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EXAMPLES OF APPLICATION OF NIL-POLYNOMIALS TO THE BIHOLOMORPHIC EQUIVALENCE PROBLEM FOR ISOLATED HYPERSURFACE SINGULARITIES

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Abstract

Let V_1, V_2 be hypersurface germs in $\mathbb{C}^m$ with $m \geq 2$, each having a quasi-homogeneous isolated singularity at the origin. In our recent article [7] we reduced the biholomorphic equivalence problem for V_1, V_2 to verifying whether certain polynomials, called nil-polynomials, that arise from the moduli algebras of V_1, V_2 are equivalent up to scale by means of a linear transformation. In this paper we illustrate the above result by the examples of simple elliptic singularities of types $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$. The examples of singularities of types $\tilde{E}_6$, $\tilde{E}_7$ motivate a conjecture that implies that just the highest-order terms of the corresponding nil-polynomials completely solve the equivalence problem in the homogeneous case. This conjecture was first proposed in our paper [5] where it was established for plane curve germs defined by binary quintics and binary sextics. In the present paper we provide further evidence supporting the conjecture for binary forms of an arbitrary degree.

1. Introduction

For a hypersurface germ V at the origin in $\mathbb{C}^m$ with $m \geq 2$ (considered with its reduced complex structure) let $\mathcal{A}(V)$ be the *moduli algebra* or *Tjurina algebra* of V. Recall that $\mathcal{A}(V)$ is the quotient of the algebra $\mathcal{O}_m$ of germs at the origin of holomorphic functions of m complex variables by the ideal generated by f and all its first-order partial derivatives, where f is any generator of the ideal $I(V)$ of elements of $\mathcal{O}_m$ vanishing on V. This definition is independent of the choice of f as well as the coordinate system

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near the origin, and the moduli algebras of biholomorphically equivalent hypersurface germs are isomorphic. It is well-known that $\dim_{\mathbb{C}} \mathcal{A}(V) < \infty$ if and only if V either is non-singular (in which case $\mathcal{A}(V)$ is trivial) or has an isolated singularity (see, e.g. [11]).

A well-known theorem due to Mather and Yau (see [18]) states that hypersurface germs V_1 and V_2 in $\mathbb{C}^m$ having isolated singularities are biholomorphically equivalent if their moduli algebras $\mathcal{A}(V_1)$ and $\mathcal{A}(V_2)$ are isomorphic. Up to isomorphism, there is only one moduli algebra of dimension 1 and only one moduli algebra of dimension 2. If $\dim_{\mathbb{C}} \mathcal{A}(V) = 1$, then V is biholomorphic to the germ of the hypersurface $\{z_1^2 + \cdots + z_m^2 = 0\}$, and if $\dim_{\mathbb{C}} \mathcal{A}(V) = 2$, then V is biholomorphic to the germ of the hypersurface $\{z_1^2 + \cdots + z_{m-1}^2 + z_m^3 = 0\}$. In general, it is not easy to tell whether two given moduli algebras are isomorphic. In our recent article [7] we obtained a criterion for algebras $\mathcal{A}(V_1)$, $\mathcal{A}(V_2)$ of dimension greater than 2 to be isomorphic provided the singularity of each of V_1, V_2 is quasi-homogeneous. In this paper we apply the criterion found in [7] to particular families of singularities.

Recall that an isolated singularity of a hypersurface germ V in $\mathbb{C}^m$ is said to be *quasi-homogeneous* if some (and therefore any) generator f of $I(V)$ in some coordinates $z = (z_1, \dots, z_m)$ near the origin is the germ of a quasi-homogeneous polynomial, where a polynomial $Q(z)$ is called quasi-homogeneous if there exist positive integers $p_1, \dots, p_m, q$ such that $Q(t^{p_1} z_1, \dots, t^{p_m} z_m) \equiv t^q Q(z)$ for all $t \in \mathbb{C}$. The singularity of V is said to be *homogeneous* if one can choose Q to be homogeneous, in which case the discriminant of Q does not vanish. By a theorem due to Saito (see [21]), the singularity of V is quasi-homogeneous if and only if f lies in the Jacobian ideal $\mathcal{J}(f)$ (i.e. the ideal in $\mathcal{O}_m$ generated by all first-order partial derivatives of f). Hence, for a quasi-homogeneous singularity, $\mathcal{A}(V)$ coincides with the *Milnor algebra* $\mathcal{O}_m/\mathcal{J}(f)$ for any generator f of $I(V)$. Therefore, if the singularity of V is quasi-homogeneous, the algebra $\mathcal{A}(V)$ is a complete intersection ring, which implies that $\mathcal{A}(V)$ is Gorenstein (see [1]). Recall that a local complex commutative associative algebra $\mathcal{A}$ with $1 < \dim_{\mathbb{C}} \mathcal{A} < \infty$ is a *Gorenstein ring* if and only if for the annihilator $\mathrm{Ann}(\mathfrak{m}) := \{u \in \mathfrak{m} : u \cdot \mathfrak{m} = 0\}$ of its maximal ideal $\mathfrak{m}$ one has $\dim_{\mathbb{C}} \mathrm{Ann}(\mathfrak{m}) = 1$ (see, e.g. [12]). The property that $\mathcal{A}(V)$ is Gorenstein characterizes quasi-homogeneous singularities (see, e.g. [17]). Next, if the singularity of V is quasi-homogeneous, the algebra $\mathcal{A}(V)$ is *(non-negatively) graded*. More precisely, one has $\mathcal{A}(V) = \oplus_{j\geq 0}\mathcal{L}_j$, where $\mathcal{L}_j$ are linear subspaces of $\mathcal{A}(V)$, with $\mathcal{L}_j\mathcal{L}_k \subset \mathcal{L}_{j+k}$ and $\mathcal{L}_0 \simeq \mathbb{C}$. The existence of such a grading on $\mathcal{A}(V)$ also characterizes quasi-homogeneous singularities (see [26]).

A criterion for two complex graded Gorenstein algebras of finite vector space dimension greater than 2 to be isomorphic was given in [7] (see also [9], [14] for results on algebras over arbitrary fields of characteristic zero). The criterion is stated in terms of certain polynomials that were first introduced in [8]. Indeed, as explained in Section 2 below, to every complex Gorenstein algebra $\mathcal{A}$ with $2 < \dim_{\mathbb{C}} \mathcal{A} < \infty$ one can associate polynomials of a special form on $\mathfrak{n} := \mathfrak{m}/\mathrm{Ann}(\mathfrak{m}) = \mathfrak{m}/\mathfrak{m}^{\nu}$, called *nil-polynomials*, of degree ν with vanishing constant and linear terms, where $\nu \geq 2$ is the nil-index of $\mathfrak{m}$. In [7] we showed that two complex graded Gorenstein algebras $\mathcal{A}_1$, $\mathcal{A}_2$ are isomorphic if and only if some (hence any) nil-polynomials P_1, P_2 arising from $\mathcal{A}_1$, $\mathcal{A}_2$, respectively, are *linearly equivalent up to scale*, that is, there exists $c \in \mathbb{C}^*$ and a linear isomorphism $L : \mathfrak{n}_1 \mapsto \mathfrak{n}_2$ such that $cP_1 = P_2 \circ L$. Applying the above isomorphism criterion to the moduli algebras of two hypersurface germs V_1, V_2 in $\mathbb{C}^m$ having quasi-homogeneous singularities, one obtains that the biholomorphic equivalence problem for V_1, V_2 reduces to the problem of linear equivalence up to scale for any nil-polynomials P_1, P_2 arising from $\mathcal{A}(V_1)$, $\mathcal{A}(V_2)$, respectively (see Theorem 2.2).

In Section 3 we show how the above criterion works for *simple elliptic hypersurface singularities*. Recall that such singularities split into three types denoted $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, and a singularity within each type is completely determined by the value of the j-invariant for the exceptional elliptic curve lying in the minimal resolution of the singularity (see [22]). The isomorphism problem for the moduli algebras of simple elliptic singularities has been extensively studied in purely algebraic terms and is now well-understood. Namely, it was explained in [2], [23] – and in a very explicit form in [4] – how one can recover the value of the j-invariant directly from the corresponding moduli algebra. In article [4] for singularities of each of the types $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$ certain forms (homogeneous polynomials) were introduced in an invariant way; we call them *Eastwood forms*. Remarkably, it has turned out that by using classical invariant theory one can extract the value of the j-invariant for the exceptional elliptic curve from the Eastwood form of the singularity.

In Section 3 we use Theorem 2.2 for providing an alternative solution to the equivalence problem for singularities of each of the types $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$. In our solution, instead of the Eastwood forms we use nil-polynomials arising from the moduli algebras. For each of the types $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, the Eastwood forms can in fact be regarded as parts of the corresponding nil-polynomials. Since the nil-polynomials contain additional terms, they should be easier to use for distinguishing biholomorphically non-equivalent singularities than the Eastwood forms. Indeed, while for singularities of types $\widetilde{E}_6$, $\widetilde{E}_7$ our arguments are similar to those in [4], for singularities of type $\widetilde{E}_8$ there is a difference. Namely, for $\widetilde{E}_8$-singularities we do not need to resort to classical invariant theory. Instead, we make elementary comparisons of some of the homogeneous components of the corresponding nil-polynomials. Utilizing components other than the Eastwood forms is essential for our arguments in the $\widetilde{E}_8$-case.

As we will see in Section 3, in order to recover the solution to the equivalence problem for simple elliptic singularities of types $\widetilde{E}_6$, $\widetilde{E}_7$, only the highest-order terms of the corresponding nil-polynomials (which essentially coincide with the Eastwood forms) need to be used. Note that singularities of type $\widetilde{E}_6$ are homogeneous and that the moduli algebra of any singularity of type $\widetilde{E}_7$ is in fact the moduli algebra of a homogeneous plane curve singularity. These observations motivate a conjecture that was first proposed in our recent paper [5] (see Conjecture 4.2 in Section 4 below). The conjecture states, in particular, that for homogeneous singularities a solution to the biholomorphic equivalence problem is encoded in the highest-order terms of nil-polynomials arising from the moduli algebras of the singularities. For a hypersurface germ V defined by a form Q of degree n with non-zero discriminant, every nil-polynomial P has degree $m(n-2)$, and one can think of its highest-order homogeneous component as a form $\widehat{P}^{[m(n-2)]}$ defined on the m-dimensional space $\mathfrak{m}(V)/\mathfrak{m}(V)^2$, where $\mathfrak{m}(V)$ is the maximal ideal of $\mathcal{A}(V)$. Any two forms constructed as above coincide up to scale, and we say that all these mutually proportional forms are *associated* to Q (note that such forms were first considered in [4] without introducing nil-polynomials). Further, two hypersurface germs defined by forms $Q_1(z)$,

$Q_2(z)$ with non-zero discriminant are biholomorphically equivalent if and only if Q_1, Q_2 are *linearly equivalent*, that is, there exists $C \in \mathrm{GL}(m, \mathbb{C})$ for which $Q_1(Cz) \equiv Q_2(z)$. It can be shown that for forms with non-zero discriminant the linear equivalence problem is solved by absolute classical invariants (see Proposition 4.1). Accordingly, Conjecture 4.2 states that one can recover all absolute invariants of forms of degree n in m variables from absolute invariants of forms of degree $m(n-2)$ in m variables by evaluating the latter for associated forms.

The treatment of simple elliptic singularities of types $\widetilde{E}_6$ and $\widetilde{E}_7$ in article [4] and in Section 3 of the present paper effectively verifies Conjecture 4.2 for binary quartics ($m = 2$, $n = 4$) and ternary cubics ($m = 3$, $n = 3$). In [5] we showed that Conjecture 4.2 also holds for binary quintics ($m = 2$, $n = 5$) and binary sextics ($m = 2$, $n = 6$). In Section 5 of this paper we give further evidence supporting the conjecture. We consider the following family of homogeneous plane curve singularities:

$$\mathcal{V}_t := \text{the germ of } \{f_t := z_1^n + t z_1^{n-1} z_2 + z_2^n = 0\},\ t \in \mathbb{C},\ n \geq 4, \qquad (1.1)$$

where the discriminant of the form f_t is non-zero, which in terms of the parameter t means $t^n \neq -n^n/(1-n)^{n-1}$ (see (5.4)). Family (1.1) first appeared in paper [16], where the plane curves $\{z_1^n + z_1 a(z_2) z_2^{\alpha} + b(z_2) z_2^{\beta} = 0\}$ were considered. Here α, β are positive integers, $n \geq 3$, and $a(z_2)$, $b(z_2)$ are holomorphic nowhere vanishing functions defined near the origin. The germs of the above curves at the origin were classified in [16] up to biholomorphic equivalence in many situations (see also [24]). However, the homogeneous case $\alpha = n-1$, $\beta = n$, $a \equiv 1$, $b \equiv \text{const}$, where the discriminant of the form $z_1^n + z_1 z_2^{n-1} + b z_2^n$ is non-zero, proved to be of substantial difficulty to the authors for $n \geq 4$. The biholomorphic equivalence problem in the homogeneous case was eventually solved in later paper [15] as stated in the theorem below (note that for $n \geq 4$ and $b \neq 0$ the form $z_1^n + z_1 z_2^{n-1} + b z_2^n$ is linearly equivalent to f_t for some t).

Theorem 1.1. *Two germs $\mathcal{V}_{t_1}$ and $\mathcal{V}_{t_2}$ are biholomorphically equivalent if and only if $t_1^n = t_2^n$.*

Theorem 1.1 was obtained in [15] by the direct substitution method, which required enormous calculations. In Section 5 below we prove Theorem 1.1 by a short elementary argument based on classical invariant theory.

Furthermore, it turns out that the solution to the biholomorphic equivalence problem for the germs $\mathcal{V}_t$ given by Theorem 1.1 can be also recovered in the spirit of Conjecture 4.2, namely by evaluating a certain absolute invariant of forms of degree $2(n-2)$ for forms associated to f_t. This last method for obtaining a solution to the equivalence problem for $\mathcal{V}_t$ is still much easier than the direct substitution method of [15].

2. A Criterion for Biholomorphic Equivalence of Quasi-Homogeneous Singularities

In this section we state some of the results of our paper [7]. Let $\mathcal{A}$ be a complex Gorenstein algebra with $2 < \dim_{\mathbb{C}} \mathcal{A} < \infty$ and $\mathfrak{m}$ the maximal ideal of $\mathcal{A}$. Further, let $\exp_2 : \mathfrak{m} \to \mathfrak{m}$ be the map

$$\exp_2(u) := \sum_{k=2}^{\infty} \frac{1}{k!} u^k, \quad u \in \mathfrak{m}.$$

By Nakayama's lemma, $\mathfrak{m}$ is a nilpotent algebra, and therefore the above sum is in fact finite, with the highest-order term corresponding to $k = \nu$, where $\nu \geq 2$ is the *nil-index* of $\mathfrak{m}$ (i.e. the largest of all integers μ for which $\mathfrak{m}^\mu \neq 0$).

Using the map $\exp_2$, one can associate to the algebra $\mathcal{A}$ a collection of polynomials of a special form. For every finite-dimensional complex vector space W we denote by $\mathbb{C}[W]$ the algebra of all $\mathbb{C}$-valued polynomials on W. Let $\mathrm{Ann}(\mathfrak{m}) := \{u \in \mathfrak{m} : u \cdot \mathfrak{m} = 0\} = \mathfrak{m}^\nu$ be the annihilator of $\mathfrak{m}$. A polynomial $P \in \mathbb{C}[W]$ is called a *nil-polynomial arising from* $\mathcal{A}$ if there exist a linear form $\omega : \mathfrak{m} \to \mathbb{C}$ and a linear isomorphism $\varphi : W \to \ker \omega$ such that $\omega(\mathrm{Ann}(\mathfrak{m})) = \mathbb{C}$ and $P = \omega \circ \exp_2 \circ \varphi$. Any nil-polynomial P has a unique decomposition

$$P = \sum_{k=2}^{\nu} P^{[k]}, \qquad P^{[k]} := \frac{1}{k!} \omega(\varphi^k),$$

where every $P^{[k]} \in \mathbb{C}[W]$ is homogeneous of degree k. The quadratic form $P^{[2]}$ is non-degenerate on W, and $P^{[\nu]} \neq 0$. Without loss of generality we may assume that $W = \mathbb{C}^K$ for $K := \dim_{\mathbb{C}} \mathfrak{m} - 1$. In this case there exists a basis $e_1, \ldots, e_K$ of $\ker \omega$ such that $\varphi(w) = \sum_{\alpha=1}^{K} w_\alpha e_\alpha$ for $w = (w_1, \ldots, w_K) \in \mathbb{C}^K$, and we write $\mathbb{C}[W] = \mathbb{C}[w_1, \ldots, w_K]$.

Further, two nil-polynomials $P_1 \in \mathbb{C}[W_1]$, $P_2 \in \mathbb{C}[W_2]$ arising from Gorenstein algebras $\mathcal{A}_1$, $\mathcal{A}_2$, respectively, are called *linearly equivalent up to scale* if there exist $c \in \mathbb{C}^*$ and a linear isomorphism $L : W_1 \to W_2$ such that $cP_1 = P_2 \circ L$. Clearly, this identity holds if and only if $cP_1^{[k]} = P_2^{[k]} \circ L$ for $k = 2, \ldots, \nu$, where ν is the nil-index of each of $\mathfrak{m}_1$, $\mathfrak{m}_2$ (note that the nil-indices of $\mathfrak{m}_1$, $\mathfrak{m}_2$ coincide since $\deg P_1 = \deg P_2$). It follows from results of [7] (see also [8], [9], [14]) that the map

$$\psi : \mathfrak{m}_1 \to \mathfrak{m}_2, \quad \psi(u+v) := \varphi_2 \circ L \circ \varphi_1^{-1}(u) + c\,\widetilde{\omega}_2^{-1}(\omega_1(v))$$

is an algebra isomorphism, where ω_1, ω_2 and $\varphi_1 : W_1 \to \ker \omega_1$, $\varphi_2 : W_2 \to \ker \omega_2$ are the linear forms and the linear isomorphisms corresponding to P_1, P_2, respectively, $\widetilde{\omega}_2 := \omega_2|_{\mathrm{Ann}(\mathfrak{m}_2)}$, $u \in \ker \omega_1$, $v \in \mathrm{Ann}(\mathfrak{m}_1)$. Thus, if P_1, P_2 are linearly equivalent up to scale, the algebras $\mathcal{A}_1$, $\mathcal{A}_2$ are isomorphic. In [7] we showed that the converse to this statement also holds if the algebras $\mathcal{A}_1$, $\mathcal{A}_2$ are graded.

Thus, we have the following theorem.

Theorem 2.1 ([7]). *Let P_1, P_2 be arbitrary nil-polynomials arising from Gorenstein algebras $\mathcal{A}_1$, $\mathcal{A}_2$ of dimension greater than 2, respectively. Suppose that the algebras $\mathcal{A}_1$, $\mathcal{A}_2$ is graded. Then $\mathcal{A}_1$, $\mathcal{A}_2$ are isomorphic if and only if P_1, P_2 are linearly equivalent up to scale.*

Applying Theorem 2.1 to the moduli algebras of quasi-homogeneous singularities, one obtains a solution to the biholomorphic equivalence problem for such singularities.

Theorem 2.2 ([7]). *Let V_1, V_2 be hypersurface germs in $\mathbb{C}^m$ each having a quasi-homogeneous singularity. Assume that* $\dim_{\mathbb{C}} \mathcal{A}(V_1) > 2$, $\dim_{\mathbb{C}} \mathcal{A}(V_2) > 2$. *Let furthermore P_1, P_2 be arbitrary nil-polynomials arising from $\mathcal{A}(V_1)$, $\mathcal{A}(V_2)$, respectively. Then the germs V_1, V_2 are biholomorphically equivalent if and only if the nil-polynomials P_1, P_2 are linearly equivalent up to scale.*

3. Application to Simple Elliptic Singularities

In this section we illustrate Theorem 2.2 by the examples of simple elliptic hypersurface singularities.

Example 3.1. Consider simple elliptic singularities of type $\widetilde{E}_6$. These are the homogeneous singularities of the following hypersurface germs at the origin in $\mathbb{C}^3$:

$$V_t := \text{the germ of } \big\{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^3+z_2^3+z_3^3+tz_1z_2z_3 = 0\big\},\ t^3 + 27 \neq 0.$$

Two germs V_{t_1}, V_{t_2} are known to be biholomorphically equivalent if and only if t_1 is obtained from t_2 by an element of the group generated by the following parameter changes:

$$t \mapsto \rho t, \quad t \mapsto \frac{3(6-t)}{t+3}, \tag{3.1}$$

where $\rho^3 = 1$ (see [2], [4], [22]).

We will now give an alternative proof of this statement using Theorem 2.2. Following [2], [4], consider the monomials

$$z_1z_2z_3,\ z_1,\ z_2,\ z_3,\ z_2z_3,\ z_1z_3,\ z_1z_2,$$

and let e_l, $l = 0, \dots, 6$, respectively, be the vectors in the maximal ideal $\mathfrak{m}(V_t)$ of $\mathcal{A}(V_t)$ arising from them. These vectors are known to form a basis of $\mathfrak{m}(V_t)$, with $\mathrm{Ann}(\mathfrak{m}(V_t))$ spanned by e_0. Then for any linear form ω on $\mathfrak{m}(V_t)$, with $\ker\omega$ spanned by e_l, $l = 1, \dots, 6$, and for $\varphi : \mathbb{C}^6 \to \ker\omega$ given by $\varphi(w) := \sum_{\alpha=1}^{6} w_\alpha e_\alpha$, with $w = (w_1, \dots, w_6)$, the corresponding nil-polynomial in $\mathbb{C}[w_1, \dots, w_6]$ is proportional to

$$P_t := -\frac{t}{18}(w_1^3 + w_2^3 + w_3^3) + w_1w_2w_3 + w_1w_4 + w_2w_5 + w_3w_6.$$

Consider the cubic terms in P_t:

$$Q_t := P_t^{[3]} = -\frac{t}{18}(w_1^3 + w_2^3 + w_3^3) + w_1w_2w_3. \tag{3.2}$$

If one regards the cubics Q_t as forms on $\mathfrak{m}(V_t)/\mathfrak{m}(V_t)^2$, then up to scale they coincide with the Eastwood forms of $\widetilde{E}_6$-singularities (see formula (3.1) in [4]). It turns out that any two non-equivalent germs are distinguished by Q_t.

Suppose that for some $t_1 \neq t_2$ the germs V_{t_1} and V_{t_2} are biholomorphically equivalent. By Theorem 2.2 there exist $c \in \mathbb{C}^*$ and $C \in \mathrm{GL}(6, \mathbb{C})$ such that $c \cdot P_{t_1}(w) \equiv P_{t_2}(Cw)$. Then we have $c \cdot Q_{t_1}(w') \equiv Q_{t_2}(C'w')$, where $w' := (w_1, w_2, w_3)$ and C' is the upper left 3×3-submatrix of the matrix C. It then follows that C' is non-degenerate and maps the zero locus of Q_{t_1} onto that of Q_{t_2}. Let $\mathcal{Z}_t$ be the curve in $\mathbb{CP}^2$ arising from the zero locus of Q_t. This curve has singularities only if either $t = 0$ or $t^3 = 216$. Hence if $t_1 = 0$, then $t_2^3 = 216$, which agrees with (3.1).

If $t \neq 0$ and $t^3 \neq 216$, then $\mathcal{Z}_t$ is an elliptic curve. The projective equivalence class of an elliptic curve is completely determined by the value of the j-invariant for the curve. The value of the j-invariant for $\mathcal{Z}_t$ is well-known (see, e.g. [2], [4], [13], [22]):

$$j(\mathcal{Z}_t) = -\frac{(t^3+27)^3}{t^3(t^3-216)^3}.$$

It then follows that t_1 and t_2 can only be related as described by (3.1).

On the other hand, if t_1 and t_2 are related as described by (3.1), one can construct a biholomorphic map between V_{t_1} and V_{t_2}. Indeed, for $\rho^3 = 1$, $\rho \neq 1$ the map

$$z_1 \mapsto \rho z_1, \quad z_2 \mapsto z_2, \quad z_3 \mapsto z_3$$

shows that V_t and $V_{\rho t}$ are equivalent, and the map

$$z_1 \mapsto z_1 + z_2 + z_3, \quad z_2 \mapsto \rho z_1 + \rho^2 z_2 + z_3, \quad z_3 \mapsto \rho^2 z_1 + \rho z_2 + z_3$$

shows that V_t and $V_{\frac{3(6-t)}{t+3}}$ are equivalent (cf. [4]).

Example 3.2. Consider simple elliptic singularities of type $\widetilde{E}_7$. These are the quasi-homogeneous singularities of the following hypersurface germs at the origin in $\mathbb{C}^3$:

$$V_t := \text{the germ of } \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^4 + t z_1^2 z_2^2 + z_2^4 + z_3^2 = 0\right\}, \ t \neq \pm 2.$$

Two germs V_{t_1}, V_{t_2} are known to be biholomorphically equivalent if and only if t_1 is obtained from t_2 by an element of the group generated by the following parameter changes:

$$t \mapsto -t, \quad t \mapsto \frac{2(6-t)}{t+2} \tag{3.3}$$

(see [4], [22], [23]).

We will now give an alternative proof of this statement using Theorem 2.2. Following [4], [23], consider the monomials

$$z_1^2 z_2^2,\ z_1,\ z_2,\ z_1^2,\ z_1 z_2,\ z_2^2,\ z_1^2 z_2,\ z_1 z_2^2,$$

and let e_l, $l = 0, \dots, 7$, respectively, be the vectors in the maximal ideal $\mathfrak{m}(V_t)$ of $\mathcal{A}(V_t)$ arising from these monomials. These vectors are known to form a basis of $\mathfrak{m}(V_t)$, with $\mathrm{Ann}(\mathfrak{m}(V_t))$ spanned by e_0. Then for any linear form ω on $\mathfrak{m}(V_t)$, with $\ker\omega$ spanned by e_l, $l = 1, \dots, 7$, and for $\varphi : \mathbb{C}^7 \to \ker\omega$ given by $\varphi(w) := \sum_{\alpha=1}^{7} w_\alpha e_\alpha$, with $w = (w_1, \dots, w_7)$, the corresponding nil-polynomial in $\mathbb{C}[w_1, \dots, w_7]$ is proportional to

$$\begin{aligned} P_t \ :=\ & -\frac{t}{48}w_1^4 + \frac{1}{4}w_1^2 w_2^2 - \frac{t}{48}w_2^4 \\ & -\frac{t}{4}w_1^2 w_3 + \frac{1}{2}w_1^2 w_5 - \frac{t}{4}w_2^2 w_5 + \frac{1}{2}w_2^2 w_3 + w_1 w_2 w_4 \\ & + w_1 w_7 + w_2 w_6 + w_3 w_5 - \frac{t}{4}w_3^2 - \frac{t}{4}w_5^2 + \frac{1}{2}w_4^2. \end{aligned}$$

Consider the fourth-order terms in P_t:

$$Q_t := P_t^{[4]} = -\frac{t}{48}w_1^4 + \frac{1}{4}w_1^2 w_2^2 - \frac{t}{48}w_2^4. \tag{3.4}$$

If one regards the quartics Q_t as forms on $\mathfrak{m}(V_t)/\mathfrak{m}(V_t)^2$, then up to scale they coincide with the Eastwood forms of $\widetilde{E}_7$-singularities (cf. formula (3.7) in [4]). It turns out that any two non-equivalent germs are distinguished by Q_t.

Suppose that for some $t_1 \neq t_2$ the germs V_{t_1} and V_{t_2} are biholomorphically equivalent. By Theorem 2.2 there exist $c \in \mathbb{C}^*$ and $C \in \mathrm{GL}(7, \mathbb{C})$ such that $c \cdot P_{t_1}(w) \equiv P_{t_2}(Cw)$. Then we have $c \cdot Q_{t_1}(w') \equiv Q_{t_2}(C'w')$, where $w' := (w_1, w_2)$ and C' is the upper left 2×2-submatrix of the matrix C. It then follows that C' is non-degenerate and maps the zero locus of Q_{t_1} onto that of Q_{t_2}. Observe that the zero locus of Q_0 consists of the complex lines $\{w_1 = 0\}$ and $\{w_2 = 0\}$, and for $t \neq 0$ the zero locus of Q_t is

$$\mathcal{Z}_t := \left\{ w' \in \mathbb{C}^2 : w_1^2 = \frac{6 + \sqrt{36 - t^2}}{t} w_2^2 \right\}.$$

Clearly, for $t \neq \pm 6$ the set $\mathcal{Z}_t$ consists of four complex lines, whereas each of $\mathcal{Z}_6$ and $\mathcal{Z}_{-6}$ is the union of two complex lines. Hence if $t_1 = 0$ then t_2 can only be ± 6, which agrees with (3.3).

Suppose now that $t_1, t_2 \neq 0, \pm 6$ and consider the Möbius transformation $m_{C'}$ of $\mathbb{CP}^1$ arising from C'. The transformation $m_{C'}$ maps the four points in $\mathbb{CP}^1$ corresponding to $\mathcal{Z}_{t_1}$ onto the four points corresponding to $\mathcal{Z}_{t_2}$. Considering the cross-ratios of these four-point sets and using the fact that cross-ratios are preserved under $m_{C'}$, it is now straightforward to see that t_1 and t_2 can only be related as described by (3.3). An alternative proof of this statement is given in [4]; it uses the invariant theory of quartics in two variables (see also Section 4 below).

On the other hand, if t_1 and t_2 are related as described by (3.3), one can construct a biholomorphic map between V_{t_1} and V_{t_2}. Indeed, the map

$$z_1 \mapsto iz_1, \quad z_2 \mapsto z_2, \quad z_3 \mapsto z_3$$

shows that V_t and V_{-t} are equivalent, and the map

$$z_1 \mapsto z_1 + z_2, \quad z_2 \mapsto z_1 - z_2, \quad z_3 \mapsto \sqrt{t+2}\, z_3$$

shows that V_t and $V_{\frac{2(6-t)}{t+2}}$ are equivalent (cf. [4]).

Remark 3.3. In Examples 3.1 and 3.2 linear equivalence of the forms Q_{t_1}, Q_{t_2} (see (3.2), (3.4)) can in fact be obtained directly, without referring to Theorem 2.2. Indeed, in each of these cases Q_t is the highest-order term of a nil-polynomial, and the highest-order terms of any two nil-polynomials arising from the same Gorenstein algebra are proportional to each other when regarded as forms on $\mathfrak{m}/\mathfrak{m}^2$.

Example 3.4. Consider simple elliptic singularities of type $\widetilde{E}_8$. These are the quasi-homogeneous singularities of the following hypersurface germs at the origin in $\mathbb{C}^3$:

$$V_t := \text{the germ of } \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^6 + t z_1^4 z_2 + z_2^3 + z_3^2 = 0\right\},\ 4t^3 + 27 \neq 0.$$

Two germs V_{t_1}, V_{t_2} are known to be biholomorphically equivalent if and only if the following holds:

$$t_1 = \rho t_2, \tag{3.5}$$

where $\rho^3 = 1$ (see [4], [22], [23]).

We will now give an alternative proof of this statement using Theorem 2.2. Following [4], [23], consider the monomials

$$z_1^4 z_2,\ z_1,\ z_2,\ z_1^2,\ z_1 z_2,\ z_1^3,\ z_1^2 z_2,\ z_1^4,\ z_1^3 z_2,$$

and let e_l, $l = 0, \dots, 8$, respectively, be the vectors in the maximal ideal $\mathfrak{m}(V_t)$ of $\mathcal{A}(V_t)$ arising from these monomials. These vectors are known to form a basis of $\mathfrak{m}(V_t)$, with $\mathrm{Ann}(\mathfrak{m}(V_t))$ spanned by e_0. Then for any linear form ω on $\mathfrak{m}(V_t)$, with $\ker\omega$ spanned by e_l, $l = 1, \dots, 8$, and for $\varphi : \mathbb{C}^8 \to \ker\omega$ given by $\varphi(w) := \sum_{\alpha=1}^{8} w_\alpha e_\alpha$, with $w = (w_1, \dots, w_8)$, the corresponding nil-polynomial in $\mathbb{C}[w_1, \dots, w_8]$ is proportional to

$$\begin{aligned} P_t :\ =\ & -\frac{t}{1080} w_1^6 + \frac{1}{24} w_1^4 w_2 - \frac{t}{36} w_1^4 w_3 + \frac{1}{6} w_1^3 w_4 - \frac{t}{9} w_1^3 w_5 + \frac{t^2}{18} w_1^2 w_2^2 \\ & + \frac{1}{2} w_1^2 w_2 w_3 - \frac{t}{6} w_1^2 w_3^2 + \frac{2t^2}{9} w_1 w_2 w_4 + w_1 w_2 w_5 + w_1 w_3 w_4 - \frac{2t}{3} w_1 w_3 w_5 \\ & + \frac{1}{2} w_1^2 w_6 - \frac{t}{3} w_1^2 w_7 - \frac{t}{18} w_2^3 + \frac{t^2}{9} w_2^2 w_3 + \frac{1}{2} w_2 w_3^2 - \frac{t}{9} w_3^3 + w_1 w_8 + w_2 w_7 \\ & + \frac{2t^2}{9} w_2 w_6 + w_3 w_6 - \frac{2t}{3} w_3 w_7 + \frac{t^2}{9} w_4^2 + w_4 w_5 - \frac{t}{3} w_5^2. \end{aligned}$$

In our arguments we will use, in particular, the third-order terms of P_t independent of w_1:

$$Q_t := -\frac{t}{18} w_2^3 + \frac{t^2}{9} w_2^2 w_3 + \frac{1}{2} w_2 w_3^2 - \frac{t}{9} w_3^3.$$

If one regards the cubics Q_t as forms on $\mathfrak{l}(V_t)/\mathfrak{m}(V_t)\mathfrak{l}(V_t)$, where

$$\mathfrak{l}(V_t) := \{u \in \mathfrak{m}(V_t) : u^4 = 0\},$$

then up to scale they coincide with the Eastwood forms of $\widetilde{E}_8$-singularities (cf. p. 308 in [4]).

Suppose that for some $t_1 \neq t_2$ the germs V_{t_1} and V_{t_2} are biholomorphically equivalent. Since 0 is the only value of t for which P_t has degree 6, we have $t_1, t_2 \neq 0$. By Theorem 2.2 there exist $c \in \mathbb{C}^*$ and $C \in \mathrm{GL}(8, \mathbb{C})$ such that

$$c \cdot P_{t_1}(w) \equiv P_{t_2}(Cw). \tag{3.6}$$

By comparing the terms of order 6 in identity (3.6), we obtain that the first row in the matrix C has the form $(\mu, 0, \dots, 0)$ and

$$c = \frac{t_2}{t_1}\mu^6. \tag{3.7}$$

Next, let $(\,*\,, \alpha, \beta, \,*\,, \dots, \,*\,)$ and $(\,*\,, \gamma, \delta, \,*\,, \dots, \,*\,)$ be the second and third rows in C, respectively, for some $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. Comparing the terms of order 4 in (3.6) that do not involve w_1^3, we see that the matrix

$$D := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is non-degenerate. Further, comparing the terms of order 5 in (3.6) we obtain

$$\beta = \frac{2}{9}\left(-3\alpha t_1 + 3\delta t_2 + 2\gamma t_1 t_2\right) \tag{3.8}$$

and

$$c = \left(\alpha - \frac{2t_2}{3}\gamma\right)\mu^4. \tag{3.9}$$

We will now compare the terms of order 3 in (3.6) that depend only on $w' := (w_2, w_3)$. We have

$$c \cdot Q_{t_1}(w') \equiv Q_{t_2}(Dw'). \tag{3.10}$$

Setting

$$D_t := \begin{pmatrix} 1/3 & 2t/3 \\ 0 & 1 \end{pmatrix}$$

one observes

$$Q_t(D_t w') = \mathsf{Q}_t(w') := \frac{t}{27}w_2^3 - 3\Delta_t w_2 w_3^2 - 4t\Delta_t w_3^3,$$

where $\Delta_t := 1 + 4t^3/27$. Hence (3.10) implies

$$c \cdot \mathsf{Q}_{t_1}(w') \equiv \mathsf{Q}_{t_2}(\widehat{D}w'), \tag{3.11}$$

where $\widehat{D} := D_{t_2}^{-1} D D_{t_1}$. By (3.8) we have

$$\widehat{D} = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix}$$

with $a := \alpha - 2t_2\gamma/3$, $b := \gamma/3$, $d := \delta + 2t_1\gamma/3$.

It follows from (3.11) and the non-degeneracy of $\widehat{D}$ that $b(a + 2t_2 b) = 0$. If $b = 0$, the comparison of the three pairs of coefficients in (3.11) yields

$$c = \frac{t_2}{t_1} a^3 = \frac{\Delta_{t_2}}{\Delta_{t_1}} a d^2 = \frac{t_2 \Delta_{t_2}}{t_1 \Delta_{t_1}} d^3.$$

Therefore $t_1^3 \Delta_{t_2} = t_2^3 \Delta_{t_1}$, and we obtain that t_1 and t_2 are related as in (3.5). Suppose now that $b \neq 0$, that is, $a = -2t_2 b$. In this situation the comparison of the three pairs of coefficients in (3.11) yields

$$c = 54 \frac{t_2}{t_1} b^3 = 2 \frac{t_2 \Delta_{t_2}}{\Delta_{t_1}} b d^2 = \frac{t_2 \Delta_{t_2}}{t_1 \Delta_{t_1}} d^3. \tag{3.12}$$

From identities (3.7), (3.9) and the first equality in (3.12) we obtain $\Delta_{t_1} = 0$, which is impossible. [We remark that identities (3.12) alone do not lead to a contradiction, they only imply $(t_1 t_2)^3 = (27/4)^2$.] Thus, if the germs V_{t_1} and V_{t_2} are biholomorphically equivalent, then t_1 and t_2 can only be related as in (3.5).

On the other hand, if t_1 and t_2 are related as in (3.5), one can construct a biholomorphic map between the germs V_{t_1} and V_{t_2}. Indeed, for $\rho^3 = 1$ the map

$$z_1 \mapsto z_1, \quad z_2 \mapsto \rho z_2, \quad z_3 \mapsto z_3$$

shows that the germs V_t and $V_{\rho t}$ are equivalent (cf. [4]).

Remark 3.5. Observe that in Example 3.4 we utilized not just the highest-order terms of the nil-polynomials but also some of their lower-order terms. Thus, in Example 3.4 we relied on Theorem 2.2 in an essential way, which is in contrast with Examples 3.1 and 3.2 (cf. Remark 3.3).

4. Homogeneous Singularities

In this section we state a conjecture that was first proposed in our recent paper [5]. The conjecture explains how a solution to the biholomorphic equivalence problem in the case of homogeneous singularities can be extracted from the corresponding nil-polynomials.

We start by recalling the definitions of relative and absolute classical invariants (see, e.g. [20] for details). Let W be a finite-dimensional complex vector space and $\mathcal{Q}_W^n$ the linear space of forms of a fixed degree $n \geq 2$ on W. Define an action of $\mathrm{GL}(W)$ on $\mathcal{Q}_W^n$ by the formula

$$(C, Q) \mapsto Q_C,\ \ Q_C(w) := Q(C^{-1}w), \text{ where } C \in \mathrm{GL}(W),\ Q \in \mathcal{Q}_W^n,\ w \in W.$$

Two forms are said to be *linearly equivalent* if they lie in the same orbit with respect to this action. An *invariant* (or *relative classical invariant*) of forms of degree n on W is a polynomial $I : \mathcal{Q}_W^n \to \mathbb{C}$ such that for any $Q \in \mathcal{Q}_W^n$ and any $C \in \mathrm{GL}(W)$ one has $I(Q) = (\det C)^k I(Q_C)$, where k is a non-negative integer called the *weight* of I. It follows that I is in fact homogeneous of degree $k \cdot \dim_{\mathbb{C}} W/n$. Finite sums of relative invariants constitute the algebra of polynomial $\mathrm{SL}(W)$-invariants of $\mathcal{Q}_W^n$, called the *algebra of invariants* (or *algebra of classical invariants*) of forms of degree n on W. By the Hilbert Basis Theorem, this algebra is finitely generated.

Next, for any two invariants I and $\widetilde{I}$ the ratio $I/\widetilde{I}$ yields a rational function on $\mathcal{Q}_W^n$ that is defined, in particular, at the points where $\widetilde{I}$ does not vanish. If I and $\widetilde{I}$ have equal weights, this function does not change under the action of $\mathrm{GL}(W)$, and we say that $I/\widetilde{I}$ is an *absolute invariant* (or *absolute classical invariant*) of forms of degree n on W. If one fixes coordinates $z_1, \dots, z_m$ in W, then any element $Q \in \mathcal{Q}_W^n$ is written as

$$Q(z_1, \dots, z_m) = \sum_{i_1+\cdots+i_m=n} \begin{pmatrix} n \\ i_1, \dots, i_m \end{pmatrix} a_{i_1,\dots,i_m} z_1^{i_1} \cdot \dots \cdot z_m^{i_m},$$

where $a_{i_1,\dots,i_m} \in \mathbb{C}$. In what follows we will introduce absolute invariants that will be defined in terms of the coefficients $a_{i_1,\dots,i_m}$. Observe that for any absolute invariant $\mathcal{I}$ so defined its value $\mathcal{I}(Q)$ is in fact independent of the choice of coordinates in W. When working in coordinates, we assume that $W = \mathbb{C}^m$ and identify $\mathrm{GL}(W)$ with $\mathrm{GL}(m, \mathbb{C})$.

We say that a non-zero form $Q \in \mathcal{Q}^n_{\mathbb{C}^m}$ is *minimal* if the germ of Q at the origin generates the ideal $I(V)$, where V is the germ of the hypersurface $\{Q = 0\}$. If Q is a binary form (i.e. $m = 2$), then it can be written as a product of non-zero linear factors, and the minimality of Q means that each of the factors has multiplicity one, i.e. Q is *square-free.* Observe that two hypersurface germs V_1, V_2 defined by minimal forms Q_1, Q_2, respectively, are biholomorphically equivalent if and only if the forms Q_1, Q_2 are linearly equivalent.

For $Q \in \mathcal{Q}^n_{\mathbb{C}^m}$, let $\Delta(Q)$ be the discriminant of Q (see Chapter 13 in [10] for the definition[1]). The discriminant is a relative classical invariant of degree $m(n-1)^{m-1}$. Set

$$X^n_m := \{Q \in \mathcal{Q}^n_{\mathbb{C}^m} : \Delta(Q) \neq 0\}.$$

Note that Q lies in X^n_m if and only if Q is minimal and the singularity of the germ of the hypersurface $\{Q = 0\}$ at the origin is isolated. If Q is a binary form, then $Q \in X^n_2$ if and only if Q is non-zero and square-free. As stated in the following proposition, for forms in X^n_m the linear equivalence problem is solved by absolute classical invariants.

Proposition 4.1 ([5]). [2] *For $n \geq 3$ the orbits of the* $\mathrm{GL}(m, \mathbb{C})$*-action on X^n_m are separated by absolute classical invariants of the kind*

$$\mathcal{I} = \frac{I}{\Delta^p}, \tag{4.1}$$

where p is a non-negative integer and I is a relative classical invariant.

In what follows the algebra of the restrictions to X^n_m of absolute invariants of the form (4.1) is denoted by $\mathcal{I}^n_m$. By the Hilbert Basis Theorem, this algebra is finitely generated. It is clear from the proof of Proposition 4.1 given in [5] that $\mathcal{I}^n_m$ is exactly the algebra of $\mathrm{GL}(m, \mathbb{C})$-invariant regular functions on the affine algebraic variety X^n_m.

The conjecture proposed in [5], which we will state below, explains how the solution to the linear equivalence problem for forms in X^n_m given

[1] The formulas for the discriminant that we use below in the cases $m = 2, 3$ differ from the one given in [10] by scalar factors.

[2] The proof of this proposition given in [5] has been suggested to us by A. Gorinov.

in Proposition 4.1 can be recovered by considering the corresponding nil-polynomials. Let V be the hypersurface germ at the origin in $\mathbb{C}^m$ defined by a form $Q \in X_m^n$ with $n \geq 3$. Observe that $\dim_{\mathbb{C}} \mathcal{A}(V) > 2$, and let $\mathfrak{m}(V)$ be the maximal ideal of $\mathcal{A}(V)$. It follows from results in [22] that $\nu = m(n-2)$. If $P = \omega \circ \exp_2 \circ \varphi$ is a nil-polynomial arising from $\mathcal{A}(V)$, then $\widetilde{P} := \omega \circ \exp_2$ is a polynomial on $\mathfrak{m}(V)$ satisfying $\widetilde{P}(v) = 0$, $\widetilde{P}(u+v) = \widetilde{P}(u)$ for all $u \in \mathfrak{m}(V)$, $v \in \mathrm{Ann}(\mathfrak{m}(V))$. Furthermore, for the highest-order homogeneous component $\widetilde{P}^{[m(n-2)]}$ of $\widetilde{P}$ one has $\widetilde{P}^{[m(n-2)]}(v) = 0$, $\widetilde{P}^{[m(n-2)]}(u+v) = \widetilde{P}^{[m(n-2)]}(u)$ for all $u \in \mathfrak{m}(V)$, $v \in \mathfrak{m}(V)^2$. Thus, $\widetilde{P}^{[m(n-2)]}$ gives rise to a form $\widehat{P}^{[m(n-2)]}$ of degree $m(n-2)$ on the m-dimensional space $\mathfrak{m}(V)/\mathfrak{m}(V)^2$. For any two nil-polynomials P, P' the corresponding forms $\widehat{P}^{[m(n-2)]}$, $\widehat{P}'^{[m(n-2)]}$ coincide up to scale (cf. Remark 3.3), and we say that any of the mutually proportional forms of degree $m(n-2)$ arising in this way is *associated* to the form Q. Clearly, for any absolute classical invariant $\mathcal{I}$ of forms of degree $m(n-2)$ on $\mathfrak{m}(V)/\mathfrak{m}(V)^2$ and any form $\mathbf{Q}$ associated to Q, the value $\mathcal{I}(\mathbf{Q})$ is a biholomorphic invariant of V. Note that invariants of this kind were first considered in article [4], where associated forms were introduced in slightly different terms. The Eastwood forms for $\widetilde{E}_6$ and $\widetilde{E}_7$ singularities are in fact forms associated to ternary cubics and binary quartics.

For convenience, we will now make a canonical choice of variables in $\mathfrak{m}(V)/\mathfrak{m}(V)^2$. Consider the factorization maps $\pi_1 : \mathcal{O}_m \to \mathcal{O}_m/\mathcal{J}(Q) = \mathcal{A}(V)$ and $\pi_2 : \mathfrak{m}(V) \to \mathfrak{m}(V)/\mathfrak{m}(V)^2$. Let e_j be the image of the germ of the coordinate function z_j under the composition $\pi_2 \circ \pi_1$, $j = 1, \dots, m$. Clearly, the vectors e_j form a basis in $\mathfrak{m}(V)/\mathfrak{m}(V)^2$, and we denote by $w_1, \dots, w_m$ the coordinates with respect to this basis. For an absolute classical invariant $\mathcal{I}$ of forms of degree $m(n-2)$ in the variables $w_1, \dots, w_m$ it is easy to observe that $\mathcal{I}(\mathbf{Q})$ is rational when regarded as a function of Q, with $\mathbf{Q}$ associated to $Q \in X_m^n$.

Let $\mathcal{R}_m^n$ denote the collection of all invariant rational functions on X_m^n obtained in this way. Further, let $\widehat{\mathcal{I}}_m^n$ be the algebra of the restrictions to X_m^n of *all* absolute invariants of forms of degree n on $\mathbb{C}^m$. Note that $\mathcal{R}_m^n$ lies in $\widehat{\mathcal{I}}_m^n$ (see Proposition 1 in [3]). We propose the following conjecture.

Conjecture 4.2. $\mathcal{R}_m^n = \widehat{\mathcal{I}}_m^n$.

Since every element of $\widehat{\mathcal{I}}_m^n$ can be represented as a ratio of two elements of $\mathcal{I}_m^n$ (see Proposition 6.2 in [19]), Conjecture 4.2 is equivalent to the statement $\mathcal{I}_m^n \subset \mathcal{R}_m^n$. It is clear from Example 3.4 that the conjecture does not hold for general quasi-homogeneous singularities.

Note that for binary quartics ($m = 2$, $n = 4$) and ternary cubics ($m = 3$, $n = 3$) one has $m(n-2) = n$, that is, in these cases $\deg \mathbf{Q} = \deg Q$ for any form $\mathbf{Q}$ associated to Q, whereas in all other situations one has $\deg \mathbf{Q} > \deg Q$. In each of these two exceptional cases Conjecture 4.2 states that every element of $\mathcal{I}_m^n$ can be recovered from some (possibly different) absolute invariant of forms of the same degree by applying them to associated forms. The above statement is essentially contained in the arguments given in [4] and in Section 3 of the present paper for singularities of types $\widetilde{E}_6$ and $\widetilde{E}_7$. In fact, Conjecture 4.2 is in part motivated by these examples.

Let $m = 2$, $n = 4$. Every non-zero square-free binary quartic is linearly equivalent to a binary quartic of the form

$$q_t(z_1, z_2) := z_1^4 + t z_1^2 z_2^2 + z_2^4, \quad t \neq \pm 2$$

(see pp. 277–279 in [6]). Note that the moduli algebra of the plane curve germ defined by q_t is isomorphic to that of the germ V_t from Example 3.2. Any form associated to q_t is again a binary quartic and is proportional to

$$\mathbf{q}_t(w_1, w_2) := t w_1^4 - 12 w_1^2 w_2^2 + t w_2^4,$$

which is the Eastwood form of the corresponding $\widetilde{E}_7$-singularity (cf. (3.4)). For $t \neq 0, \pm 6$ the quartic $\mathbf{q}_t$ is square-free; in this case the original quartic q_t is associated to $\mathbf{q}_t$, and it is reasonable to say that for $t \neq 0, \pm 2, \pm 6$ the quartics q_t and $\mathbf{q}_t$ are dual to each other.

The algebra of classical invariants of binary quartics is generated by certain invariants I_2 and I_3, where the subscripts indicate the degrees (see, e.g. pp. 101–102 in [6]). For a binary quartic of the form

$$Q(z_1, z_2) = a_4 z_1^4 + 6 a_2 z_1^2 z_2^2 + a_0 z_2^4$$

the values of the invariants I_2 and I_3 are computed as follows:

$$\mathrm{I}_2(Q) = a_0 a_4 + 3 a_2^2, \quad \mathrm{I}_3(Q) = a_0 a_2 a_4 - a_2^3, \tag{4.2}$$

and $\Delta(Q) = \mathrm{I}_2(Q)^3 - 27\,\mathrm{I}_3(Q)^2$. Define an absolute invariant of binary quartics as

$$\mathrm{J} := \frac{\mathrm{I}_2^3}{\Delta}. \tag{4.3}$$

The restriction $\mathrm{J}|_{X_2^4}$ generates the algebra $\mathcal{I}_2^4$, and we have

$$\mathrm{J}(q_t) = \frac{(t^2+12)^3}{108(t^2-4)^2}$$

(cf. [4] and Example 3.2 above).

Consider another absolute invariant of binary quartics:

$$\mathrm{K} := \frac{\mathrm{I}_2^3}{27\,\mathrm{I}_3^2}. \tag{4.4}$$

Then one obtains $\mathrm{K}(\mathbf{q}_t) = \mathrm{J}(q_t)$, and therefore $\mathrm{K}(\mathbf{Q}) = \mathrm{J}(Q)$ for any $Q \in X_2^4$ and any $\mathbf{Q}$ associated to Q. Thus, the absolute invariant K evaluated for associated quartics yields a generator of $\mathcal{I}_2^4$, which agrees with Conjecture 4.2.

Let $m = 3$, $n = 3$. Every ternary cubic with non-zero discriminant is linearly equivalent to a ternary cubic of the form

$$c_t(z_1, z_2, z_3) := z_1^3 + z_2^3 + z_3^3 + t z_1 z_2 z_3, \quad t^3 + 27 \neq 0$$

(see p. 401 in [25]). Note that the hypersurface germ defined by c_t is exactly the germ V_t from Example 3.1. Any form associated to c_t is again a ternary cubic and is proportional to

$$\mathbf{c}_t(w_1, w_2, w_3) := t w_1^3 + t w_2^3 + t w_3^3 - 18 w_1 w_2 w_3,$$

which is the Eastwood form of the corresponding $\widetilde{E}_6$-singularity (cf. (3.2)). For $t \neq 0$, $t^3 - 216 \neq 0$ one has $\Delta(\mathbf{c}_t) \neq 0$; in this case the original cubic c_t is associated to $\mathbf{c}_t$, and it is reasonable to say that for $t \neq 0$, $t^3 + 27 \neq 0$, $t^3 - 216 \neq 0$ the cubics c_t and $\mathbf{c}_t$ are dual to each other.

The algebra of classical invariants of ternary cubics is generated by certain invariants $\mathtt{I}_4$ and $\mathtt{I}_6$, where, as before, the subscripts indicate the degrees (see pp. 381–389 in [6]). For a ternary cubic of the form

$$Q(z_1, z_2, z_3) = az_1^3 + bz_2^3 + cz_3^3 + 6dz_1z_2z_3$$

the values of the invariants $\mathtt{I}_4$ and $\mathtt{I}_6$ are computed as follows:

$$\mathtt{I}_4(Q) = abcd - d^4, \quad \mathtt{I}_6(Q) = a^2b^2c^2 - 20abcd^3 - 8d^6,$$

and $\Delta(Q) = \mathtt{I}_6^2 + 64\mathtt{I}_4^3$. Define an absolute invariant of ternary cubics as

$$\mathtt{J} := \frac{\mathtt{I}_4^3}{\Delta}.$$

The restriction $\mathtt{J}|_{X_3^3}$ generates the algebra $\mathcal{I}_3^3$, and we have

$$\mathtt{J}(c_t) = -\frac{t^3(t^3 - 216)^3}{110592(t^3 + 27)^3}$$

(cf. [4] and Example 3.1 above). Observe that $\mathtt{J}(c_t) = j(Z_t)/110592$, where $j(Z_t)$ is the value of the j-invariant for the elliptic curve Z_t in $\mathbb{CP}^2$ defined by the cubic c_t (see, e.g. [13]).

Consider another absolute invariant of ternary cubics:

$$\mathtt{K} := \frac{1}{4096\,\mathtt{J}}.$$

Then one obtains $\mathtt{K}(\mathbf{c}_t) = \mathtt{J}(c_t)$, and therefore $\mathtt{K}(\mathbf{Q}) = \mathtt{J}(Q)$ for any $Q \in X_3^3$ and any $\mathbf{Q}$ associated to Q. Thus, the absolute invariant $\mathtt{K}$ evaluated for associated cubics yields a generator of $\mathcal{I}_3^3$, which again agrees with Conjecture 4.2.

As we have seen, verification of Conjecture 4.2 for binary quartics and ternary cubics is not hard. In [5] the conjecture was established for binary quintics ($m = 2$, $n = 5$) and binary sextics ($m = 2$, $n = 6$), which was much more involved computationally. The proof of Theorem 1.1 in the next section will provide additional evidence supporting Conjecture 4.2 for binary forms of an arbitrary degree.

5. The Family $\mathcal{V}_t$

Observe that the "if" implication in Theorem 1.1 is trivial since the curve $\{f_{\rho t}(z_1, z_2) = 0\}$, with $\rho^n = 1$, is biholomorphically equivalent to the curve $\{f_t(z_1, z_2) = 0\}$ by means of the map $z_1 \mapsto \rho z_1$, $z_2 \mapsto z_2$. In this section we will give two proofs of the "only if" implication. Our first proof is based on applying classical invariants directly to the forms f_t defined in (1.1), whereas our second proof proceeds along the lines of Conjecture 4.2 and is based on applying classical invariants to forms associated to f_t. Both our proofs are much easier than the proof by the direct substitution method given in [15].

We will now introduce the invariants required for our proofs. Let $Q \in \mathcal{Q}^n_{\mathbb{C}^2}$ be a binary form of any degree $n \geq 2$ written as

$$Q(z_1, z_2) = \sum_{i=0}^{n} \binom{n}{i} a_i z_1^i z_2^{n-i},$$

where $a_i \in \mathbb{C}$. The form Q can be represented as a product of linear terms

$$Q(z_1, z_2) = \prod_{\nu=1}^{n} (b_\nu z_1 - c_\nu z_2),$$

for some $b_\nu, c_\nu \in \mathbb{C}$. The discriminant of Q is then given by

$$\Delta(Q) = \frac{(-1)^{n(n-1)/2}}{n^n} \prod_{1 \leq \alpha < \beta \leq n} (b_\alpha c_\beta - b_\beta c_\alpha)^2$$

(see pp. 97–101 in [6]). The discriminant is a relative invariant of degree $2(n-1)$ which is non-zero if and only if Q is non-zero and square-free. Furthermore, if $a_n \neq 0$, the discriminant $\Delta(Q)$ can be computed as

$$\Delta(Q) = \frac{\mathbf{R}(Q, \partial Q/\partial z_1)}{n^n a_n}, \tag{5.1}$$

where for two forms P and S we denote by $\mathbf{R}(P, S)$ their resultant (see p. 36 in [20]).

Next, define the nth *transvectant* as

$$(Q,Q)^{(n)} := (n!)^2 \sum_{i=0}^{n} (-1)^i \binom{n}{i} a_i a_{n-i}$$

(see Chapter 5 in [20]). The transvectant $(Q,Q)^{(n)}$ is an invariant of degree 2. It is identically zero if n is odd, thus for any odd n we consider the invariant $\left(Q^2,Q^2\right)^{(2n)}$, which has degree 4. Observe that for the relative invariant I_2 of binary quartics defined in (4.2) one has $\mathrm{I}_2(Q) = (Q,Q)^{(4)}/1152$.

We now introduce an absolute invariant of binary forms of degree n as follows:

$$J(Q) := \begin{cases} \dfrac{\left[(Q,Q)^{(n)}\right]^{n-1}}{\Delta(Q)} & \text{if } n \text{ is even,} \\[2ex] \dfrac{\left[\left(Q^2,Q^2\right)^{(2n)}\right]^{(n-1)/2}}{\Delta(Q)} & \text{if } n \text{ is odd.} \end{cases} \tag{5.2}$$

Notice that for the absolute invariant J of binary quartics defined in (4.3) one has $\mathrm{J} = J/1152^3$. Next, for even values of n we introduce the following absolute invariant of binary forms of degree n:

$$K(Q) := \frac{(\mathbf{H}(Q),\mathbf{H}(Q))^{(2(n-2))}}{\left[(Q,Q)^{(n)}\right]^2},$$

where $\mathbf{H}(Q)$ is the Hessian of Q. Note that $\mathbf{H}(Q) \in \mathcal{Q}_{\mathbb{C}^2}^{2(n-2)}$, and the relative invariant $(\mathbf{H}(Q),\mathbf{H}(Q))^{(2(n-2))}$ has degree 4.

***Proof* 1.** In our first proof of the "only if" implication in Theorem 1.1 we find $J(f_t)$, where f_t is the binary form defined in (1.1). A straightforward computation yields

$$\begin{aligned} (f_t,f_t)^{(n)} &= 2(n!)^2 && \text{if } n \text{ is even,} \\ (f_t^2,f_t^2)^{(2n)} &= 2(2n)!\Big((2n)! - 2(n!)^2\Big) && \text{if } n \text{ is odd.} \end{aligned} \tag{5.3}$$

Therefore, the numerators in (5.2) do not depend on t and are non-zero. We will now compute the discriminant $\Delta(f_t)$. Since for f_t we have $a_n = 1$, one can apply formula (5.1). The resultant $\mathbf{R}(f_t, \partial f_t/\partial z_1)$ can be easily found by using cofactor expansions, and we get $\mathbf{R}(f_t, \partial f_t/\partial z_1) = (1-n)^{n-1}t^n + n^n$. Hence

$$\Delta(f_t) = (1-n)^{n-1}t^n/n^n + 1. \tag{5.4}$$

Formulas (5.3) and (5.4) imply

$$J(f_t) = \frac{1}{\mu t^n + \nu}$$

for some $\mu, \nu \in \mathbb{C}$ with $\mu \neq 0$, which yields the desired result. □

***Proof* 2.** In our second proof of the "only if" implication in Theorem 1.1 we first assume that $n \geq 5$ and find the value of the absolute invariant K for forms associated to f_t. Any such form has degree $2(n-2)$ and is proportional to

$$\begin{aligned}
\mathbf{f}_t(w_1, w_2) \; := & \sum_{j=n-1}^{2(n-2)} \binom{2(n-2)}{j} \left(\frac{(1-n)t}{n}\right)^{j+2-n} w_1^j w_2^{2(n-2)-j} \\
& + \frac{(n-1)t^2}{n^2} \sum_{j=n-1}^{2(n-2)} \binom{2(n-2)}{j} \left(\frac{(1-n)t}{n}\right)^{2(n-2)-j} w_1^{2(n-2)-j} w_2^j \\
& + \binom{2(n-2)}{n-2} w_1^{n-2} w_2^{n-2}.
\end{aligned}$$

A straightforward computation yields

$$\begin{aligned}
(\mathbf{f}_t, \mathbf{f}_t)^{(2(n-2))} &= ((2(n-2))!)^2 \binom{2(n-2)}{n-2} \Delta(f_t), \\
(\mathbf{H}(\mathbf{f}_t), \mathbf{H}(\mathbf{f}_t))^{(2(2n-6))} &= \Delta(f_t)^2(\rho\Delta(f_t) + \sigma)
\end{aligned} \tag{5.5}$$

for some $\rho, \sigma \in \mathbb{C}$ with $\rho \neq 0$. The expressions in (5.5) imply

$$K(\mathbf{f}_t) = \rho'\Delta(f_t) + \sigma' = \rho'' t^n + \sigma'' \tag{5.6}$$

for some $\rho', \sigma', \rho'', \sigma'' \in \mathbb{C}$ with $\rho', \rho'' \neq 0$, which leads to the desired result for $n \geq 5$.

Finally, as shown in Section 4, for $n = 4$ we have $1152^3\, \mathrm{K}(\mathbf{f}_t) = J(f_t)$, where K is the absolute invariant of binary quartics defined in (4.4). This completes the proof. □

Remark 5.1. It is clear from (5.6) that for $n \geq 5$ and suitable $a, b \in \mathbb{C}$ the absolute invariant

$$K'(Q) := \frac{\Big[(Q,Q)^{(2(n-2))}\Big]^2}{a(\mathbf{H}(Q), \mathbf{H}(Q))^{(2(2n-6))} + b\Big[(Q,Q)^{(2(n-2))}\Big]^2}$$

of forms of degree $2(n-2)$ has the property $K'(\mathbf{f}_t) = J(f_t)$. Thus, as claimed above, the example of the family $\mathcal{V}_t$ indeed supports Conjecture 4.2.

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