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TOWER OF COVERINGS AND COMPLEX STRUCTURES ON FOUR DIMENSIONAL MANIFOLDS

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Abstract

We study a tower of coverings of a real four dimensional manifold and its relations to properties such as complex structures and existence of meromorphic functions. This allows us revisit the question of existence of a complex structure on an almost complex surface from a perspective different from the usual one. A consequence is the existence of an almost complex surface with any prescribed set of Chern numbers c_1^2 and c_2 satisfying the Noether relation but supporting no integrable almost complex structure.

1. Introduction

1.1. A natural question in complex geometry is to understand the difference between existence of almost complex structures and existence of complex structures on a differentiable manifold of even dimension. It follows from the well-known work of Van de ven [7] that there exist lots of four dimensional differentiable manifolds supporting almost complex structure but carrying no complex structure, which means integrable almost complex structure in this article. The situation is essentially completely unknown in higher dimensions. One of the main difficulties dealing with higher dimensional varieties is that there may not be any meromorphic functions and we do not have Kähler conditions in general for a hypothetical complex structure, which make the usual algebraic geometric or differential geometric methods difficult to apply.

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The article grows out of a modest attempt to understand such a problem. We explore some approach different from the usual one to the problem, via the study of a tower of coverings, which allows us to come up with a sufficient condition to construct meromorphic functions on complex manifolds, which by itself is of independent interest. However for the problem related to the existence of complex structure mentioned above, we have concrete results only in real dimension four.

Here is a brief summary of our results. We give a new criterion on existence of non-trivial meromorphic functions in real dimension four. In particular, we study from the point of view of a tower of coverings, when they exist. This is stated as Theorem 1. A criterion such as Theorem 1 allows us to produce rather easily examples of almost complex surfaces with no integrable almost complex structure. In fact, we use the approach to produce for each pair of admissible numbers (p, q) an almost complex manifold of complex dimension two with $c_1^2 = p, c_2 = q$ but carries no integrable almost complex structure. We refer the readers to Theorem 2 for details and remarks. Though probably not to be unexpected, to the knowledge of the author, the result has not been stated before. For a detailed description as stated in Theorem 2, we need results from classification of surfaces.

1.2. A differentiable manifold M of even dimension is said to carry an almost complex structure if the tangent bundle carries a complex structure which is differentially compatible but may not be holomorphically compatible with respect to an atlas of coordinate charts. A differentiable manifold of real dimension $2n$ is said to carry a complex structure if it can be covered by differentially compatible local charts which are homeomorphic to an open sets in $\mathbb{C}^n$ and the transition functions are holomorphic with respect to the local coordinates. The restriction of a complex structure to the tangent bundle yields naturally an almost complex structure. If an almost complex structure is the restriction of a complex structure from the manifold, we say that the almost complex structure is integrable. In such a case, we call the manifold a complex manifold. A complex manifold is projective algebraic, or in short, projective, if it can be realized as a complex analytic subvariety of some projective space $P^N_{\mathbb{C}}$.

A classifical result of Kodaira (cf. [1]) states that a smooth complex surface M supporting a holomorphic line bundle L with $c_1(L)^2 > 0$ is projective algebraic. The first result of this article is to give a somewhat different criterion to conclude positive algebraic dimension for a complex surface different from the one of Kodaira, in terms of Todd genus and existence of a tower of coverings of the manifold.

Definition 1. We say that M supports an infinite tower of coverings if there exists an infinite sequence of finite, non-identity, unramified coverings $\cdots \to M_{i+1} \to M_i \cdots \to M_2 \to M_1$ with $M_1 = M$.

Remark.

(1) Note that the definition here is more general than the usual definition of a tower of coverings as in [9]. We do not assume that the sequence M_i has to approach to the universal covering $\widetilde{M}$ as i tends to infinity.

(2) There are plenty of examples of M supporting an infinite tower of coverings. Here are some explicit ones.

   (a) The fundamental group $\pi_1(M)$ is residually finite. In such a case, we just take the coverings corresponding to an infinite tower of normal subgroups of $\pi_1(M)$. The intersection of the fundamental groups regarded as a group of deck transformations on the universal covering approaches to the trivial group..

   (b) The first Betti number $b_1(M) > 0$. In such a case, there is a non-trivial homomorphism $\rho : \pi_1(M) \to \mathbb{Z}$. We just take normal coverings corresponding to the kernel of $\rho_i : \pi_1(M) \to \mathbb{Z} \to \mathbb{Z}_{p_i}$, where $\mathbb{Z}_{p_i}$ is reduction mod(p_i) of a nested sequence of ideals p_i, $p_{i+1} \subset p_i$, $i = 1, 2, \ldots$.

Theorem 1. *Let M be a complex surface with Todd genus $c_1^2 + c_2 \neq 0$ supporting an infinite tower of coverings. Then the algebraic dimension of M is positive.*

1.3. Theorem 1 allows us to derive some simple criterion to construct examples of four dimensional manifolds supporting almost complex structure but no complex structure by attaching one or two standard compact manifolds.

Theorem 2. *Let (p, q) be any pair of integers satisfying $p+q \equiv 0 \pmod{12}$. Then there exists a compact connected differentiable four-manifold M supporting an almost complex structure with the first and second Chern numbers given by p and q respectively, but does not support any complex structure.*

More precisely, M can be chosen as follows. Let $M_{a,b,c,\alpha} = (\sharp_a P^2_{\mathbb{C}})\sharp(\sharp_b(\overline{P}^2_{\mathbb{C}})$ $\sharp(\sharp_c(P^1_{\mathbb{C}} \times R))\sharp(\sharp_\alpha[(S^1 \times S^3)\sharp(P^1_{\mathbb{C}} \times P^1_{\mathbb{C}})])$ be the direct sum of a copies of $P^2_{\mathbb{C}}$, b copies of $\overline{P}^2_{\mathbb{C}}$ which as a differentiable manifold is $P^2_{\mathbb{C}}$ with opposite orientation, c copies of $P^1_{\mathbb{C}} \times R$ with R a Riemann surface of genus two, and α copies of $(S^1 \times S^3)\sharp(P^1_{\mathbb{C}} \times P^1_{\mathbb{C}})$. Then for an appropriate choice of positive integers $a \geqslant 1, b \geqslant 1, c \geqslant 2$, and $\alpha \geqslant 1$, $M_{a,b,c,\alpha}$ does not carry any complex structure but carries some almost complex structure $\mathcal{C}$, so that the Chern numbers of the resulting almost complex surface $M_{a,b,\alpha,\mathcal{C}}$ satisfies $c_1^2(M_{a,b,\alpha,\mathcal{C}}) = p$ and $c_2(M_{a,b,\alpha,\mathcal{C}}) = q$.

Remark. We recall that it is already proved in [7] that given any pair of integers (p, q) with $p + q \equiv 0 \pmod{12}$, there exists a compact connected almost complex manifold with Chern numbers (c_1^2, c_2) given by the pair. Moreover, suppose $p \leqslant 2q$, there is a compact complex manifold with Chern numbers given by the pair. Theorem 2 above shows that the same underlying differentiable manifold satisfying both properties exists for each admissible pair of integers satisfying $p + q \equiv 0 \pmod{12}$.

1.4. The organization of the paper goes as follows. In §2, a criterion for a complex manifold to have positive algebraic dimension is proved. Theorem 1 in terms of non-vanishing of the Todd genus is a consequence of the criterion. To prepare for a proof of Theorem 2, we prove Proposition 4 in §3, which gives a criterion to modify an existing complex manifold to another real four manifold with almost complex structure. In this case, appropriate disjoint union allows us to avoid the constraint that the Todd genus has to be non-trivial. In §4, we apply the discussions in the earlier sections to construct almost complex manifolds with no integrable complex structure and prescribed Chern numbers, leading to a proof of Theorem 2.

1.5. It is a pleasure for the author to thank the referee for helpful comments on the article.

2. Criterion for Positivity of the Algebraic Dimension

2.1. We consider first a general observation on the algebraic dimension of a complex manifold.

Proposition 1. *Let M be a complex manifold. Let $\cdots \to M_i \to \cdots \to M_2 \to M_1 = M$ be an infinite tower of normal coverings of a complex manifold M of complex dimension $n > 0$. Let $N_i = [M_i : M_1]$ be the index of the covering. Suppose that $h^0(M_i, K_{M_i}) \geq cN_i$ for some constant $c > 0$ independent of i. Then M has positive algebraic dimension. Suppose furthermore that $m = 2$. Then M is either projective algebraic or is an elliptic fibration.*

2.2. From an infinite tower of coverings $\cdots \to M_3 \to M_2 \to M_1 = M$, we let $\cap_i \pi_1(M_i) = \Gamma_\infty$ and denote by $M_\infty = \widetilde{M}/\Gamma_\infty$.

Lemma 1. *Let $N_i = [M_i : M_1]$ be the index of the coverings. Suppose that $h^0(M_i, K_{M_i}) \geq cN_i$ for some constant $c > 0$ independent of i. Then the reduced L^2 cohomology group $H^0_{(2)}(M_\infty, K_{M_\infty}) \neq \emptyset$.*

Proof. The covering M_∞ is a geodesically complete non-compact Riemannian manifold which is a covering of M_i for all i. In general, let $f_l, l = 1, \ldots, p_g(X)$ be an orthonormal base of $H^0(X, K_X)$ of a manifold X. Let $B_X(x) = \sum_{l=1}^{p_g(X)} |f_l(x)|^2$ be the trace of the Bergman kernel. Let F be a fundamental domain of M on M_i. It follows that

$$\begin{aligned}\int_F \sup_{f \in H^0(M_i, K_{M_i}), \|f\|=1} |f(x)|^2 &= \frac{1}{N_i} \int_{M_i} \sup_{f \in H^0(M_i, K_{M_i}), \|f\|=1} |f(x)|^2 \\ &= \frac{1}{N_i} \int_{M_i} B_{M_i}(x) \\ &= \frac{1}{N_i} h^0(M_i, K_{M_i}) \\ &\geq c\end{aligned}$$

Hence there exists $x_i \in F$ and $f_i \in H^0(M_i, K_{M_i})$ of L^2-norm 1 such that $|f_i(x_i)| \geq c$. Standard uniform convergence on compacta argument implies that some subsequence of f_i converges to give a non-trivial L^2 $(2, 0)$-form f_∞ on M_∞. Regularity results for elliptic operators implies that f_∞ is actually smooth (cf. [9, p.203]). □

2.3.

Lemma 2. *Suppose that the space $H^0_{(2)}(M_\infty, K_{M_\infty})$ of L^2-holomorphic sections of the canonical line bundle K_{M_∞} on M_∞ is non-trivial. Then the algebraic dimension of M is positive.*

Sketch of Proof. We recall the well-known approach to construct a non-trivial meromorphic function on M. Non-emptiness of $H^0_{(2)}(M_\infty, K_{M_\infty})$ implies that it is actually infinite dimension, by considering $\gamma^* f$ for $\gamma \in \Gamma_\infty$ and $f \in H^0_{(2)}(M_\infty, K_{M_\infty})$. Hence $f^2 \in L^1(M_\infty)$. By considering the Poincaré series of f_i^2 for some $f_i \in H^0_{(2)}(M_\infty, K_{M_\infty})$, we get a holomorphic section of $H^0(M, K_M^2)$. Since M is compact, there exists a constant r such that for each point p on M_∞, there exists a geodesic ball of geodesic radius r centered at p. It follows easily that there exists a constant $c > 0$ such that for a holomorphic section $f \in H^0_{(2)}(M_\infty, K_{M_\infty})$, $|f(p)|^2 \leq c\int_{B(p,r)} |f|^2 \leq c\|f\|^2$, the L^2-norm of f (cf. [9]). Hence f is bounded pointwise on M_∞ and therefore $f^l \in H^0_{(2)}(M_\infty, K^l_{M_\infty})$ for all $l \geq 1$. It follows that for any $f_1, \ldots, f_k \in H^0_{(2)}(M_\infty, K_{M_\infty})$, $f_1^2 \cdots f_k^2$ is an L^1-holomorphic section of $K^{2k}_{M_\infty}$. The Poincaré series $\sum_{\gamma\in\Gamma} \gamma^*(f_1^2 \cdots f_k^2)$ is then a holomorphic section of $H^0(M, K_M^{2k})$. It is known that for k sufficiently large, we may choose $g_1, \cdots, g_k \in H^0_{(2)}(M_\infty, K_{M_\infty})$ such that $\sum_{\gamma\in\Gamma} \gamma^*(g_1^2 \cdots g_k^2) \neq 0$ and is not proportional to $\sum_{\gamma\in\Gamma} \gamma^*(f_1^2 \cdots f_k^2)$ by a constant, cf. [4]. In such case, the quotient $\frac{\sum_{\gamma\in\Gamma} \gamma^*(f_1^2 \cdots f_k^2)}{\sum_{\gamma\in\Gamma} \gamma^*(g_1^2 \cdots g_k^2)}$ is a meromorphic function on M. This concludes the proof of Lemma 2. □

2.4.

***Proof of Proposition* 1.** From the lemmas above, we know that the algebraic dimension of M is positive. Hence it suffices for us to consider the case that M has complex dimension 2. Note that $h^0_{(2)}(\widetilde{M})$ is either trivial or infinite dimensional. In this case, the algebraic dimension of M is either 1 or 2.

Suppose that the algebraic dimension is 1. Then there exists a fibration $\pi : M \to C$ onto a algebraic curve. According to Proposition 3.1 in Chapter VI of [1], we know that M is a minimal properly elliptic surface, $p : M \to C$, where C is an algebraic curve of genus g.

In the case that the dimension of M is 2, It follows from a well-known result of Chow-Kodaira (cf. [1, p.139]) that existence of two algebraically independent meromorphic functions on a compact complex surface implies that the surface is projective algebraic.

This concludes the proof of Proposition 1. □

2.5. The results above essentially help us to handle the case of $c_1^2(M) + c_2(M) > 0$, see **2.6** below. We need another criterion for the projective algebraicity of surfaces in case that the characteristic number is negative.

Proposition 2. *Suppose that a complex surface M supports an infinite tower of coverings and satisfies the condition that $c_1^2(M) + c_2(M) < 0$, Then M has positive algebraic dimension.*

Proof. Assume for the sake of proof by contradiction that M is not algebraic. The Riemann-Roch Theorem tells us that

$$h^0(M, \mathcal{O}) - h^1(M, \mathcal{O}) + h^2(M, \mathcal{O}) = \frac{1}{12}(c_1(M)^2 + c_2(M)).$$

Since the characteristic number on the right hand side is multiplicative with respective to the index of an unramified covering, we know that

$$h^0(M', \mathcal{O}) - h^1(M', \mathcal{O}) + h^2(M', \mathcal{O}) = \frac{[M' : M]}{12}(c_1(M)^2 + c_2(M))$$

for an unramified covering M' of M. It follows that

$$h^1(M', \mathcal{O}) = -\frac{[M' : M]}{12}(c_1(M)^2 + c_2(M)) + h^2(M') - 1.$$

Suppose that $[M' : M] \geq 36$. We have $h^1(M') \geq 3$ since $c_1(M)^2 + c_2(M) < 0$. For a complex surface M, we have the well-known relation that $h^1(M, \mathcal{O}) = h^{0,1}(M) = h^{1,0}(M) + \epsilon$, where $\epsilon = 0$ if M is Kähler and $\epsilon = 1$ if otherwise (cf. [1]). We conclude that $h^{1,0}(M') \geq c[M' : M]$ for some constant $c > 0$ independent of M'. Applying the discussions of Lemma 1 to the tower of coverings, we conclude that on M_∞, the space of L^2-holomorphic one forms $H^0_{(2)}(M, \Omega)$ is non-trivial and hence infinite dimensional. It follows from the arguments of Gromov [4] as in Lemma 2 again that M has positive algebraic dimension. □

2.6. We can now conclude the proof of Theorem 1.

***Proof of Theorem* 1.** Suppose that $(c_1(M)^2 + c_2(M)) > 0$, the hypothesis of Theorem 1 and Riemann-Roch shows that the conditions of Proposition 1 are satisfied. Similarly in the case of $(c_1(M)^2 + c_2(M)) < 0$, Proposition 2 applies. This concludes the proof of Theorem 1. □

3. Construction of Non-Integrable Almost Complex Structures

3.1. We recall the following well-known criterion of Wu (cf. [1]), where we denote by $\chi(M), \tau(M)$ and $w^2(M) \in H^2(M, \mathbb{Z}_2)$ the Euler characteristic, topological index and second Stiefel-Whitney class of a four dimensional manifold M respectively.

Proposition 3 (Wu). *Let M be a four dimensional orientable manifold. Let θ be a cohomology class in $H^2(M, \mathbb{Z})$ satisfying $\theta \equiv w^2(M) \pmod 2$ and $\theta^2 = 3\tau(M) + 2\chi(M)$. Then there exists an almost complex structure $\mathcal{C}$ on M with $c_1((M, \mathcal{C})) = \theta$.*

3.2. The following result is a preparation for the proof of Theorem 2 in the next section.

Proposition 4. *Let M be a complex surface with characteristic numbers $c_1^2(M)$ and $c_2(M)$. Let M_α denote the direct sum of M, α copies of $S^1 \times S^3$ and also $(P^1_{\mathbb{C}} \times P^1_{\mathbb{C}})$, representing $M\sharp(\sharp_\alpha[(S^1 \times S^3)\sharp(P^1_{\mathbb{C}} \times P^1_{\mathbb{C}})])$. Then*

(a) *M_α has the same Euler number and index as M.*

(b) *M_α supports a non-trivial almost complex structure for all α.*

(c) *Suppose α is chosen such that $b_1(M_\alpha)$ is odd. Then an integrable complex structure on M_α has to come from an elliptic surface.*

3.3. Proof of Proposition 4

In our notation, $M_\alpha = M\sharp(\sharp_\alpha[(S^1 \times S^3)\sharp(P^1_{\mathbb{C}} \times P^1_{\mathbb{C}})])$. The two cocycle $c_1(M)$ on M induces a two cycle θ on M_α.

Recall that for a connected sum $X = M_1\sharp M_2$ of connected manifolds M_1 and M_2, where $\dim(M_1) = \dim(M_2) = n$, the Betti numbers satisfy $h^i(X) = h^i(M_1) + h^i(M_2)$ for $1 \le i \le n-1$ and $h^0(X) = h^n(X) = 1$. It follows easily that

$$\begin{aligned}
b_1(M_\alpha) &= b_1(M) + \alpha b_1(S^1 \times S^3) = b_1(M) + \alpha, \\
b_2(M_\alpha) &= b_2(M) + \alpha b_2(P^1_{\mathbb{C}} \times P^1_{\mathbb{C}}) = b_2(M) + 2\alpha, \\
b_3(M_\alpha) &= b_3(M) + \alpha b_3(S^1 \times S^3) = b_3(M) + \alpha,
\end{aligned}$$

Hence

$$\chi(M_\alpha) = \sum_i (-1)^i b_i(M_\alpha) = \sum_i (-1)^i b_i(M) = \chi(M).$$

On the other hand, $\tau(S^1 \times S^3) = 0$ as $b^2(S^1 \times S^3) = 0$, and $\tau(P^1_{\mathbb{C}} \times P^1_{\mathbb{C}}) = 0$ as the bilinear form on $H^2(P^1_{\mathbb{C}} \times P^1_{\mathbb{C}})$ has signature $(1,1)$. It follows that $\tau(M_\alpha) = \tau(M)$. Hence (a) is proved.

Since $w^i(S^n) = 0$ for $1 \le i \le n$ and $n = 1, 2, 3$, we know that $w^2(M_\alpha) = w^2(M)$. Hence from the fact that $c_1(M) \equiv w^2(M)$, we conclude that $\theta \equiv w^2(M_\alpha) \pmod 2$. From Proposition 3, there exists an almost complex structure on M_α with $c_1(M_\alpha) = \theta$. This concludes the proof of parts (b).

Consider now part (c). Assume that $\mathcal{C}$ is an almost complex structure on M_α. For simplicity, we write $M_\alpha = M_{\alpha-1}\sharp(S^1 \times S^3 \sharp P^1_{\mathbb{C}} \times P^1_{\mathbb{C}})$. Let $\rho : \pi_1(M_\alpha) \to \mathbb{Z}$ be the natural homomorphism obtained projection into the π_1 of the last factor $(S^1 \times S^3)\sharp(P^1_{\mathbb{C}} \times P^1_{\mathbb{C}})$. Let A_i be the kernel the homomorphism ρ_i obtained from the composition of ρ and projection map $\mathbb{Z} \to p^i\mathbb{Z}$ for some prime ideal p of $\mathbb{Z}$. There exists an infinite tower of coverings $M_{\alpha,i}$ with $\pi_1(M_{\alpha,i}) = \pi_1(M_{\alpha,i})/\ker(\rho_i)$ associated to the homomorphism ρ_i as mentioned in the Remark 2 after Theorem 1. If $c_1^2(M_\alpha) + c_2(M_\alpha) \neq 0$, since $b_1(M_\alpha)$ is odd, Theorem 1 and Proposition 1 imply that M_α is not integrable unless it is an elliptic surface.

Hence we need only to consider the case that $c_1^2(M_\alpha) + c_2(M_\alpha) = 0$ in the following discussions. In this case, we have on $M_{\alpha,i}$,

$$\begin{aligned} b^+(M_{\alpha,i}) - b^-(M_{\alpha,i}) &= \frac{[M_{\alpha,i}, M_\alpha]}{3}(c_1^2(M_\alpha) - 2c_2(M_\alpha)) = -[M_{\alpha,i}, M_\alpha]c_2(M_\alpha) \\ b^+(M_{\alpha,i}) + b^-(M_{\alpha,i}) &= [M_{\alpha,i}, M_\alpha]c_2(M_\alpha) - 2 + 2b_1(M_{\alpha,i}). \end{aligned}$$

It follows that $b^+ = b_1(M_{\alpha,i}) - 1$ and hence

$$p_g = \frac{1}{2}(b^+ - \epsilon) = \frac{1}{2}(b_1(M_{\alpha,i}) - 1 - \epsilon),$$

where ϵ is either 1 or 0. Since $\alpha \geqslant 5$, the proportion of b_1 on $M_{\alpha,i}$ coming from homology classes in the pull-back of $M_{\alpha-1}$ to the one from M_α is the same as the ratio $\frac{b_1(M_{\alpha-1})}{b_1(M_\alpha)}$ and is greater than 0. Hence we conclude that

$$b_1(M_{\alpha,i}) \ \geq \ \frac{b_1(M_{\alpha-1})}{b_1(M_\alpha)} b_1(M_{\alpha,i})$$

$$= \frac{b_1(M_{\alpha-1})}{b_1(M_\alpha)} b_1(M_\alpha)[M_{\alpha,i} : M_\alpha].$$

It follows that there exists a positive constant $c > 0$ such that $p_g(M_{\alpha,i}) \geq c[(M_{\alpha,i}) : M_\alpha]$. Again, Theorem 1 and Proposition 1 imply that M_α is not integrable unless M is a elliptic surface. This concludes the proof of (c). □

4. Geography of Almost Complex Surfaces with No Integrable Complex Structure

4.1. The purpose of this section is to give a proof of Theorem 2, which allows us to find a fourfold equipped with an almost complex structure of any admissible prescribed pair of Chern numbers but does not support an integrable complex structure. Note that the Chern numbers of any almost complex surface satisfy the Noether relation $\theta \cdot \theta + \chi(M) \equiv 0 \pmod{12}$.

4.2. Proof of Theorem 2

Since the argument is a bit long, we summarize our construction as follows. Given any pair of integers p, q satisfying $p+q \equiv 0 \pmod{12}$, we choose c a positive integer such that $a = \frac{1}{6}(p+q)+3c-1$ and $b = \frac{1}{6}(5q-p)+3c-1$ are positive integers. Let $M_{a,b,c} = (\sharp_a P^2_{\mathbb{C}})\sharp(\sharp_b \overline{P}^2_{\mathbb{C}})\sharp(\sharp_c(P^1_{\mathbb{C}} \times R))$. Here R is a genus two algebraic curve, $\overline{P}^2_{\mathbb{C}}$ is the underlying differentiable manifold of $P^2_{\mathbb{C}}$ with opposite orientation, and p_i for $i = 1, 2$ denotes projection into the i-th factor of a product. We have also used the following notation. Denote by H_N the positive generator of the Neron-Severi group on a Riemann surface N. In the case of $\overline{P}^2_{\mathbb{C}}$, we use the same notation to denote the two cycle corresponding to the hyperplane line bundle on the underlying surface $P^2_{\mathbb{C}}$. We also use $H_{p_1(P^1_{\mathbb{C}}\times R)}$ and $H_{p_2(P^1_{\mathbb{C}}\times R)}$ to denote the two cycles of the Chern classes of the hyperplane line bundles on the first factor $P^1_{\mathbb{C}}$ and the second factor R of the product $P^1_{\mathbb{C}} \times R$ respectively, where we choose $P^1_{\mathbb{C}} \times R$ to be the first component of the connected sum $\sharp_c(P^1_{\mathbb{C}} \times R)$. Note that similar type of manifolds have already been utilized in Van der Ven (cf. [7], [1]). For a positive integer α as used in Proposition 4, let $M_{a,b,c,\alpha} = M_{a,b,c}\sharp(\sharp_\alpha[(S^1 \times S^3))\sharp(P^1_{\mathbb{C}} \times P^1_{\mathbb{C}})])$. Note that $M_{a,b,c,\alpha}$ has the same Euler characteristic, index and the second Stiefel-Whitney class as $M_{a,b,c}$. Let θ be the 2-cocycle on $M_{a,b,c,\alpha}$ induced by $\sum_{i=1}^{a/2}[H_{P^2_{\mathbb{C},2i-1}} + 3H_{P^2_{\mathbb{C},2i}}] + \sum_{j=1}^{b}[H_{\overline{P}^2_{\mathbb{C},j}}] + [2H_{p_1(P^1_{\mathbb{C}}\times R)} + (1-3c)H_{p_2(P^1_{\mathbb{C}}\times R)}]$ defined on $M_{a,b,c}$. Then $\theta^2 = 3\tau(M_{a,b,c,\alpha}) + 2\chi(M_{a,b,c,\alpha})$ and $\theta \equiv w^2(M_{a,b,c,\alpha})$

(mod 2). In this way, we will show that $M_{a,b,c,\alpha}$ supports an almost complex structure with Chern numbers given by p and q according to Proposition 3, and it will not support any integrable complex structure according to Theorem 1, Proposition 4 and their consequences.

4.3. In the following we will implement the outline above. We begin with some computations. It is easy to see that in terms of the positive generator of the Neron-Severi group,

$$c_1(P^2_{\mathbb{C}}) = 3H_{P^2_{\mathbb{C}}}, c_2(P^2_{\mathbb{C}}) = 3H_{P^2_{\mathbb{C}}} \cdot H_{P^2_{\mathbb{C}}}.$$

Hence

$$\chi(P^2_{\mathbb{C}}) = 3, \tau(P^2_{\mathbb{C}}) = 1.$$

Since $\overline{P}^2_{\mathbb{C}}$ is just the same underlying differentiable manifold with opposite orientation, it is clear that

$$\chi(\overline{P}^2_{\mathbb{C}}) = 3, \tau(\overline{P}^2_{\mathbb{C}}) = -1.$$

For $P^1_{\mathbb{C}} \times R$, the total Chern class is given by $(1 + 2H_{P^1_{\mathbb{C}}})(1 - 2H_R)$. Hence

$$\begin{aligned} c_1(P^1_{\mathbb{C}} \times R) &= 2(H_{P^1_{\mathbb{C}}} - H_R), \\ c_2(P^1_{\mathbb{C}} \times R) &= -4H_{P^1_{\mathbb{C}}} \cdot H_R. \end{aligned}$$

We know that

$$\chi(P^1_{\mathbb{C}} \times R) = -4, \tau(P^1_{\mathbb{C}} \times R) = 0.$$

Hence for $M = (\sharp_a P^2_{\mathbb{C}})\sharp(\sharp_b \overline{P}^2_{\mathbb{C}})\sharp(\sharp_c (P^1_{\mathbb{C}} \times R))$,

$$\begin{aligned} \chi(M_{a,b,c}) &= a\chi(P^2_{\mathbb{C}}) + b\chi(\overline{P}^2_{\mathbb{C}}) + c\chi(P^1_{\mathbb{C}} \times R) - 2(a+b+c-1) = a+b-6c+2, \\ \tau(M_{a,b,c}) &= a\tau(P^2_{\mathbb{C}}) + b\tau(\overline{P}^2_{\mathbb{C}}) + c\tau(P^1_{\mathbb{C}} \times R) = a - b. \end{aligned}$$

Suppose p, q are integers satisfying $p + q \equiv 0 \pmod{12}$. We claim that there exists a $M_{a,b,c}$ with $\chi(M_{a,b,c}) = q$ and a two cocycle θ on $M_{a,b,c}$ satisfying $\theta = w^2(M_{a,b,c})$ and $\theta^2 = p \equiv 3\tau(M_{a,b,c}) + 2\chi(M_{a,b,c})$, so that there exists an almost complex structure on $M_{a,b,c}$.

In fact, $3\tau(M_{a,b,c}) + 2\chi(M_{a,b,c}) = 5a - b - 12c + 4$. Hence we require

$$\begin{aligned} p &= 5a - b - 12c + 4, \\ q &= a + b - 6c + 2. \end{aligned}$$

This is equivalent to

$$\begin{aligned} a &= \frac{1}{6}(p+q) + 3c - 1 \\ b &= \frac{1}{6}(5q-p) + 3c - 1. \end{aligned}$$

As $p + q \equiv 0 \pmod{12}$, we may assume that $p + q = 12k$. Hence

$$a = 2k + 3c - 1, b = q - 2k + 3c - 1.$$

If c is sufficiently large, we obviously have a solution in positive integers a, b. We choose c to be an odd positive integer and a to be an even positive integer. Then on $M_{a,b,c} = (\sharp_a P^2_{\mathbb{C}})\sharp(\sharp_b \overline{P}^2_{\mathbb{C}})\sharp(\sharp_c(P^1_{\mathbb{C}} \times R))$, we choose

$$\theta = \sum_{i=1}^{a/2}[H_{P^2_{\mathbb{C},2i-1}} + 3H_{P^2_{\mathbb{C},2i}}] + \sum_{j=1}^{b}[H_{\overline{P}^2_{\mathbb{C},j}}] + [2H_{p_1(P^1_{\mathbb{C}}\times R)} + (1-3c)H_{p_2(P^1_{\mathbb{C}}\times R)}].$$

Note that

$$\begin{aligned} (\sum_{i=1}^{a/2}(H_{P^2_{\mathbb{C},2i-1}} + 3H_{P^2_{\mathbb{C},2i}}))^2 &= 5a \\ (\sum_{j=1}^{b}(H_{\overline{P}^2_{\mathbb{C},j}}))^2 &= -b \\ (2H_{p_1(P^1_{\mathbb{C}}\times R)} + (1-3c)H_{p_2(P^1_{\mathbb{C}}\times R)})^2 &= 4 - 12c. \end{aligned}$$

From our choice of a, b, c and θ, it follows that $\chi(M_{a,b,c}) = q$ and

$$\theta^2 = 5a - b + 12c + 4 = p = 3\tau(M_{a,b,c}) + 2\chi(M_{a,b,c}).$$

Furthermore, we have

$$
\begin{aligned}
\theta &= \sum_{i=1}^{a/2}[H_{P^2_{\mathbb{C},2i-1}}+3H_{P^2_{\mathbb{C},2i}}]+\sum_{j=1}^{b}[H_{\overline{P}^2_{\mathbb{C},j}}]+[2H_{p_1(P^1_{\mathbb{C}}\times R)}+(1-3c)H_{p_2(P^1_{\mathbb{C}}\times R)}]\\
&\equiv \sum_{i=1}^{a/2}[H_{P^2_{\mathbb{C},2i-1}}+3H_{P^2_{\mathbb{C},2i}}]+\sum_{j=1}^{b}[H_{\overline{P}^2_{\mathbb{C},j}}] \pmod 2\\
&\equiv \sum_{i=1}^{a/2}[H_{P^2_{\mathbb{C},2i-1}}+H_{P^2_{\mathbb{C},2i}}]+\sum_{j=1}^{b}[H_{\overline{P}^2_{\mathbb{C},j}}] \pmod 2\\
&\equiv \sum_{i=1}^{a/2}[w^2(P^2_{\mathbb{C},2i-1})+w^2(P^2_{\mathbb{C},2i})]+\sum_{j=1}^{b}[w^2(\overline{P}^2_{\mathbb{C},j})]+\sum_{j=1}^{b}w^2(P_{\mathbb{C},j}\times R_j) \pmod 2,
\end{aligned}
$$

where we use the fact $(1-3c)$ is even from our choice of c, and the fact that $w^2(P_{\mathbb{C},j}\times R_j)\equiv 0 \pmod 2$.

Hence we manage to construct θ on an appropriate $M_{a,b,c}$ such that θ satisfies the criterion of Wu. Consider now the four-fold $M_{a,b,c,\alpha} = M_{a,b,c}\sharp(\sharp_\alpha[(S^1 \times S^3)\sharp(P^1_{\mathbb{C}}\times P^1_{\mathbb{C}})])$. As argued in Proposition 4, $M_{a,b,c,\alpha}$ satisfies the criterion of Wu in Proposition 3.

4.4. We are going to show that for appropriate choice of α, $M_{a,b,c,\alpha}$ supports an almost complex structure, but does not support any integrable almost complex structure.

First of all, choose α such that $b_1(M_{a,b,c,\alpha})$ is odd. Then there exists a non-trivial tower of coverings as in Theorem 1 and hence the algebraic dimension $a(M_{a,b,c,\alpha})$ of $M_{a,b,c,\alpha}$ is either 1 or 2. From Proposition 1, we know that $a(M_{a,b,c,\alpha}) = 2$ implies that $M_{a,b,c,\alpha}$ is projective algebraic, contradicting the fact that $b_1(M_{a,b,c,\alpha})$ is odd from our choice of the parameters. Hence we are left with the case that $a(M_{a,b,c,\alpha}) = 1$. As in Proposition 4, $M_{a,b,c,\alpha}$ is then an elliptic surface. Consulting Table 10 in Chapter VI of [1], we conclude that unless $b_1(M_{a,b,c,\alpha}) \leqslant 4$, $M_{a,b,c,\alpha}$ has to be a minimal proper elliptic surface. Choose constants $\alpha > 7$. We know that $b_1(M_{a,b,c,\alpha}) > 7$ and hence $M_{a,b,c,\alpha}$ is a minimal proper elliptic surface. Hence there is an elliptic fibration $p: M_{a,b,c,\alpha} \to C$. In this case, we conclude that

$$\pi_1(F) \xrightarrow{\beta} \pi_1(M_{a,b,c,\alpha}) \xrightarrow{\gamma} \pi_{1,orb}(C) \to 0.$$

Here F denotes a generic fiber of p, and $\pi_{1,orb}(C)$ denotes the orbifold fundamental group of C. Note that C has orbifold structure near the points corresponding to the singular fibers, cf. Theorem 2.9 in [2].

Since $\alpha > 7$ and F is an elliptic curve, we conclude from the above exact sequence that $b_1(C) > \alpha - 2 > 5$. Hence genus $g(C) \geqslant 3$. Note however from construction that $\pi_1(M_{a,b,c,\alpha})$ is the free product of c copies of $\pi_1(R)$, and α copies of $\mathbb{Z}$.

Let $R_1, \ldots, R_c$ be the copies of R in the c copies of $P^1_{\mathbb{C}} \times R$ in the definition of $M_{a,b,c,\alpha}$. Recall that $c \geqslant 2$. We claim that $\gamma(\pi_1(R_i)) = 0$ for at least one index i. Assume on the contrary that $\gamma(\pi_1(R_i)) \neq 0$ for all i. Then there exists a non-trivial smooth mapping from the Riemann surface R_i of genus 2 to another Riemann surface C of genus $g(C) \geqslant 3$. The mapping can be deformed to a non-trivial harmonic map f_i from the classical result of Eells and Sampson [3] with respect to the Poincaré metrics. There are two cases to consider, $\mathrm{rank}_{\mathbb{R}} f_i = 1$ or 2.

If $\mathrm{rank}_{\mathbb{R}} f_i = 1$, it is known that the image of f_i is simply a geodesic curve γ in C from a result of Sampson [8]. From the exact sequence of fundamental groups above, as the genus of R_i is 2, it follows that the kernel of $\gamma|_{P^1_{\mathbb{C}} \times R_i}$ must be accounted for from $\gamma(\pi_1(F))$. This however is possible only for one of the i's, since F is a connected elliptic curve.

Hence as $c \geqslant 2$, there exists at least one i with $\mathrm{rank}_{\mathbb{R}} f_i \geqslant 2$. For such an i, the absolute of the degree d_i of f_i is at least 1. From the result of Eells-Wood [EW], as

$$\chi(R_i) + |d_i| \cdot |\chi(C)| > 0,$$

we conclude that f_i is either holomorphic or anti-holomorphic. This however contradicts the usual Hurwitz Formula. Note that the non-existence of the mapping in the case of non-zero degree also follows from the classical result of Kneser [5]. The claim is proved.

It follows from the claim that for some $i = 1, \ldots, c$, $\pi_1(R_i)$ has to come from $\beta(\pi_1(F))$ in the above exact sequence of fundamental groups. However this contradicts the fact that F is an elliptic curve, which implies that $\pi_1(R_i)$ cannot be contained in $\beta(\pi_1(F))$. Note that $\beta(\pi_1(F))$ can only take the value of $\mathbb{Z}, \mathbb{Z}^2$ or the trivial group.

We conclude that by choosing $\alpha > 7$ and c sufficiently large and a, b as described earlier in **4.3**, the surface $M_{a,b,c,\alpha}$ cannot support any integrable almost complex structure. ☐

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