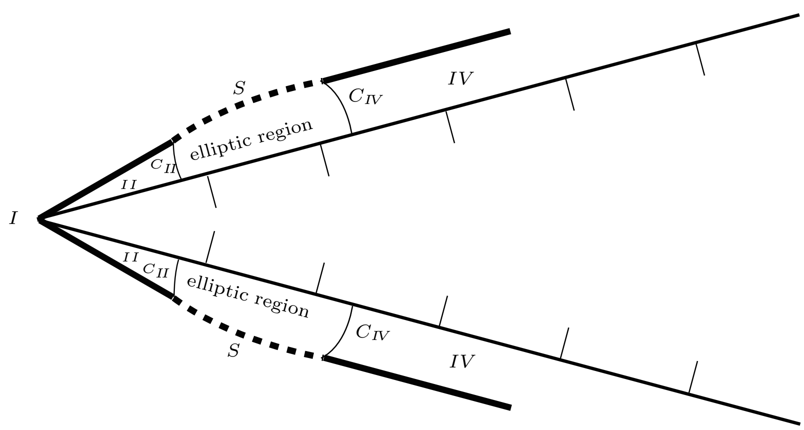

Figure 12: Time $t = 1$.

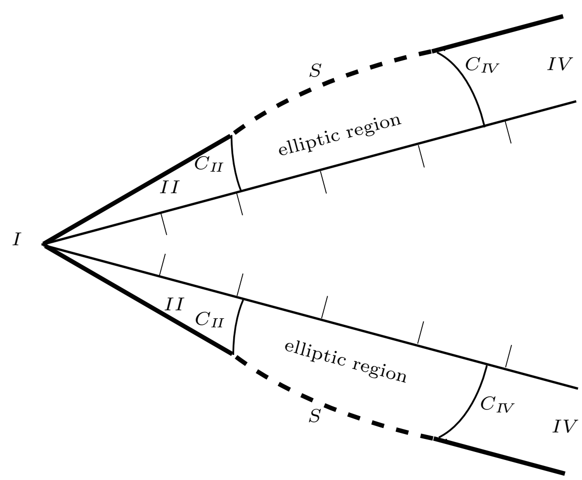

Figure 13: Time $t = 2$.

Remark 5.2. For the precise statement of Theorem 5.1, see [8]. It is proved using the Leray-Schauder degree theory. It is a *global method* of using the homotopy method, making use of the global estimate of the Ellipticity Principle, Theorem 3.2. Such a global consideration is an important new development in the theory of multi-dimensional gas flow.

6. Boundary Condition

Continuing the discussions, Heisenberg raises the basic point that the models of partial differential equations in the fluid dynamics may not be applicable in various physical situations.

Heisenberg:

I have one question in connection with these applications of the hydrodynamical equations. Should one assume from the beginning that these equations actually could be used to such a large extent? If we take the case of the gas expanding into a vacuum, the density at the front is so low that the mean free path becomes larger than the distance to the assumed front. Should one not start from the kinetic picture and say that at the front the molecules will sort themselves out according to their velocities?

Heisenberg:

Then the physical front would be formed by a selection of those molecules which had the highest velocities and did not suffer a collision for a long time. One should expect that there, especially, we have a velocity distribution different from the normal one, and therefore we should not apply the ordinary concepts like temperature and so on. I do not know how big the actual difference is, but I have tried to estimate it. One feels at least that there is a rather large region in which ordinary hydrodynamics cannot be applied, simply because the concepts of temperature and so on would be rather useless.

von Neumann:

Therefore, while it is certainly not rigorously true, don't you think it is sensible, first of all, to apply hydrodynamic theory, and get a solution? If you then discuss in what portions of the field the mean free path is small compared to the distances over which all essential changes occur (one of the most important portions is that where the distance from the boundary *is small), it is reasonable to assume that the hydrodynamical equations may at least be used in such regions. When one has to deal with the boundary regions, the Maxwell–Boltzmann theory should be called upon.*

von Neumann:

Now what I have to say is that if one accepts this, and if one estimates how large these extraordinary regions are, in the cases which are of interest in the present context, they turn out to be fairly small. Properly speaking, in the case of the Riemann expansion into vacuum, the region where you have to be careful is quite large but it involves very little substance and very little energy. Hence, in many cases, the correction of the hydrodynamical solution in that region need not be discussed.

Heisenberg:

I certainly agree chiefly with what you say. I only would like to observe that the failures of hydrodynamical solutions determine the boundary conditions. The boundary conditions react back on the solutions of the hydrodynamic equations, and since these boundary conditions cannot be determined from hydrodynamics and require a detailed study of molecular processes, the two things are interconnected. With you, I believe that on the whole we can talk about hydrodynamical equations and their solutions, but the selection of the solutions to be used depends on the boundary conditions and to this extent we get these non-hydrodynamical parts of the field into our problem.

von Neumann:

The boundary layer theory for a fluid of low viscosity certainly furnishes a monumental warning. The naive and yet prima facie seemingly reasonable procedure would be to apply the ordinary equations of the ideal fluid and then to expect that viscosity will somehow take care of itself in a narrow region along the wall. We have learned that this procedure may lead to great errors; a complete theory of the boundary layer may give you completely different conditions also for the flow in the bulk of the field. It is possible that the same discipline will be necessary for the boundary with a vacuum. All I would like to say now is that there is yet no evidence for this.

Remark 6.1. The question of finding the boundary conditions for the fluid dynamics equations from the consideration of the kinetic theory is an important one. It has been a very active field of study since the time of the above exchanges between Heisenberg and von Neumann. It is also a very rich field.

The boundary condition and the appropriate corresponding fluid dynamics equations depend on the *physical situation*. There are the classical *Hilbert and Chapman-Enskog expansions*. It is important to adjust the expansion for the particular physical situation that one considers. Thus the Hilbert expansion should be performed depending on the Knudsen number, the Mach number, the Reynold number, the degree of nonlinearity, the geometry of the solid boundary and other factors. In the case of flow near vacuum, for instance, the situation is different between the evaporation toward the vacuum from a sphere and from a cylinder. The study of fluid dynamics from the kinetic point of view has given rise to *modern fluid dynamics*. Interested readers are recommended the book by Sone, [19].

The companion article [18] to [17] discusses the interesting problem of the transition of regular to Mach reflections. These are self-similar solutions with initial data consisting of a shock at the tip of the ramp, Figure 14. For *sufficiently large* wedge angle, the solution at later time is a *regular reflection*, Figure 15. The construction of the basic wave pattern of the regular reflection solution has been considered for quite sometime and is finally done for the potential flow equation in [3]. It is interesting that the two papers [8] and [3] have proposed two different approaches for problems of similar analytical nature. The technique in [8] has also been applied to this problem of regular reflection, [6]. As the angle of the wedge decreases, more complicated wave patterns appear, including the *Mach reflection*, Figure 16. There is the question of the *transition criterion* of the termination of the regular reflection. There are the sonic, stability, geometric and other transition criteria, [4]. In view of the above discussions on the boundary behaviors, the transition criteria on the level of inviscid models would provide a good starting point for the real flows with the viscous effects, as studied from the kinetic theory. These are important topics for future research.

Acknowledgment

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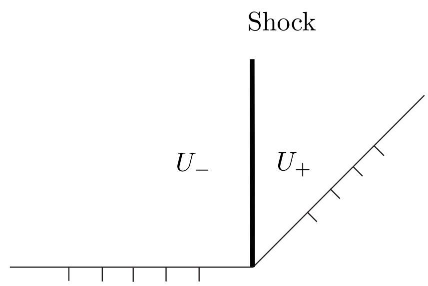

Figure 14: Initial data with shock

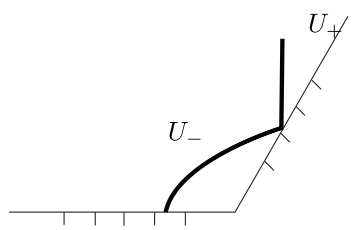

Figure 15: Regular reflection

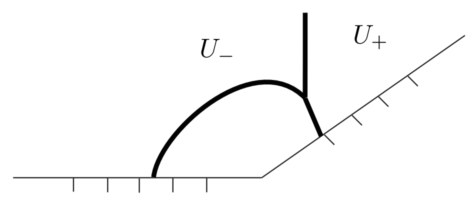

Figure 16: Mach reflection

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