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BI-HAMILTONIAN SYSTEMS OF MINIMAL POISSON W-ALGEBRAS

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Abstract

In this note, we discuss bi-Hamiltonian structures of Poisson W-algebras of minimal nilpotent elements and show that there always exist two kinds of bi-Hamiltonian pairs, which we call "KdV type" and "HD type" since they are generalizations of the KdV and the Harry-Dym hierarchies associated to the Virasoro Poisson vertex algebra.

1. Poisson Vertex Algebra and the Lenard Scheme

A Poisson vertex algebra (PVA) is a super-commutative associative algebra V which is a $\mathbf{C}[\partial]$-module, where ∂ is a derivation on V, equipped with a bilinear map

$$\{\ {}_\lambda\ \} : V \otimes V \longrightarrow V \otimes \mathbf{C}[\lambda]$$

satisfying the following conditions

(i) $\{\partial P_\lambda Q\} = -\lambda\{P_\lambda Q\}$

(ii) $\{P_\lambda \partial Q\} = (\partial+\lambda)\{P_\lambda Q\}$

(iii) $\{Q_\lambda P\} = -(-1)^{|P||Q|}{}_{\leftarrow}\{P_{-\partial-\lambda} Q\}$

(iv) $\{P_\lambda\{Q_\mu R\}\} - (-1)^{|P||Q|}\{Q_\mu\{P_\lambda R\}\} = \{\{P_\lambda Q\}_{\lambda+\mu} R\}$

(v) $\{P_\lambda QR\} = \{P_\lambda Q\}R + (-1)^{|P||Q|}Q\{P_\lambda R\}$

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for any $P, Q, R \in V$, where $|P|$ denotes the parity of P, namely $|P| = 0$ (resp. 1) if P is an even (resp. odd) element. A bilinear map $\{\ {}_\lambda\ \}$ satisfying the above conditions (i) $\sim$ (v) is called a Poisson λ-bracket on V. Following [1] and [2], we use notations ${}_{\leftarrow}\{\ {}_x\ \}$ and $\{\ {}_x\ \}_{\rightarrow}$ to indicate that x should be put to the left and to the right respectively in each terms of expression of $\{\ {}_x\ \}$.

For each $u, v \in V$, $\{u_\lambda v\}$ is a polynomial of λ with coefficients in V, so it is written in the form $\{u_\lambda v\} = \sum_{0 \le j \le N} P_j\, \lambda^j$ $(P_j \in V)$. Then we put

$$\{u_\partial v\}_{\rightarrow} := \{u_\lambda v\}_{\rightarrow}|_{\lambda=\partial} := \sum_{0 \le j \le N} P_j\, \partial^j$$

which is a differential operator on V.

Let u_1, u_2, $\cdots$ be a set of generators of V, and put $H_{i,j} := \{u_{j\,\partial} u_i\}_{\rightarrow}$. The matrix $H := \big(H_{i,j}\big)_{i,j=1,2,\cdots}$ is called the Hamiltonian. Also define the covariant differentiation

$$\frac{\delta f}{\delta u_i} := \sum_{p=0}^{\infty} (-1)^p\, \partial^p \Big(\frac{\partial f}{\partial u_i^{(p)}} \Big)$$

and denote the column vector $\begin{pmatrix} \frac{\delta f}{\delta u_1} \\ \frac{\delta f}{\delta u_2} \\ \vdots \end{pmatrix}$ by $\frac{\delta f}{\delta u}$, where $u_i^{(p)} := \partial^p u_i$.

Next, we consider the situation where two λ-brackets $\{\ {}_\lambda\ \}_K$ and $\{\ {}_\lambda\ \}_H$ are given. A pair of two Poisson λ-brackets $(\{\ {}_\lambda\ \}_K, \{\ {}_\lambda\ \}_H)$ is called a bi-Hamiltonian pair if $\alpha\{\ {}_\lambda\ \}_K + \beta\{\ {}_\lambda\ \}_H$ is a Poisson λ-bracket for any $\alpha, \beta \in \mathbf{C}$. Let H and K be the Hamiltonians corresponding to $\{\ {}_\lambda\ \}_H$ and $\{\ {}_\lambda\ \}_K$ respectively. A bi-Hamiltonian pair is called integrable if there exist sequences $\{\xi_n\}_{n \in \mathbf{Z}_{\ge 0}}$ and $\{h_n\}_{n \in \mathbf{Z}_{\ge 0}}$, called the Lenard scheme, consisting of linearly independent elements in V satisfying the conditions $K\xi_0 = 0$, $H\xi_n = K\xi_{n+1}$ and $\xi_n = \frac{\delta h_n}{\delta u}$ for all $n \in \mathbf{Z}_{\ge 0}$.

Then, regarding u_1, u_2, $\cdots$ as functions of (t, x), we consider the following equations for $u_j(t,x)$'s, called the n-th evolution equation ($n \in \mathbf{Z}_{\geq 0}$):

$$\begin{pmatrix} \frac{\partial u_1}{\partial t_n} \\ \frac{\partial u_2}{\partial t_n} \\ \vdots \end{pmatrix} = H\xi_n (= K\xi_{n+1}).$$

We note that $\int h_n dx$ ($n \in \mathbf{Z}_{\geq 0}$) are integral of motion for these equations.

Here we look at the simplest examples.

Example 1. (cf. [1]) The Virasoro PVA is the commutative associative algebra $V = \mathbf{C}[u, u', u'', \cdots]$ with the Virasoro Poisson λ-bracket

$$\{u_\lambda u\} = (\partial + 2\lambda)(\alpha u + \beta) + \lambda^3 c \qquad (\alpha, \beta, c \in \mathbf{C}),$$

which is a linear combination of three λ-brackets $\{u_\lambda u\}_1 := (\partial + 2\lambda)u$, $\{u_\lambda u\}_2 := \lambda$ and $\{u_\lambda u\}_3 := \lambda^3$. So this PVA produces two kinds of bi-Hamiltonian pairs such that $\deg K < \deg H$:

$$\begin{cases} \{\ {}_\lambda\ \}_K = \{\ {}_\lambda\ \}_1 \\ \{\ {}_\lambda\ \}_H = \{\ {}_\lambda\ \}_2 + \{\ {}_\lambda\ \}_3 \end{cases} \quad \text{and} \quad \begin{cases} \{\ {}_\lambda\ \}_K = \{\ {}_\lambda\ \}_2 \\ \{\ {}_\lambda\ \}_H = \{\ {}_\lambda\ \}_1 + \{\ {}_\lambda\ \}_3. \end{cases}$$

In each case, the Lenard scheme looks as follows:

$$\text{Case 1:} \begin{cases} \{u_\lambda u\}_K := \{u_\lambda u\}_2 = \lambda \\ \{u_\lambda u\}_H := \{u_\lambda u\}_1 + c\{u_\lambda u\}_3 = (\partial + 2\lambda)u + \lambda^3 c \end{cases}.$$

Their Hamiltonians are $K = \partial$ and $H = u' + 2u\partial + c\partial^3$.
Letting $\xi_0 := 1$, we have

$$\begin{aligned} \xi_0 &= 1 \\ \xi_1 &= u \\ \xi_2 &= cu'' + \frac{3}{2}u^2 \\ \xi_3 &= c^2 u^{(4)} + 5cuu'' + \frac{5}{2}cu'^2 + \frac{5}{2}u^3 \\ &\vdots \end{aligned}$$

and

$$\begin{aligned} h_0 &= u \\ h_1 &= \frac{1}{2}u^2 \\ h_2 &= -\frac{c}{2}\,u'^{\,2} + \frac{1}{2}\,u^3 \\ h_3 &= \frac{c^2}{2}\,u''^{\,2} - \frac{5}{2}\,cuu'^{\,2} + \frac{5}{8}\,u^4 \\ &\vdots \qquad . \end{aligned}$$

Then the 2nd evolution equation is

$$\frac{\partial u}{\partial t_1} = K\xi_2 = cu''' + 3uu' : \text{KdV equation.}$$

Case 2: $\begin{cases} \{u_\lambda u\}_K = \{u_\lambda u\}_1 = (\partial + 2\lambda)u \\ \{u_\lambda u\}_H = \alpha\{u_\lambda u\}_2 + c\{u_\lambda u\}_3 = \lambda\alpha + \lambda^3 c \end{cases}$

Then their Hamiltonians are $K = u' + 2u\partial$ and $H = \alpha\partial + c\partial^3$. Letting $\xi_0 := u^{-1/2}$ so that $K\xi_0 = 0$, we have

$$\begin{aligned} \xi_0 &= u^{-1/2} \\ \xi_1 &= \frac{\alpha}{4}\,u^{-3/2} - \frac{c}{4}\,u^{-5/2}u'' + \frac{5c}{16}\,u^{-7/2}u'^{\,2} \\ \xi_2 &= \frac{48}{512}\alpha^2 u^{-5/2} + \frac{\alpha c}{512}\Big\{-160u^{-7/2}u'' + 280u^{-9/2}u'^{\,2}\Big\} + \frac{c^2}{512}\Big\{-64u^{-7/2}u^{(4)} \\ &\quad +448u^{-9/2}u'u''' + 336u^{-9/2}u''^{\,2} - 1848u^{-11/2}u'^{\,2}u'' + 1155u^{-13/2}u'^{\,4}\Big\} \\ &\vdots \end{aligned}$$

and

$$\begin{aligned} h_0 &= 2u^{1/2} \\ h_1 &= -\frac{\alpha}{2}\,u^{-1/2} + \frac{c}{8}\,u^{-5/2}u'^{\,2} \\ h_2 &= \frac{1}{256}\cdot\frac{\sqrt{u}}{u^6}\,\Big\{-16\alpha^2u^4 + 40\alpha cu^2u'^{\,2} - 16c^2u^2u''^{\,2} + 35c^2u'^{\,4}\Big\} \\ &\vdots \qquad . \end{aligned}$$

Then the 1st evolution equation is the Harry-Dym equation:

$$\frac{\partial u}{\partial t_0} = H\xi_0 = \alpha\big(u^{-1/2}\big)' + c\big(u^{-1/2}\big)'''.$$

Example 2. Consider the $N = 1$ Poisson superconformal algebra (SCA), which is generated by an even element L and an odd element G with λ-brackets

	L	G
L	$(\partial + 2\lambda)L + \lambda^3 c$	$(\partial + \frac{3}{2}\lambda)G$
G	$(\frac{1}{2}\partial + \frac{3}{2}\lambda)G$	$L + 2\lambda^2 c$

Applying the deformation $L \longmapsto L + a$ $(a \in \mathbf{C})$ to the above λ-bracket, we get

	L	G
L	$(\partial + 2\lambda)L + 2a\lambda + \lambda^3 c$	$(\partial + \frac{3}{2}\lambda)G$
G	$(\frac{1}{2}\partial + \frac{3}{2}\lambda)G$	$L + a + 2\lambda^2 c$

Since a and c are arbitrary complex numbers, this λ-bracket is a linear combination of three λ-brackets:

$$\begin{aligned} \{\ {}_\lambda\ \}_1 &:= \text{ terms of } a^0c^0 \\ \{\ {}_\lambda\ \}_2 &:= \text{ terms of } a \\ \{\ {}_\lambda\ \}_3 &:= \text{ terms of } c. \end{aligned}$$

Then, just as in Example 1, this PVA produces two kinds of bi-Hamiltonian pairs such that $\deg K < \deg H$:

$$\begin{cases} \{\ {}_\lambda\ \}_K := \{\ {}_\lambda\ \}_1 \\ \{\ {}_\lambda\ \}_H := \{\ {}_\lambda\ \}_2 + \{\ {}_\lambda\ \}_3 \end{cases} \quad \text{and} \quad \begin{cases} \{\ {}_\lambda\ \}_K := \{\ {}_\lambda\ \}_2 \\ \{\ {}_\lambda\ \}_H := \{\ {}_\lambda\ \}_1 + \{\ {}_\lambda\ \}_3. \end{cases}$$

$$\text{Case 1}: \begin{cases} \{\ {}_\lambda\ \}_K := \{\ {}_\lambda\ \}_2 = \text{terms of } a \\ \{\ {}_\lambda\ \}_H := \{\ {}_\lambda\ \}_1 + c\{\ {}_\lambda\ \}_3 \end{cases}$$

In this case, their Hamiltonians are

$$K = \begin{pmatrix} 2\partial & 0 \\ 0 & 1 \end{pmatrix}$$

$$H = \begin{pmatrix} L' + 2L\partial + c\,\partial^3 & \frac{1}{2}\,G' + \frac{3}{2}\,G\partial \\ G' + \frac{3}{2}\,G\partial & L + 2c\,\partial^2 \end{pmatrix}$$

and the Lenard scheme is as follows:

$$\begin{aligned} \xi_0 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \xi_1 &= \begin{pmatrix} \frac{1}{2}\,L \\ G' \end{pmatrix} \\ \xi_2 &= \begin{pmatrix} \frac{3}{8}\,L^2 + \frac{c}{4}\,L'' + \frac{3}{4}\,GG' \\ \frac{3}{2}\,LG' + \frac{3}{4}\,L'G + 2c\,G''' \end{pmatrix} \\ &\vdots \end{aligned}$$

and

$$\begin{aligned} h_0 &= L \\ h_1 &= \frac{1}{4}\,L^2 + \frac{1}{2}\,GG' \\ h_2 &= \frac{1}{8}\,L^3 - \frac{c}{8}\,L'^{\,2} + \frac{3}{4}\,LGG' - c\,G'G'' \\ &\vdots \qquad . \end{aligned}$$

Then the 2nd evolution equation $\frac{\partial}{\partial t_1}\begin{pmatrix} L \\ G \end{pmatrix} = K\xi_2$ is the super KdV equation:

$$\begin{cases} \frac{\partial L}{\partial t_1} = \frac{3}{2}\,LL' + \frac{c}{2}\,L''' + \frac{3}{2}\,GG'' \\ \frac{\partial G}{\partial t_1} = \frac{3}{2}\,LG' + \frac{3}{4}\,L'G + 2c\,G'''. \end{cases}$$

Case 2 : $\begin{cases} \{\ \lambda\ \}_K := \{\ \lambda\ \}_1 = \text{terms of } a^0c^0 \\ \{\ \lambda\ \}_H := \{\ \lambda\ \}_3 = \text{terms of } c \end{cases}$

In this case, their Hamiltonians K and H are as follows:

$$K = \begin{pmatrix} L' + 2L\partial & \frac{1}{2}\,G' + \frac{3}{2}\,G\partial \\ G' + \frac{3}{2}\,G\partial & L \end{pmatrix}$$

$$H = \begin{pmatrix} \partial^3 & 0 \\ 0 & 2\partial^2 \end{pmatrix}$$

Taking

$$\xi_0 := \begin{pmatrix} L^{-1/2} + \frac{3}{4} L^{-5/2} G G' \\ -L^{-3/2} G' + \frac{3}{4} L^{-5/2} L' G \end{pmatrix}$$

so that $K\xi_0 = 0$, we have

$$\begin{aligned} h_0 &= 2u^{1/2} - \frac{1}{2} u^{-3/2} G G' \\ h_1 &= \frac{1}{8} L^{-5/2} L'^{\,2} + \frac{105}{32} L^{-9/2} L'^{\,2} G G' - \frac{3}{4} L^{-5/2} G G''' + \frac{1}{4} L^{-5/2} G' G'' \\ &\vdots \end{aligned}$$

Then the 1st evolution equation $\frac{\partial}{\partial t_0} \begin{pmatrix} L \\ G \end{pmatrix} = H\xi_0$ is the super Harry-Dym equation:

$$\begin{cases} \frac{\partial L}{\partial t_0} = \left(L^{-1/2} + \frac{3}{4} L^{-5/2} G G' \right)''' \\ \frac{\partial G}{\partial t_0} = 2\left(- L^{-3/2} G' + \frac{3}{4} L^{-5/2} L' G \right)''. \end{cases}$$

2. Poisson W-Algebras via Quantum Reduction

Let $\mathfrak{g}$ be a finite-dimensional simple Lie superalgebra with Cartan subalgebra $\mathfrak{h}$ and non-degenerate invariant super-symmetric even bilinear form $(|)$ and f be a nilpotent element in the even part of $\mathfrak{g}$. Then the Poisson W-algebra $W(\mathfrak{g}, f, k)$, where k is an arbitrary complex number called the level, is constructed by quantum reduction as follows.

Let Δ (resp. Δ_+) be the set of all non-zero (resp. positive) roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and put $\mathfrak{n}_\pm := \sum_{\alpha \in \pm\Delta_+} \mathfrak{g}_\alpha$. Let $\tilde{S}(\mathfrak{g}, k)$ denote the super-commutative associative algebra generated by all elements in $\mathfrak{g}$ and their derivatives, which is a PVA with the λ-bracket:

$$\{X_\lambda Y\} := [X, Y] + \lambda k (X|Y) \qquad (X, Y \in \mathfrak{g}).$$

Choose elements $x \in \mathfrak{h}$ and $e \in \mathfrak{n}_{+,even}$ such that $[x, e] = e$, $[x, f] =$

$-f$, $[e,f]=x$ and $\alpha(x)\geq 0$ for all $\alpha\in\Delta_+$, and put

$$\begin{aligned}
\mathfrak{g}^f &:= \{Y\in\mathfrak{g} | [f,Y]=0\} & \\
S_j &:= \{\alpha\in\Delta | \alpha(x)=j\} & (j\in\tfrac{1}{2}\mathbf{Z}), \\
\mathfrak{g}_j &:= \{Y\in\mathfrak{g} | [x,Y]=jY\} & (j\in\tfrac{1}{2}\mathbf{Z}), \\
S_+ &:= \bigcup_{j>0} S_j = \{\alpha\in\Delta \,|\, \alpha(x)>0\}. &
\end{aligned}$$

Introduce the ghost particles $\varphi_\alpha, \varphi_\alpha^* \,(\alpha\in S_+)$ and $\phi_\alpha \,(\alpha\in S_{1/2})$ such that φ_α and φ_α^* are bosons (resp. fermions) if and only if α is an odd (resp. even) root, and ϕ_α is a boson (resp. fermion) if and only if α is an even (resp. odd) root. Let Φ be the super-commutative associative algebra generated by these elements and their derivatives, which is a PVA with Poisson λ-bracket

$$\{\varphi_{\alpha\,\lambda}\,\varphi_\beta^*\} := \delta_{\alpha,\beta}, \quad \{\phi_{\alpha\,\lambda}\,\phi_\beta\} := (f|[e_\alpha,\, e_\beta]) \quad \text{and} \quad \text{all others} := 0,$$

where e_α's are root vectors which are arbitrarily chosen and fixed such that $(e_\alpha|e_{-\alpha})=1$ for $\alpha\in\Delta_+$.

Put $C(\mathfrak{g},f,k) := \tilde{S}(\mathfrak{g},k)\otimes\Phi$ and define the "charge" in it by

$$\begin{cases}
\text{charge}(u) := 0 & (u\in\tilde{S}(\mathfrak{g})) \\
\text{charge}(\varphi_\alpha) := 1 & (\alpha\in S_+) \\
\text{charge}(\varphi_\alpha^*) := -1 & (\alpha\in S_+) \\
\text{charge}(\phi_\alpha) := 0 & (\alpha\in S_{1/2}).
\end{cases}$$

Letting $C_j(\mathfrak{g},f,k)$ denote the space spanned by all elements of charge j in $C(\mathfrak{g},f,k)$, we have a direct sum decomposition $C(\mathfrak{g},f,k)=\bigoplus_{j\in\mathbf{Z}} C_j(\mathfrak{g},f,k)$.

Consider an element $d\in C_{-1}(\mathfrak{g},f,k)$ defined by

$$\begin{aligned}
d \;:=\; & \sum_{\alpha\in S_+} (-1)^{|\alpha|} e_\alpha \varphi_\alpha^* - \frac{1}{2}\sum_{\alpha\in S_+} (-1)^{|\alpha||\gamma|} c_{\alpha,\beta}^\gamma \varphi_\gamma\varphi_\alpha^*\varphi_\beta^* \\
& + \sum_{\alpha\in S_+} (f|e_\alpha)\varphi_\alpha^* + \sum_{\alpha\in S_{1/2}} \phi_\alpha\varphi_\alpha^*
\end{aligned}$$

where $c_{\alpha,\beta}^\gamma$ are structure constants: $[e_\alpha, e_\beta] = \sum_\gamma c_{\alpha,\beta}^\gamma e_\gamma$. This element d satisfies

$$\{d_\lambda d\} = 0. \tag{2.1}$$

Put

$$W(\mathfrak{g},f,k) := \{u \in C_0(\mathfrak{g},f,k) \mid \{d_\lambda u\}|_{\lambda=0} = 0\}. \tag{2.2}$$

Then it is easy to see that $W(\mathfrak{g},f,k)$ is a Poisson vertex subalgebra of $C(\mathfrak{g},f,k)$, called the Poisson W-algebra.

To describe the structure of this Poisson W-algebra $W(\mathfrak{g},f,k)$, we put

$$J^{(X)} := X + \sum_{\alpha,\beta\in S_+} (-1)^{|\alpha|} c^{e_\alpha}_{X,e_\beta} \varphi_\alpha \varphi^*_\beta \tag{2.3}$$

for each $X \in \mathfrak{g}$. Then, by the same arguments as in the proof of Theorem 4.1 in [8], we obtain

Theorem 2.1. (1) *For each* $v \in \mathfrak{g}^f_{-j} := \mathfrak{g}^f \cap \mathfrak{g}_{-j}$ $(j \in \frac{1}{2}\mathbf{Z}_{\geq 0})$, *there exists an element*

$$J^{\{v\}} = J^{(v)} + \cdots \qquad \in W(\mathfrak{g},f,k)$$

where "$\cdots$"-part is written by $J^{(u)}$ $(u \in \sum_{-j<i\leq 0} \mathfrak{g}_i)$ *and* ϕ_α $(\alpha \in S_{1/2})$.
(2) $W(\mathfrak{g},f,k)$ *is generated by* $J^{\{v\}}$ $(v \in \mathfrak{g}^f)$.

3. Minimal Poisson W-algebras

We consider the case where the highest root θ in Δ_+ is an even root, and normalize the inner product by $(\theta|\theta) = 2$, and put $f := e_{-\theta}$. Then the sl_2-triple (x,e,f) is

$$x = \frac{1}{2}\theta, \quad e = \frac{1}{2}e_\theta, \quad f = e_{-\theta} \tag{3.1}$$

and the ad(x)-gradation of $\mathfrak{g}$ is as follows:

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1 \tag{3.2a}$$

where

$$\mathfrak{g}_1 = \mathbf{C}e, \qquad \mathfrak{g}_0 = \mathbf{C}x \oplus \mathfrak{g}^f_0, \qquad \mathfrak{g}_{-1} = \mathbf{C}f. \tag{3.2b}$$

So the centralizer $\mathfrak{g}^f$ of f is a direct sum

$$\mathfrak{g}^f = \mathbf{C}f \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}^f_0 \tag{3.3}$$

and the Poisson W-algebra $W(\mathfrak{g}, e_{-\theta}, k)$ is generated by the following elements:

$$L := \frac{-1}{k} J^{\{f\}}, \quad J^{\{u\}} \, (=: G^{\{u\}}) \, (u \in \mathfrak{g}_{-1/2}), \quad J^{\{u\}} \, (u \in \mathfrak{g}_0^f). \tag{3.4}$$

Here and after we write $G^{\{u\}}$ in place of $J^{\{u\}}$ in particular for $u \in \mathfrak{g}_{-1/2}$.

In order to write down the λ-brackets of these generators explicitly, we take bases of $\mathfrak{g}_0^f$ and $\mathfrak{g}_{1/2}$ as follows:

- $\{v_i\}_{i\in\tilde{S}_0^f}, \{v^i\}_{i\in\tilde{S}_0^f}$: bases of $\mathfrak{g}_0^f$ such that $(v_i| \, v^j) = \delta_{i,j}$,
- $\{u_i\}_{i\in S_{1/2}}, \{u^i\}_{i\in S_{1/2}}$: bases of $\mathfrak{g}_{1/2}$ such that $[u_i, \, u^j] = \delta_{i,j} e_\theta$.

Then we have

Theorem 3.1. *Formulas for $J^{\{f\}}$ and $J^{\{u\}}$ $(u \in \mathfrak{g}_0^f)$ and $G^{\{u\}}$ $(u \in \mathfrak{g}_{-1/2})$ are explicitly given as follows:*

$$\begin{aligned}
J^{\{f\}} &= J^{(f)} + \sum_{j\in S_{1/2}} (-1)^{|u_j|} \phi^j J^{([f,\, u_j])} - \frac{1}{2} \sum_{i\in\tilde{S}_0^f} J^{(v^i)} J^{(v_i)} - k\partial J^{(x)} \\
J^{\{u\}} &= J^{(u)} - \frac{1}{2} \sum_{i,j\in S_{1/2}} (-1)^{|u_i|} c^{u_i}_{u,u_j} \phi_i \phi^j \\
G^{\{u\}} &= J^{(u)} + \sum_{i\in S_{1/2}} \phi_i J^{([u^i,\, u])} + k \sum_{i\in S_{1/2}} (u^i|u)\partial\phi_i \\
&\quad - \frac{(-1)^{|u|}}{3} \sum_{i,j,k\in S_{1/2}} \big(f| \, [u_k, \, [u_j, \, [u_i, \, u]]]\big)\phi^i\phi^j\phi^k
\end{aligned}$$

where $\phi_i := \phi_{u_i}$ *and* $\phi^i := \phi_{u^i}$ *namely* $\{\phi_{i\lambda}\phi^j\} = \delta_{i,j}$.

By a long calculation using these explicit formulas, we obtain their λ-brackets as follows:

Theorem 3.2. *The λ-brackets in $W(\mathfrak{g}, e_{-\theta}, k)$ are given as follows:*

(1) $\{L_\lambda L\} = (\partial + 2\lambda)L - \frac{k}{2}\lambda^3$

$\{L_\lambda J^{\{v\}}\} = (\partial + \lambda)J^{\{v\}}$

$\{L_\lambda G^{\{v\}}\} = \big(\partial + \frac{3}{2}\lambda\big)G^{\{v\}}$

(2) $\{J^{\{u\}}{}_\lambda J^{\{v\}}\} = (\partial + \lambda)J^{\{[u,v]\}} + \lambda k(u|v)$

$\{J^{\{u\}}{}_\lambda G^{\{v\}}\} = G^{\{[u,v]\}}$

(3) $\{G^{\{u\}}{}_\lambda G^{\{v\}}\} = -kg_{u,v}L + \frac{g_{u,v}}{2}\sum_{i\in\tilde{S}_0^f} J^{\{v^i\}}J^{\{v_i\}}$

$$+\sum_{i\in S_{1/2}} J^{\{[u,u^i]^\natural\}}J^{\{[u_i,v]^\natural\}} + 2k(\partial+2\lambda)J^{\{[[e,u],v]^\natural\}} + \lambda^2k^2g_{u,v}$$

where $g_{u,v} := (e_\theta|[u,v]) \quad (u,v\in\mathfrak{g}_{-1/2})$ *and*

$$\natural : \mathfrak{g}_0 = \mathbf{C}x \oplus \mathfrak{g}_0^f \longrightarrow \mathfrak{g}_0^f : \textit{orthogonal projection.}$$

Remark 1. (1) Quadratic terms in $\{G^{\{u\}}{}_\lambda G^{\{v\}}\}$ do not appear in the following cases:

$\mathfrak{g}$	$W(\mathfrak{g}, e_{-\theta}, k)$
$sl(2,\mathbf{C})$	Virasoro algebra
$osp(1\|2)$	N=1 SCA
$sl(2\|1)$	N=2 SCA
$sl(2\|2)$/center	N=4 SCA

(2) Quadratic terms in $\{G^{\{u\}}{}_\lambda G^{\{v\}}\}$ can be removed by adding suitable free fermions in the following cases:

$\mathfrak{g}$	$W(\mathfrak{g}, e_{-\theta}, k)$
$osp(3\|2)$	N=3 SCA
$D(2,1;\alpha)$	big N=4 SCA

In the above cases, bi-Hamiltonian pairs can be constructed in a similar way as in the case of N=1 SCA. In all other cases, we can do the following “linearlization of parameters” to construct bi-Hamiltonian pairs.

For this sake, we introduce the super-commutative associative algebra F generated by particles $\{\sigma_{-\alpha}\}_{\alpha\in S_{1/2}}$ and their derivatives such that $\sigma_{-\alpha}$ is a boson (resp. fermion) if and only if α is an even (resp. odd) root, with λ-brackets

$$\{\sigma_{-\alpha}\,{}_\lambda\sigma_{-\beta}\} := (e_\theta|[e_{-\alpha}, e_{-\beta}]) = g_{e_{-\alpha},e_{-\beta}}. \tag{3.5}$$

Let $\widetilde{W}(\mathfrak{g}, e_{-\theta}, k)$ denote the Poisson vertex algebra generated by $W(\mathfrak{g}, e_{-\theta}, k)$ and F. Then, by a suitable choice of generators of this PVA $\widetilde{W}(\mathfrak{g}, e_{-\theta}, k)$, one can construct a bi-Hamiltonian pair, as is shown below.

First, to construct a bi-Hamiltonian pair of KdV type, we choose and fix an element $h_0 \in \mathfrak{h}^f$ such that $\alpha(h_0) \neq 0$ for all $\alpha \in S_0$, and put

$$\tilde{J}^{\{u\}} := J^{\{u\}} - (h_0|u)t \qquad (u \in \mathfrak{g}_0^f) \tag{3.6a}$$

$$\tilde{G}^{\{e_{-\alpha}\}} := G^{\{e_{-\alpha}\}} + ta_\alpha\sigma_{-\alpha} \qquad (\alpha \in S_{1/2}) \tag{3.6b}$$

where t is an arbitrary complex number and

$$a_\alpha := \sqrt{(-1)^{|\alpha|}(\alpha|h_0)^2 - \frac{(h_0|h_0)}{2}}. \tag{3.7}$$

Then by Theorem 3.2, we have the following

Theorem 3.3. *Take generators L, $\tilde{J}^{\{u\}}$ $(u \in \mathfrak{g}_0^f)$, $\tilde{G}^{\{u\}}$ $(u \in \mathfrak{g}_{-1/2})$ and $\sigma_{-\alpha}$'s for $\widetilde{W}(\mathfrak{g}, e_{-\theta}, k)$. Then their Poisson λ-brackets are as follows:*

(1) $\{L_\lambda L\} = (\partial + 2\lambda)L - \frac{k}{2}\lambda^3$

$\{L_\lambda \tilde{J}^{\{v\}}\} = (\partial + \lambda)\tilde{J}^{\{v\}} + t\lambda(h_0|u)$

$\{L_\lambda \tilde{G}^{\{e_{-\alpha}\}}\} = \big(\partial + \frac{3}{2}\lambda\big)(\tilde{G}^{\{e_{-\alpha}\}} - ta_\alpha\sigma_{-\alpha}) \qquad (\alpha \in S_{1/2})$

(2) $\{\tilde{J}^{\{u\}}{}_\lambda \tilde{J}^{\{v\}}\} = (\partial + \lambda)\tilde{J}^{\{[u,v]\}} + t\lambda(h_0|[u,v]) + \lambda k(u|v)$

$\{\tilde{J}^{\{u\}}{}_\lambda \tilde{G}^{\{v\}}\} = \tilde{G}^{\{[u,v]\}} - t\sum_{\gamma \in S_{1/2}} c_{u,v}^{e_{-\gamma}} a_\gamma\sigma_{-\gamma}$

(3) $\{\tilde{G}^{\{u\}}{}_\lambda \tilde{G}^{\{v\}}\} = -kg_{u,v}L + \frac{g_{u,v}}{2}\sum_{i \in \tilde{S}_0^f} \tilde{J}^{\{v^i\}}\tilde{J}^{\{v_i\}} + tg_{u,v}\tilde{J}^{\{h_0\}}$

$$+ \sum_{i \in S_{1/2}} \tilde{J}^{\{[u, u^i]^\natural\}}\tilde{J}^{\{[u_i, v]^\natural\}} - 2t\Big(\tilde{J}^{\{[[h_0,u],[e,v]]^\natural\}} + \tilde{J}^{\{[[e,u],[h_0,v]]^\natural\}}\Big)$$

$$+ 2k(\partial + 2\lambda)\tilde{J}^{\{[[e,u],v]^\natural\}} + 4tk\lambda(h_0|[[e,u],v]) + \lambda^2k^2g_{u,v}$$

(4) $\{L_\lambda\sigma_{-\beta}\} = \{\tilde{J}^{\{u\}}{}_\lambda\sigma_{-\beta}\} = 0$

$\{\tilde{G}^{\{e_{-\alpha}\}}{}_\lambda\sigma_{-\beta}\} = -ta_\alpha g_{e_{-\alpha},e_{-\beta}}$

$\{\sigma_{-\alpha\lambda}\sigma_{-\beta}\} = g_{e_{-\alpha},e_{-\beta}}$.

Since this is a Poisson λ-bracket for any $t \in \mathbf{C}$, we get a bi-Hamiltonian pair $(\{\ {}_\lambda\ \}_K, \{\ {}_\lambda\ \}_H)$ by putting

$$\{\ {}_\lambda\ \}_H := t^0\text{-terms} \quad \text{and} \quad \{\ {}_\lambda\ \}_K := t^1\text{-terms}.$$

Namely we have

Corollary 3.4. *The following pair* $(\{\ {}_\lambda\ \}_K, \{\ {}_\lambda\ \}_H)$ *is a bi-Hamiltonian pair:*

(1)
$$\{L_\lambda L\}_H := (\partial + 2\lambda)L - \tfrac{k}{2}\lambda^3\,,$$
$$\{L_\lambda J^{\{v\}}\}_H := (\partial + \lambda)J^{\{v\}}\,,$$
$$\{L_\lambda G^{\{v\}}\}_H := \big(\partial + \tfrac{3}{2}\lambda\big)G^{\{v\}}\,,$$
$$\{J^{\{u\}}{}_\lambda J^{\{v\}}\}_H := (\partial + \lambda)J^{\{[u,v]\}} + \lambda k(u|v)\,,$$
$$\{J^{\{u\}}{}_\lambda G^{\{v\}}\}_H := G^{\{[u,v]\}}\,,$$
$$\{G^{\{u\}}{}_\lambda G^{\{v\}}\}_H := -kg_{u,v}L + \tfrac{g_{u,v}}{2}\sum_{i \in \tilde{S}_0^f} J^{\{v^i\}}J^{\{v_i\}}$$
$$+ \sum_{i \in S_{1/2}} J^{\{[u,\,u^i]^\natural\}}J^{\{[u_i,\,v]^\natural\}} + 2k(\partial + 2\lambda)J^{\{[[e,u],v]^\natural\}} + \lambda^2k^2g_{u,v}\,,$$
$$\{L_\lambda\sigma_{-\beta}\}_H := \{J^{\{u\}}{}_\lambda\sigma_{-\beta}\}_H := \{G^{\{u\}}{}_\lambda\sigma_{-\beta}\}_H := 0\,,$$
$$\{\sigma_{-\alpha\lambda}\sigma_{-\beta}\}_H := g_{e_{-\alpha},e_{-\beta}}.$$

(2)
$$\{L_\lambda L\}_K := 0$$
$$\{L_\lambda J^{\{v\}}\}_K := \lambda(h_0|u)$$
$$\{L_\lambda G^{\{e_{-\alpha}\}}\}_K := -a_\alpha(\partial + \tfrac{3}{2}\lambda)\sigma_{-\alpha} \qquad (\alpha \in S_{1/2})$$
$$\{J^{\{u\}}{}_\lambda J^{\{v\}}\}_K := \lambda(h_0|[u,v])$$
$$\{J^{\{u\}}{}_\lambda G^{\{v\}}\}_K := -\textstyle\sum_{\gamma \in S_{1/2}} c_{u,v}^{e_{-\gamma}} a_\gamma\sigma_{-\gamma}$$
$$\{G^{\{u\}}{}_\lambda G^{\{v\}}\}_K := g_{u,v}\tilde{J}^{\{h_0\}} - 2\big(J^{\{[[h_0,u],\,[e,v]]^\natural\}} + J^{\{[[e,u],\,[h_0,v]]^\natural\}}\big)$$
$$\{L_\lambda\sigma_{-\beta}\}_K := \{J^{\{u\}}{}_\lambda\sigma_{-\beta}\}_K := \{\sigma_{-\alpha\lambda}\sigma_{-\beta}\}_K = 0$$
$$\{G^{\{e_{-\alpha}\}}{}_\lambda\sigma_{-\beta}\}_K := -a_\alpha g_{e_{-\alpha},e_{-\beta}}\ .$$

Next, to construct bi-Hamiltonian pair of HD-type, we put

$$\hat{G}^{\{e_{-\alpha}\}} := G^{\{e_{-\alpha}\}} + k\,\partial\sigma_{-\alpha} \qquad (\alpha \in S_{1/2}) \tag{3.8}$$

and take the generators

$$L, \quad J^{\{u\}}(u \in \mathfrak{g}_0^f), \quad \hat{G}^{\{u\}}(u \in \mathfrak{g}_{-1/2}), \quad \sigma_{-\alpha}(\alpha \in S_{1/2}) \tag{3.9}$$

for $\widetilde{W}(\mathfrak{g}, e_{-\theta}, k)$. Then, by Theorem 3.2 and the formula (3.5), we see that the Poisson λ-brackets of these elements (3.9) consist only of terms of k^0 and k^1 and are given by the following:

Theorem 3.5. (1) *The Poisson λ-brackets of the elements in* (3.9) *is given by*

$$\{\ {}_\lambda\ \} = \{\ {}_\lambda\ \}_K^{\mathrm{HD}} + k\{\ {}_\lambda\ \}_H^{\mathrm{HD}}$$

where $\{\ {}_\lambda\ \}_K^{\mathrm{HD}}$ and $\{\ {}_\lambda\ \}_H^{\mathrm{HD}}$ are defined as follows:

(i) $\{L_\lambda L\}_K^{\mathrm{HD}} := (\partial + 2\lambda)L$

$\{L_\lambda J^{\{v\}}\}_K^{\mathrm{HD}} := (\partial + \lambda)J^{\{v\}}$

$\{L_\lambda \hat{G}^{\{v\}}\}_K^{\mathrm{HD}} := \big(\partial + \frac{3}{2}\lambda\big)\hat{G}^{\{v\}}$

$\{J^{\{u\}}{}_\lambda J^{\{v\}}\}_K^{\mathrm{HD}} := (\partial + \lambda)J^{\{[u,v]\}}$

$\{J^{\{u\}}{}_\lambda \hat{G}^{\{v\}}\}_K^{\mathrm{HD}} := \hat{G}^{\{[u,v]\}}$

$\{\hat{G}^{\{u\}}{}_\lambda \hat{G}^{\{v\}}\}_K^{\mathrm{HD}} := \frac{g_{u,v}}{2}\sum_{i\in\tilde{S}_0^f} J^{\{v^i\}}J^{\{v_i\}} + \sum_{i\in S_{1/2}} J^{\{[u,u^i]^\natural\}}J^{\{[u_i,v]^\natural\}}$

$\{L_\lambda \sigma_{-\beta}\}_K^{\mathrm{HD}} := \{J^{\{u\}}{}_\lambda \sigma_{-\beta}\}_K^{\mathrm{HD}} := \{\hat{G}^{\{u\}}{}_\lambda \sigma_{-\beta}\}_K^{\mathrm{HD}} := 0$

$\{\sigma_{-\alpha\lambda}\sigma_{-\beta}\}_K^{\mathrm{HD}} := g_{e_{-\alpha},e_{-\beta}}$

(ii) $\{L_\lambda L\}_H^{\mathrm{HD}} := -\frac{1}{2}\lambda^3$

$\{L_\lambda J^{\{v\}}\}_H^{\mathrm{HD}} := 0$

$\{L_\lambda \hat{G}^{\{e_{-\alpha}\}}\}_H^{\mathrm{HD}} := -\big(\partial + \frac{3}{2}\lambda\big)\partial\sigma_{-\alpha}$

$\{J^{\{u\}}{}_\lambda J^{\{v\}}\}_H^{\mathrm{HD}} := \lambda(u|v)$

$\{J^{\{u\}}{}_\lambda \hat{G}^{\{v\}}\}_H^{\mathrm{HD}} := -\sum_{\gamma\in S_{1/2}} c_{u,v}^{e_{-\gamma}}\,\partial\sigma_{-\gamma}$

$\{\hat{G}^{\{u\}}{}_\lambda \hat{G}^{\{v\}}\}_H^{\mathrm{HD}} := -g_{u,v}L + 2(\partial + 2\lambda)J^{\{[[e,u],v]^\natural\}}$

$\{L_\lambda \sigma_{-\beta}\}_H^{\mathrm{HD}} := \{J^{\{u\}}{}_\lambda \sigma_{-\beta}\}_H^{\mathrm{HD}} := \{\sigma_{-\alpha\lambda}\sigma_{-\beta}\}_H^{\mathrm{HD}} := 0$

$\{\hat{G}^{\{e_{-\alpha}\}}{}_\lambda \sigma_{-\beta}\}_H^{\mathrm{HD}} := -\lambda g_{e_{-\alpha},e_{-\beta}}$.

(2) *The pair $(\{\ {}_\lambda\ \}_K^{\mathrm{HD}}, \{\ {}_\lambda\ \}_H^{\mathrm{HD}})$ is a bi-Hamiltonian pair.*

Conjecture 1. *Bi-Hamiltonian pairs in Corollary* 3.4 *and Theorem* 3.5 *are integrable.*

4. The Example of $W(sl_3, e_{-\theta}, k)$

For $\mathfrak{g} = sl(3, \mathbf{C})$, take a basis

$$\begin{cases} e_{\alpha_1} := E_{1,2}, & e_{\alpha_2} := E_{2,3}, & e_{\alpha_1+\alpha_2} := E_{1,3}, \\ e_{-\alpha_1} := E_{2,1}, & e_{-\alpha_2} := E_{3,2}, & e_{-\alpha_1-\alpha_2} := E_{3,1}, \\ \alpha_1 := E_{1,1} - E_{2,2}, & \alpha_2 := E_{2,2} - E_{3,3}, & \end{cases}$$

and an sl_2-triple

$$x := \frac{1}{2}(\alpha_1 + \alpha_2), \quad e := \frac{1}{2} e_{\alpha_1+\alpha_2}, \quad f := e_{-\theta} = e_{-\alpha_1-\alpha_2},$$

where $E_{i,j}$ is the 3×3-matrix which has 1 in the (i,j)-entry and 0 everywhere else. Then

$$\begin{cases} \mathfrak{g}_0^f = \mathbf{C}\cdot(\alpha_1 - \alpha_2), \\ \mathfrak{g}_{-1/2} = \mathbf{C}\, e_{-\alpha_1} \oplus \mathbf{C}\, e_{-\alpha_2}, \\ \mathfrak{g}_{-1} = \mathbf{C}\, e_{-\alpha_1-\alpha_2} = \mathbf{C}\, f. \end{cases}$$

So $W(sl_3, e_{-\theta}, k)$ is generated by

$$\begin{cases} L := \frac{-1}{k} J^{\{f\}} \\ J := J^{\{\alpha_1-\alpha_2\}} \\ G^{\{f_i\}} := G^{\{e_{-\alpha_i}\}} \qquad (i = 1, 2) \end{cases}$$

with λ-brackets

	L	J	$G^{\{f_1\}}$	$G^{\{f_2\}}$
L	$(\partial + 2\lambda)L - \frac{\lambda^3}{2}k$	$(\partial+\lambda)J$	$(\partial + \frac{3}{2}\lambda)G^{\{f_1\}}$	$(\partial + \frac{3}{2}\lambda)G^{\{f_2\}}$
J	λJ	$6\lambda k$	$-3G^{\{f_1\}}$	$3G^{\{f_2\}}$
$G^{\{f_1\}}$	$(\frac{\partial}{2} + \frac{3}{2}\lambda)G^{\{f_1\}}$	$3G^{\{f_1\}}$	0	$kL - \frac{1}{3}J^2 + \frac{k}{2}(\partial + 2\lambda)J - \lambda^2 k^2$
$G^{\{f_2\}}$	$(\frac{\partial}{2} + \frac{3}{2}\lambda)G^{\{f_2\}}$	$-3G^{\{f_2\}}$	$-kL + \frac{1}{3}J^2 + \frac{k}{2}(\partial + 2\lambda)J + \lambda^2 k^2$	0

In order to construct bi-Hamiltonian pairs, we introduce bosonic elements σ, τ such that

$$\{\sigma_\lambda \tau\} = 1, \qquad \{\sigma_\lambda \sigma\} = \{\tau_\lambda \tau\} = 0$$

and consider the Poisson vertex algebra $\widetilde{W}(sl_3, e_{-\theta}, k)$ generated by $W(sl_3, e_{-\theta}, k)$ and σ and τ.

Case 1: (bi-Hamiltonian pair of KdV-type):

We put

$$\begin{cases} \tilde{J} & := J - a \\ G_1 & := G^{\{f_1\}} + \frac{a}{3}\sigma \\ G_2 & := G^{\{f_2\}} + a\tau \end{cases} \qquad (a \in \mathbf{C}).$$

Then the λ-brackets of these elements are as follows:

	L	$\tilde{J}$	G_1	G_2	σ	τ
L	$(\partial + 2\lambda)L - \frac{\lambda^3}{2}k$	$(\partial + \lambda)\tilde{J} + \lambda a$	$(\partial + \frac{3}{2}\lambda)G_1 - \frac{a}{3}(\partial + \frac{3}{2}\lambda)\sigma$	$(\partial + \frac{3}{2}\lambda)G_2 - a(\partial + \frac{3}{2}\lambda)\tau$	0	0
J	$\lambda J + \lambda a$	$6\lambda k$	$-3G_1 + a\sigma$	$3G_2 - 3a\tau$	0	0
G_1	$(\frac{\partial}{2} + \frac{3}{2}\lambda)G_1 - \frac{a}{3}(\frac{\partial}{2} + \frac{3}{2}\lambda)\sigma$	$3G_1 - a\sigma$	0	$kL - \frac{1}{3}\tilde{J}^2 - \frac{2}{3}a\tilde{J} + \frac{k}{2}(\partial + 2\lambda)\tilde{J} + \lambda ak - \lambda^2 k^2$	0	$\frac{a}{3}$
G_2	$(\frac{\partial}{2} + \frac{3}{2}\lambda)G_2 - a(\frac{\partial}{2} + \frac{3}{2}\lambda)\tau$	$-3G_2 + 3a\tau$	$-kL + \frac{1}{3}\tilde{J}^2 + \frac{2}{3}a\tilde{J} + \frac{k}{2}(\partial + 2\lambda)\tilde{J} + \lambda ak + \lambda^2 k^2$	0	$-a$	0
σ	0	0	0	a	0	1
τ	0	0	$-\frac{a}{3}$	0	-1	0

Putting

$$\{\ {}_\lambda\ \}_K := a^0\text{-terms} \quad \text{and} \quad \{\ {}_\lambda\ \}_H := a^1\text{-terms},$$

we get a bi-Hamiltonian pair. By simply using J and c for $\tilde{J}$ and k, their

Hamiltonians K and H are as follows:

$$K = \begin{pmatrix} 0 & \partial & -\frac{1}{6}\sigma' - \frac{1}{2}\sigma\partial & -\frac{1}{2}\tau' - \frac{3}{2}\tau\partial & 0 & 0 \\ \partial & 0 & -\sigma & 3\tau & 0 & 0 \\ -\frac{1}{3}\sigma' - \frac{1}{2}\sigma\partial & \sigma & 0 & \frac{2}{3}J + c\partial & 0 & \frac{-1}{3} \\ -\tau' - \frac{3}{2}\tau\partial & -3\tau & -\frac{2}{3}J + c\partial & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \end{pmatrix}$$

$$H = \begin{pmatrix} \begin{pmatrix} L' + 2L\partial \\ -\frac{c}{2}\partial^3 \end{pmatrix} & J\partial & \frac{1}{2}G_1' + \frac{3}{2}G_1\partial & \frac{1}{2}G_2' + \frac{3}{2}G_2\partial & 0 & 0 \\ J' + J\partial & 6c\partial & 3G_1 & -3G_2 & 0 & 0 \\ G_1' + \frac{3}{2}G_1\partial & -3G_1 & 0 & \begin{pmatrix} -cL + \frac{1}{3}J^2 \\ +\frac{c}{2}(J' + 2J\partial) \\ +c^2\partial^2 \end{pmatrix} & 0 & 0 \\ G_2' + \frac{3}{2}G_2\partial & 3G_2 & \begin{pmatrix} cL - \frac{1}{3}J^2 \\ +\frac{c}{2}(J' + 2J\partial) \\ -c^2\partial^2 \end{pmatrix} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

To calculate the Lenard scheme for this bi-Hamiltonian pair, we take

$$\xi_0 := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \tau' \\ -\sigma' \end{pmatrix} \qquad \text{so that} \qquad K\xi_0 = 0.$$

Then we have

$$\xi_1 = \begin{pmatrix} J+3\sigma\tau \\ L+\frac{3}{2}(\sigma\tau'-\sigma'\tau) \\ 3\tau' \\ -\sigma' \\ G_2'+3L\tau+3J\tau'+\frac{3}{2}J'\tau+12\sigma\tau\tau'-3c\tau'' \\ 3\big(-G_1'+L\sigma-J\sigma'-\frac{1}{2}J'\sigma-4\sigma\sigma'\tau-c\sigma''\big) \end{pmatrix}$$

and

$$\begin{aligned} h_0 &= L+\sigma\tau', \\ h_1 &= L(J+3\sigma\tau)+\frac{3}{2}J(\sigma\tau'-\sigma'\tau)+3G_1\tau'-G_2\sigma'+6\sigma^2\tau\tau'-3c\sigma\tau''. \end{aligned}$$

The 1st evolution equation $\frac{\partial u}{\partial t_0}=H\xi_0$ is

$$\frac{\partial}{\partial t_0}\begin{pmatrix} L \\ J \\ G_1 \\ G_2 \\ \sigma \\ \tau \end{pmatrix} = \begin{pmatrix} L' \\ J' \\ G_1' \\ G_2' \\ \sigma' \\ \tau' \end{pmatrix}$$

and the 2nd evolution equation $\frac{\partial u}{\partial t_1}=H\xi_1$ is as follows:

$$\begin{aligned} \frac{\partial L}{\partial t_1} &= 2(LJ)'-\frac{c}{2}J'''+3L'\sigma\tau+6L(\sigma\tau)'+\frac{3}{2}J(\sigma\tau''-\sigma''\tau) \\ &\quad +\frac{3}{2}(G_1'\tau'+3G_1\tau'')-\frac{1}{2}(G_2'\sigma'+3G_2\sigma'')-\frac{3}{2}c(\sigma\tau)''' \\ \frac{\partial J}{\partial t_1} &= 6cL'+2JJ'+3(J\sigma\tau)'+9G_1\tau'+3G_2\sigma'+9c(\sigma\tau''-\sigma''\tau) \\ \frac{\partial G_1}{\partial t_1} &= -3LG_1+JG_1'+\frac{3}{2}J'G_1+3G_1'\sigma\tau+9G_1\sigma'\tau \\ &\quad +\Big(cL-\frac{1}{3}J^2-\frac{c}{2}J'\Big)\sigma'-cJ\sigma''-c^2\sigma''' \\ \frac{\partial G_2}{\partial t_1} &= 3LG_2+JG_2'+\frac{3}{2}J'G_2+3G_2'\sigma\tau+9G_2\sigma\tau' \end{aligned}$$

$$
\begin{aligned}
&+\Big(3cL - J^2 + \frac{3}{2}cJ'\Big)\tau' + 3cJ\tau'' - 3c^2\tau''' \\
\frac{\partial\sigma}{\partial t_1} &= 3\Big(G_1' - L\sigma + J\sigma' + \frac{1}{2}J'\sigma + 4\sigma\sigma'\tau + c\sigma''\Big) \\
\frac{\partial\tau}{\partial t_1} &= G_2' + 3\Big(L\tau + J\tau' + \frac{1}{2}J'\tau + 4\sigma\tau\tau' - c\tau''\Big). \qquad (4.1)
\end{aligned}
$$

Case 2: (bi-Hamiltonian pair of HD-type):

We put

$$G_1 := G^{\{f_1\}} + k\,\partial\sigma\,, \qquad G_2 := G^{\{f_2\}} + k\,\partial\tau.$$

Then the λ-brackets for these elements are as follows:

	L	J	G_1	G_2	σ	τ
L	$(\partial+2\lambda)L$ $-\frac{\lambda^3}{2}k$	$(\partial+\lambda)J$	$(\partial+\frac{3}{2}\lambda)G_1$ $-k(\partial+\frac{3}{2}\lambda)\partial\sigma$	$(\partial+\frac{3}{2}\lambda)G_2$ $-k(\partial+\frac{3}{2}\lambda)\partial\tau$	0	0
J	λJ	$6\lambda k$	$-3G_1+3k\partial\sigma$	$3G_2-3k\partial\tau$	0	0
G_1	$(\frac{\partial}{2}+\frac{3}{2}\lambda)G_1$ $-k(\frac{\partial}{2}+\frac{3}{2}\lambda)\partial\sigma$	$3G_1-3k\partial\sigma$	0	$kL-\frac{1}{3}J^2$ $+\frac{k}{2}(\partial+2\lambda)J$	0	$-\lambda k$
G_2	$(\frac{\partial}{2}+\frac{3}{2}\lambda)G_2$ $-k(\frac{\partial}{2}+\frac{3}{2}\lambda)\partial\tau$	$-3G_2+3k\partial\tau$	$-kL+\frac{1}{3}J^2$ $+\frac{k}{2}(\partial+2\lambda)J$	0	λk	0
σ	0	0	0	λk	0	1
τ	0	0	$-\lambda k$	0	-1	0

Putting $\{\ {}_\lambda\ \}_K := k^0$-terms and $\{\ {}_\lambda\ \}_H := k^1$-terms, we get a bi-Hamiltonian pair, whose Hamiltonians K and H are as follows:

$$
K = \begin{pmatrix}
L'+2L\partial & J\partial & \frac{1}{2}G_1'+\frac{3}{2}G_1\partial & \frac{1}{2}G_2'+\frac{3}{2}G_2\partial & 0 & 0 \\
J'+J\partial & 0 & 3G_1 & -3G_2 & 0 & 0 \\
G_1'+\frac{3}{2}G_1\partial & -3G_1 & 0 & \frac{1}{3}J^2 & 0 & 0 \\
G_2'+\frac{3}{2}G_2\partial & 3G_2 & -\frac{1}{3}J^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
$$

$$H = \begin{pmatrix} \frac{-1}{2}\,\partial^3 & 0 & \frac{-1}{2}\sigma'' - \frac{3}{2}\sigma'\partial & \frac{-1}{2}\tau'' - \frac{3}{2}\tau'\partial & 0 & 0 \\ 0 & 6\,\partial & -3\,\sigma' & 3\,\tau' & 0 & 0 \\ -\sigma'' - \frac{3}{2}\sigma'\partial & 3\,\sigma' & 0 & -L + \frac{1}{2}J' + J\partial & 0 & -\partial \\ -\tau'' - \frac{3}{2}\tau'\partial & -3\tau' & L + \frac{1}{2}J' + J\partial & 0 & \partial & 0 \\ 0 & 0 & 0 & \partial & 0 & 0 \\ 0 & 0 & -\partial & 0 & 0 & 0 \end{pmatrix}$$

To compute the Lenard scheme, we take

$$\xi_0 := \begin{pmatrix} 0 \\ J^2\,\omega \\ 9G_2\,\omega \\ 9G_1\,\omega \\ 0 \\ 0 \end{pmatrix} \qquad \text{so that} \qquad K\xi_0 = 0$$

where $\omega := (J^3 + 27\,G_1G_2)^{-2/3}$. Then

$$\begin{aligned} h_0 &= \omega^{-1/2} = (J^3 + 27G_1G_2)^{1/3}, \\ h_1 &= 3LJ\omega + \frac{1}{2}\,J^2\Big(\frac{G_2'}{G_2} - \frac{G_1'}{G_1}\Big)\omega - 9(G_1\omega)'\tau - 9(G_2\omega)'\sigma. \end{aligned}$$

The first evolution equation $\frac{\partial u}{\partial t_0} = H\xi_0$ is as follows:

$$\frac{\partial}{\partial t_0}\begin{pmatrix} L \\ J \\ G_1 \\ G_2 \\ \sigma \\ \tau \end{pmatrix} = \begin{pmatrix} -\frac{9}{2}(G_1\tau'' + G_2\sigma'')\omega - \frac{27}{2}\big\{(G_1\omega)'\tau' + (G_2\omega)'\sigma'\big\} \\ 6(J^2\omega)' + 27(G_1\tau' - G_2\sigma')\omega \\ 3J^2\omega\sigma' + 9J(G_1\omega)' + 9\Big(-L + \frac{1}{2}J'\Big)G_1\omega \\ -3J^2\omega\tau' + 9J(G_2\omega)' + 9\Big(L + \frac{1}{2}J'\Big)G_2\omega \\ 9(G_1\omega)' \\ -9(G_2\omega)' \end{pmatrix} \qquad (4.2)$$

Our conjecture in this case is that these equations (4.1) and (4.2) will be integrable.

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