Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **8** (2013), No. 3, pp. 323-350

ISSUES IN DECONVOLUTION, FROM EUCLIDEAN SPACE TO THE HEISENBERG GROUP

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Abstract

We review some of the important results on deconvolution, particularly the multi-channel deconvolution problem, for the setting of Euclidean space, focusing on the central role of the Hörmander strongly coprime condition in this area of analysis. We then address the problem of deconvolution in the Heisenberg group setting, beginning with the results of [16]. We also extend the results of [16] to three solid tori, a higher dimensional analogue of the three squares considered in [9]. The work of [16] on multi-channel deconvolution is ongoing research, with several important issues still to be fully explored. We address some of these issues, with particular attention to extension of the strongly coprime condition. We also recall the related result of [16] providing a means to extend a deconvolution from a complex space to the Heisenberg group setting and consider a few extensions of this result.

1. Introduction

The Pompeiu problem is an area of integral geometry dealing with the issue of characterizing the conditions when a function is uniquely determined by integral averages over a set S or collection of sets $\{S_i\}_{i=1}^m$, assuming integrals over all translations and rotations. Consider a given set $S \subset X$, and let $\mathcal{J}$ represent the group of translations and rotations in X. If the vanishing of the integrals

$$\int_{\gamma \cdot S} f(x) d\mu_{\gamma S}(x) = 0 \qquad \text{for all } \gamma \in \mathcal{J}$$

Received February 06, 2013 and in revised form May 12, 2013.

AMS Subject Classification: Primary: 45E10, 45F05; Secondary: 43A80.

Key words and phrases: Pompeiu problem, deconvolution, Heisenberg group, Laguerre functions, Bessel functions, Gelfand transform, inversion.

where μ_S is area measure on the set S, implies that $f \equiv 0$, then the set S is said to possess the Pompeiu property. Variations in the conditions for the Pompeiu problem include the space of functions for f, the ambient space X, the group action used for the translations of the given set, and potential alternative weights on the set S. The Pompeiu problem seeks to characterize the conditions under which the Pompeiu property holds. In certain cases, especially where the set S or each set $\{S_i\}_{i=1}^m$ is rotation invariant, the integral conditions reduce to only translation of the given set(s). In this paper we deal with the cases where the ambient space X is Euclidean space, usually $\mathbf{C}^n$, in which case translations are by $\mathbf{w} \in \mathbf{C}^n$, or the Heisenberg group $\mathbf{H}^n$, where the translations are described below in Section 3. Here we define the Pompeiu transform $\mathcal{P}$ as

$$(\mathcal{P}f)(\gamma) = \int_{\gamma \cdot S} f(x) d\mu_{\gamma S}(x) \qquad \gamma \in \mathcal{J}.$$

In many of the cases we consider, we utilize only translations of a collection of sets $\{S_i\}_{i=1}^m$, in which case the transform reduces to

$$\begin{aligned} (\mathcal{P}f)(y) &= \left(\int_{\check{S}_1} f(y-x) d\mu_1(x), \ldots, \int_{\check{S}_m} f(y-x) d\mu_m(x) \right) \\ &= ((f * T_1)(y), \ldots, (f * T_m)(y)) \end{aligned}$$

where the T_j are defined by $\langle \phi, T_j \rangle = \int_{\check{S}_j} \phi(x) d\mu_j(x)$. Since the transform has reduced to translations by members $y \in X$, we have also given the expression in terms of convolutions. Looking at this transform as a system of convolution equations also gives insight into the approaches used for this type of problem and ties in to many interesting issues arising in this area of study. See also the survey [8] for additional insight and connections to interesting related topics. In this context, the Pompeiu property refers to injectivity of this integral transform and thus corresponds to unique characterization of the function f in terms of the given collection of integral information. For an injective integral transform, the inverse will exist, and our goal is to find a means to represent the inverse operator. As this Pompeiu transform has been represented as a collection of convolutions, $(f * T_1, \ldots, f * T_m)$, this issue becomes a question of deconvolution. The goal is to produce deconvolvers $(\nu_1, \ldots, \nu_m)$ that will allow recovery of the function f through deconvolution, i.e.

$$f * T_1 * \nu_1 + \cdots + f * T_m * \nu_m = f.$$

The multi-sensor deconvolution problem addresses conditions under which such $\nu_1, \ldots, \nu_m$ exist and also how to determine them for given $T_1, \ldots, T_m$. In the next section we briefly review some of the work on the multi-sensor deconvolution problem in Euclidean space and then in Section 3 we address the extension of this issue to the Heisenberg groups setting. Section 4 recalls the Heisenberg deconvolution results [16] for the cases of a ball and a sphere or of two spheres of appropriate radii, while in Section 5 this type of deconvolution result is extended to the case of three solid tori. Section 6 recalls the result of [16] for extension of a given deconvolution of radial distributions from the space $\mathbf{C}^n$ to $\mathbf{H}^n$, and this result is also extended to the three solid tori. This result on extensions of deconvolutions may assume the radial distributions $T_1, \ldots, T_m$ satisfy the Hörmander strongly coprime condition on $\mathbf{C}^n$ in order to imply deconvolution on $\mathbf{C}^n$. In Section 7 we discuss the relation to an extension of the Hörmander strongly coprime condition and current work we are doing in this direction. Finally Section 8 discusses some of the additional issues for deconvolution in the Heisenberg group setting.

2. Basics of Deconvolution and Hörmander Strongly Coprime Condition

We assume we are given a collection of convolutions $(f * T_1, \ldots, f * T_m)$ representing the signal received on multiple channels, $1, \ldots, m$, and the goal is to recover the function f representing the original signal. This problem reduces to finding deconvolvers $(\nu_1, \ldots, \nu_m)$ satisfying the conditions

$$\widehat{T}_1 \widehat{\nu}_1 + \cdots + \widehat{T}_m \widehat{\nu}_m \equiv 1, \tag{2.1}$$

called the analytic Bezout equation. In addition we would like these distributions $(\nu_1, \ldots, \nu_m)$ to be in the space of compactly supported distributions. This condition (2.1) is equivalent to $T_1 * \nu_1 + \cdots + T_m * \nu_m = \delta$. Assuming we find such $\nu_1, \dots, \nu_m$, then we can recover the signal f by convolving the received signals with these deconvolvers:

$$\begin{aligned} f * T_1 * \nu_1 + \cdots + f * T_m * \nu_m &= f * (T_1 * \nu_1 + \cdots + T_m * \nu_m) \\ &= f * \delta = f \end{aligned}$$

Assuming the existence of $(\nu_1, \dots, \nu_m)$ satisfying the analytic Bezout equation (2.1), we can apply the Paley-Weiner estimates to each of the compactly supported distributions $|\nu_1|, \dots, |\nu_m|$ to attain the estimate

$$|\widehat{T}_1(\xi)| + \cdots + |\widehat{T}_m(\xi)| \geq C \frac{1}{(1+|\xi|)^N} e^{-B|Im(\xi)|}$$

for some $C, N, B > 0$. Such a set of convolvers $(T_1, \dots, T_m)$ is said to be strongly coprime. A theorem of Hörmander [23] implies the existence of compactly supported distributions $(\nu_1, \dots, \nu_m)$ satisfying the analytic Bezout equation (2.1)

$$\widehat{T}_1 \widehat{\nu}_1 + \cdots + \widehat{T}_m \widehat{\nu}_m \equiv 1,$$

or $T_1 * \nu_1 + \cdots + T_m * \nu_m = \delta$, if and only if the convolvers $(T_1, \dots, T_m)$ are strongly coprime. Whereas the deconvolution problem for a single convolution operator $C_\mu(f) = f * \mu$ is shown in [13] to always be ill-posed in the sense of Hadamard, the multichannel deconvolution problem for a set of strongly coprime system of convolution equations is then well-posed. Furthermore this result of Hörmander settles the issue of existence of a set of compactly supported deconvolvers $(\nu_1, \dots, \nu_m)$. The deconvolution paper of Berenstein and Yger [10] provided impetus as well as important methods and results for the modern research in multi-channel deconvolution. In addition to extending work on the Pompeiu problem to the problem of deconvolution, this paper provided constructive results in a number of particular cases. They are able to give explicit formulas to determine deconvolvers $\nu_1, \dots, \nu_m$ from the compactly supported distributions $T_1, \dots, T_m$ using the operations of derivation, integration, convolution, and summation.

We also mention that recent research on deconvolution has included a different approach based on non-periodic sampling and frames. This work of Casey and Walnut includes a method for deconvolution for systems of convolution equations using nonperiodic sampling [12, 13, 25, 26]. A recent result addresses a local version of the three squares result [22].

Our focus here is extension of the work on the Pompeiu problem in the Heisenberg setting to the naturally associated questions in inversion of the Pompeiu transform through deconvolution. We begin by transference of the Pompeiu problem and the problem of deconvolution into the Heisenberg group setting.

3. Heisenberg Group Setting of Pompeiu Problem and Deconvolution

We consider the standard representation of the Heisenberg group $\mathbf{H}^n$ as the boundary of the Siegel upper half space in $\mathbf{C}^{n+1}$, as described, for instance, in [3]. Standard results on the injectivity of the Pompeiu transform have been given for cases of balls, spheres, and solid tori in [1, 2, 3]. These results rely on analytic methods based on the Gelfand transform on integrable radial or polyradial functions on $\mathbf{H}^n$, i.e. $L^1_0(\mathbf{H}^n) = \{f \in L^1(\mathbf{H}^n) : f(U\cdot\mathbf{g}) = f(\mathbf{g})$ for all $U \in U(n)$ and $\mathbf{g} \in \mathbf{H}^n\}$ and $L^1_{\mathbf{0}}(\mathbf{H}^n) = \{f \in L^1(\mathbf{H}^n) : f(\tau\cdot\mathbf{g}) = f(\mathbf{g})$ for all $\tau \in \mathbf{T}^n$ and $\mathbf{g} \in \mathbf{H}^n\}$. In this Heisenberg setting, the translations for the Pompeiu problem can be defined based on the Heisenberg group action $[\mathbf{z}, t]\cdot[\mathbf{w}, s] = [\mathbf{z}+\mathbf{w}, t+s+2\mathrm{Im}\,\mathbf{z}\cdot\bar{\mathbf{w}}]$, also written as $\mathbf{g}\cdot\mathbf{h}$ where $\mathbf{g} = [\mathbf{z}, t]$, $\mathbf{h} = [\mathbf{w}, t]$, and $\mathbf{z}, \mathbf{w} \in \mathbf{C}^n$, $t, s \in \mathbf{R}$. We note the space of left-invariant vector fields on $\mathbf{H}^n$ are spanned by $Z_j = \frac{\partial}{\partial z_j} + i\bar{z}_j\frac{\partial}{\partial t}$ and $\bar{Z}_j = \frac{\partial}{\partial \bar{z}_j} - iz_j\frac{\partial}{\partial t}$, for $j = 1, \ldots, n$, together with the extra direction $T = \frac{\partial}{\partial t}$, generated by the commutators $[\bar{Z}_j, Z_k] = 2i\delta_{j,k}T$. The translations of the set $S \subset \mathbf{H}^n$ by an element of $\mathbf{g} \in \mathbf{H}^n$ are given by $\mathbf{g}\cdot S = \{\mathbf{g}\cdot\mathbf{h} : \mathbf{h} \in S\}$. Because of the nature of the analytic methods applied to this problem, we usually consider $S \subset \mathbf{C}^n \times \{0\}$. This approach relates to Strichartz's definition of the Radon transform for $\mathbf{H}^n$ [24] and yields integral conditions of the form $\int_S L_{\mathbf{g}} f(\mathbf{z}, 0)\,d\mu_S(\mathbf{z})$, where $L_{\mathbf{g}}$ is left translation by $\mathbf{g} \in \mathbf{H}^n$ and μ_S is Lebesgue measure on S. We note that it is also possible to consider a set $S \subset \mathbf{H}^n$ of the same dimension as the ambient space, such as several versions of the Heisenberg ball considered in [19], by extending the above techniques.

The problem of deconvolution for the Heisenberg setting is a direct extension of the integral conditions for the Pompeiu problem. Given a collection of sets $S_1, \ldots, S_m \subset \mathbf{C}^n \times \{0\}$ similarly define the Pompeiu transform

$$(\mathcal{P}f)(\mathbf{g}) = ((f * T_1)(\mathbf{g}), \ldots, (f * T_m)(\mathbf{g}))$$

where T_j are defined by $\langle \phi, T_j \rangle = \int_{S_j} \phi(\mathbf{z}, 0) d\mu_{S_j}(\mathbf{z})$. In most cases these sets $S_1, \ldots, S_m$ are either radial or polyradial, although we are also interested in the more general case in which radial symmetry is not assumed. In general we will assume the sets $S_1, \ldots, S_m$ are chosen so the transform $\mathcal{P}$ is injective, and the goal is to invert this transform. It is also possible to vary this definition to address other issues, such as variations in the dimension of the set or including the use of rotations. The issue of rotations is particularly important, and we address some aspects of it in this paper. It is conventional to use a continuum of rotations, as done in the results in [11, 17]. However when considering the problem of deconvolution, it is helpful to reduce to finite number of rotations. In this case $T_1, \ldots, T_m$ may represent m rotations of the same set S. The ususal approach will be to find deconvolvers $\mu_1, \ldots, \mu_m$ satisfying the analytic Bezout equation (2.1), as described above. The appropriate version of the analytic Bezout equation is given by using the Gelfand transform, which extends the Fourier transform to the setting of radial or polyradial functions on $\mathbf{H}^n$. The first steps in this analysis are demonstrating the existence of the deconvolvers for specific cases, and ideally providing an appropriate extension of the strongly coprime condition to the Heisenberg setting. It is then also possible to consider construction of the deconvolvers $\mu_1, \ldots, \mu_m$ from the given distribution $T_1, \ldots, T_m$, or other issues in deconvolution.

For radial and polyradial functions the Fourier transform extends into the Heisenberg setting as the Gelfand transform, based either upon bounded $U(n)$-spherical funtions for $L^1_0(\mathbf{H}^n)$ or bounded $\mathbf{T}^n$-spherical functions for $L^1_{\mathbf{0}}(\mathbf{H}^n)$. In line with the approach to harmonic analysis for $\mathbf{H}^n$ outlined in [24], these are based on the joint eigenfuntions of the Heisenberg subLaplacian

$$\mathcal{L} = -\frac{1}{2}\sum_{j=1}^{n} Z_j \bar{Z}_j + \bar{Z}_j Z_j$$

and iT, representing the extra direction. These bounded $U(n)$-spherical functions are given by

$$\Psi^{\lambda}_{k,k} = \binom{k+n-1}{k}^{-1} e^{2\pi i\lambda t} e^{-2\pi|\lambda||\mathbf{z}|^2} L_k^{(n-1)}(4\pi|\lambda||\mathbf{z}|^2) \quad (\lambda, k) \in \mathbf{R}^* \times \mathbf{Z}_+$$

and

$$\mathcal{J}^{\rho}_{n-1} = (n-1)!2^{n-1}\frac{J_{n-1}(\rho|\mathbf{z}|)}{(\rho|\mathbf{z}|)^{n-1}} \quad \rho \in \mathbf{R}_{+}.$$

The spectrum for the Gelfand transform for $U(n)$-spherical functions can be represented by the Heisenberg fan $\mathcal{H}$ composed of a central Bessel ray $\mathcal{H}_\rho$ and infinitely many Laguerre rays $\mathcal{H}_{k,\pm}$ converging to the Bessel ray. These may be denoted by

$$\begin{aligned}\mathcal{H} &= \mathcal{H}_\rho \cup (\cup_{k=1}^{\infty}\mathcal{H}_{k,+} \cup \mathcal{H}_{k,-}) \\ &= \{(0,\rho) : \rho \geq 0\} \cup \left(\cup_{k=1}^{\infty}\left\{\left(\lambda, 4|\lambda|\left(k+\frac{n}{2}\right)\right) : \lambda \in \mathbf{R}^*\right\}\right).\end{aligned}$$

Here the Laguerre rays correspond to $(\lambda, k) \in \mathbf{R}^* \times \mathbf{Z}_+$, also indexing the $\Psi^{\lambda}_{k,k}$, while the Bessel ray corresponds to $\rho \in \mathbf{R}_+$, also indexing the $\mathcal{J}$. The $U(n)$-spherical Gelfand transform is then defined for $f \in L^1_0(\mathbf{H}^n)$, yielding $\widetilde{f}(p)$, defined for each $p \in \mathcal{H}$ and given by

$$\widetilde{f}(\lambda, k) = \int_{\mathbf{H}^n} f(\mathbf{g})\overline{\Psi^{\lambda}_{k,k}(\mathbf{g})}dm(\mathbf{g}) \quad \text{and} \quad \widetilde{f}(0;\rho) = \int_{\mathbf{H}^n} f(\mathbf{g})\overline{\mathcal{J}^{\rho}_{n-1}(\mathbf{g})}dm(\mathbf{g}).$$

The radial and polyradial agree for $n = 1$, and for higher dimensions n the polyradial can be formed based on a Cartesian product. These bounded $\mathbf{T}^n$-spherical functions are given by

$$\psi^{\lambda}_{\mathbf{k},\mathbf{k}}(\mathbf{z},t) = e^{2\pi i\lambda t}e^{-2\pi|\lambda||\mathbf{z}|^2}\prod_{j=1}^{n} L^{(0)}_{k_j}(4\pi|\lambda||z_j|^2) \quad \text{for } (\lambda, \mathbf{k}) \in \mathbf{R}^* \times (\mathbf{Z}_+)^n,$$

and

$$\mathcal{J}_\rho(\mathbf{z}) = \prod_{j=1}^{n} J_0(\rho_j|z_j|) \quad \text{for } \rho \in (\mathbf{R}^*)^n.$$

Thus for $f \in L^1_{\mathbf{0}}(\mathbf{H}^n)$, the Gelfand transform $\tilde{f}$ is defined by

$$\tilde{f}(\lambda, \mathbf{k}) = \int_{\mathbf{H}^n} f(\mathbf{g})\psi^{\lambda}_{\mathbf{k},\mathbf{k}}(\mathbf{g})dm(\mathbf{g}), \quad \text{and} \quad \tilde{f}(\mathbf{0};\rho) = \int_{\mathbf{H}^n} f(\mathbf{g})\mathcal{J}_\rho(\mathbf{g})dm(\mathbf{g}).$$

This transform $\tilde{f}$ is a function on the Heisenberg brush,

$$\mathcal{H}_b = \cup_{\mathbf{k}\in(\mathbf{Z}_+)^n}\{(\lambda, |\lambda|(4k_1+2), \ldots |\lambda|(4k_n+2)) : \lambda \in \mathbf{R}^*\}$$

$$\cup\{(0, \rho_1^2, \dots, \rho_n^2) : \rho \in (\mathbf{R}_+)^n\},$$

where the point $(\lambda, |\lambda|(4k_1+2), \dots, |\lambda|(4k_n+2))$ coresponds to $\psi_{\mathbf{k}}^{\lambda}$ and the point $(0, \rho_1^2, \dots, \rho_n^2)$ corresponds to $\mathcal{J}_\rho$. We note this is the appropriate version of the Heisenberg brush for our purposes, as found in [3, p. 204]. It corresponds to the version of the Heisenberg fan for $n = 1$ extended to many variables, and its importance for us relates to convergence on the Heisenberg brush in the relative subspace topology carrying over to convergence in the associated Gelfand transform. This follows from the formula [20, p. 199]

$$e^{-|\tau||z_i|^2} L_{k_i}^{(0)}(2|\tau| \cdot |z_i|) = J_0\left(\sqrt{|\tau|(4k_i+2)}|z_i|\right) + O(k_i^{-3/4}).$$

In application of these transforms to radial or polyradial domains, it is also helpful to define functions

$$\Psi_k^{(n-1)}(x) = \int_0^x e^{-t/2} L_k^{(n-1)}(t) t^{n-1} dt$$

and

$$j_n(x) = \frac{J_n(x)}{x^n}$$

.

In some cases it is helpful to consider these issues from the perspective of the group Fourier transform and the Weyl calculus. Here the transforms will be operator-valued functions of the operators of position $\mathbf{P} = (P_1, \dots, P_n)$ and momentum $\mathbf{Q} = (Q_1, \dots, Q_n)$, where $P_j u(\mathbf{x}) = x_j u(\mathbf{x})$ and $Q_j u(\mathbf{x}) = \frac{1}{i}\frac{\partial u}{\partial x_j}(\mathbf{x})$. The group Fourier transform on $\mathbf{H}^n$ is based on the infinite-dimensional representations

$$\pi_{\pm\lambda} = e^{2\pi i(\pm\lambda t \pm \lambda^{1/2}\mathbf{x}\cdot P + \lambda^{1/2}\mathbf{y}\cdot Q)} \quad \text{for } \lambda \in \mathbf{R}_+ \setminus \{0\},$$

and the one-dimensional representations

$$\pi_{(\xi,\eta)} = e^{2\pi i(x\cdot\xi + y\cdot\eta)} \quad \text{for } (\xi,\eta) \in \mathbf{R}^n \times \mathbf{R}^n,$$

attained in the limit as $\lambda \to 0$. For additional details we refer to [3, 21]. In the Pompeiu context this perspective has been used to unify the Bessel and Laguerre parts of the spectrum [2, 3, 18], as is also done in the conditions for deconvolution.

4. Deconvolution Results of Chang, Eby, and Grinberg

Recent work in the article [16] gives a good introduction to the deconvolution problem in the Heisenberg group setting. We recall here the main results of this paper in which deconvolving sequences $\{\nu_{1,j}\}$ and $\{\nu_{2,j}\}$ are constructed for each of two cases: a ball and a sphere of the same radius and two balls of appropriate radii. For the first case of a ball and sphere of the same radii, we first define the distributions S_r and T_r as

$$\langle \phi, S_r \rangle = \int_{|\mathbf{z}|=r} f(\mathbf{z},0)d\sigma_r(\mathbf{z}) \quad \text{and} \quad \langle \phi, T_r \rangle = \int_{|\mathbf{z}|\leq r} f(\mathbf{z},0)d\mu_r(\mathbf{z}),$$

where σ_r and μ_r are the Lebesgue measures on the sphere and ball. We also define compact sets $\{K_j\}$, based on the Bessel zeros of the relevant Gelfand transforms, forming

$$K_j = \{p = (x,y) \in \mathcal{H} : x^2 + y^2 \leq N_j^2\},$$

where $(x,y) = (\lambda, |\lambda|(4k+2))$ or $(x,y) = (0,\rho^2)$, and where N_j is three quarters of the distance between the jth zero of $\widetilde{T}_r(0;\rho)$ and the $(j+1)$st zero of $\widetilde{S}_r(0;\rho)$.

Theorem 4.1 ([16], Theorem 1)**.** *Let S_r and T_r be the distributions defined above. Consider the sequence of compact sets $\{K_j\} \subset \mathcal{H}$, which forms a compact exhaustion of the Heisenberg fan $\mathcal{H}$, as given above. There exist sequences of functions $\{\nu_{1,j}\}$ and $\{\nu_{2,j}\}$ with the property that*

$$\widetilde{S}_r\widetilde{\nu}_{1,j} + \widetilde{T}_r\widetilde{\nu}_{2,j} \equiv 1 \qquad \text{on } K_j.$$

It is also true that

$$\widetilde{S}_r\widetilde{\nu}_{1,j} + \widetilde{T}_r\widetilde{\nu}_{2,j} \equiv 0 \qquad \text{on } V_j^c,$$

where each V_j is an open set defined above such that $K_j \subset V_j \subset K_{j+1}$.

Here we briefly outline the proof, which is based on the idea of inverting each of the distributions, yielding $\frac{1}{\widetilde{S}_r}$, each valid on a set that stays away from the associated zeros. The majority of the "work" in the proof relates to setting up the appropriate sets on which to invert $\widetilde{S}_r$ and $\widetilde{T}_r$, then patching these together in an organized fashion. A fundamental role is also played by the convergence of points in the Heisenberg fan $\mathcal{H}$, based on the subspace topology, and the corresponding convergence in the transform. The first step is formation of appropriate neighborhood systems separating Bessel zeros. This yields neighborhoods $\{C_i\}$ separated from the Bessel zeros of $\widetilde{T}_r$ and neighborhoods $\{C_i^{'}\}$ separated from the Bessel zeros of $\widetilde{S}_r$. Using the subspace topology for the Heisenberg fan $\mathcal{H}$ these neighborhoods are then extended to neighborhoods $\{C_{i,j_i}\}$ and $\{C^{'}_{i,j_i^{'}}\}$ in $\mathcal{H}$ that also cover parts of the Laguerre rays. We also fill in additional neighborhoods $\{D_i\}$ and $\{D_i^{'}\}$ of the remaining Laguerre zeros, in order to cover the entire Heisenberg fan $\mathcal{H}$. The next step is construction of "local identities" and "local inverses" on each of these neighborhoods, away from the zero sets. Finally, the inverses on each of these neighborhoods are all put together in organized fashion for the compact exhaustion $\{K_j\}$. Beginning with the inverse on K_{i-1} we paste on the inverses on the neighborhoods covering the additional zeros in K_i. This pasting is done using the construction $\rho = \rho_1 + \rho_2 - \rho_1 * \rho_2$.

In the next case of two balls of different radii, we now define the distributions T_1 and T_2 as

$$\langle \phi, T_1\rangle = \int_{|\mathbf{z}|\le r_1} f(\mathbf{z},0)d\mu_{r_1}(\mathbf{z}) \quad \text{and} \quad \langle \phi, T_2\rangle = \int_{|\mathbf{z}|\le r_2} f(\mathbf{z},0)d\mu_{r_2}(\mathbf{z}),$$

where μ_{r_1} and μ_{r_2} are the Lebesgue measures on the balls.

Theorem 4.2 ([16], Theorem 2)**.** *We assume that r_1 and r_2 satisfy the conditions*

1. $\left(\frac{r_1}{r_2}\right) \notin \mathcal{Q}(J_n) = \{\frac{\gamma x}{\gamma y} : J_n(x) = J_n(y) = 0, \gamma \in \mathbf{R}^*\}$,
2. $\left(\frac{r_1}{r_2}\right)^2 \notin \mathcal{Q}(\Psi_k^{(n-1)}) = \{\frac{\gamma x}{\gamma y} : \Psi_k^{(n-1)}(x) = \Psi_k^{(n-1)}(y) = 0, \gamma \in \mathbf{R}^*\}$, *for all $k \in \mathbf{Z}_+$.*

Then $\widetilde{T}_1$ and $\widetilde{T}_2$ do not have any common zeros. Consider the sequence of compact sets $\{K_j\} \subset \mathcal{H}$ given below, which forms a compact exhaustion of the Heisenberg fan $\mathcal{H}$. There exist sequences of functions $\{\nu_{1,j}\}$ and $\{\nu_{2,j}\}$ with the property that

$$\widetilde{S}_r\widetilde{\nu}_{1,j} + \widetilde{T}_r\widetilde{\nu}_{2,j} \equiv 1 \qquad \text{on } K_j.$$

It is also true that

$$\widetilde{S}_r \widetilde{\nu}_{1,j} + \widetilde{T}_r \widetilde{\nu}_{2,j} \equiv 0 \qquad on\ V_j^c,$$

where each V_j is an open set defined above such that $K_j \subset V_j \subset K_{j+1}$.

Here the compact exhaustion $\{K_j\}$ is formed similarly to the above case, based on separation of consecutive zeros of $\widetilde{T}_1(0;\rho)$ and $\widetilde{T}_2(0;\rho)$. The same overall outline is used for the proof in this case, however we must give closer attention to a few additional issues. We still have simple zeros along the Bessel ray $\mathcal{H}_\rho$, but they are not necessarily interlacing. However they can still be grouped together in appropriate neighborhoods, and they are still isloated, allowing for separation. In addition the zeros may coalesce as j becomes infinite, requiring us to adjust the neighborhood sizes for different sets K_j in the compact exhaustion. However, separation of the zeros is still provided, and the rest of the procedure goes through.

Although these deconvolving sequences are not a complete solution of the problem of deconvolution in the sense given in [10], they do provide a means for recovery of the function f from the signals received by the deconvolvers $f * T_1, \ldots, f * T_m$. Observe that in these two cases we have

$$\begin{aligned} f * T_1 * \nu_{1,j} + f * T_2 * \nu_{2,j} &= f * (T_1 * \nu_{1,j} + T_2 * \nu_{2,j}) \\ &= f_j \end{aligned}$$

where $\widetilde{f}_j$ agrees with $\widetilde{f}$ on K_j but vanishes outside of V_j. Clearly f_j approaches f as j goes to infinity. For the more complete solution it is still possible to consider the issue of convergence of each deconvolving sequence to a single compactly supported distribution. In addition it is possible to consider explicit construction of the deconvolvers from $T_1, \ldots, T_m$ using the operations of derivation, integration, convolution, and summation.

These two cases, Theorem 4.1 and 4.2 are representative of two possible behaviors for the zero sets in the case of simple zeros, either a uniform separation or have a coalescing of zeros as j approaches infinity. It is significant that the procedure allows construction of the deconvolving sequences in both cases. However, for purposes of convergence the case of uniform separation between the zero sets is much easier. In this case it is easy to show convergence of the deconvolving sequences to a compactly supported distribution. For the other case, the sequence converges to a distribution, but the space in which it converges appears to vary depending upon the closeness of these zero sets as j becomes large [16]. This relates to the issue of N-well approximation of the ratio r_1/r_2 by ratios of zeros of the relevant zero sets, for J_1. Also see [16] for more information on this point. In addition we are continuing to explore further this issue of convergence of the sequence of distributions and its relation to the joint distribution for the zero sets of the convolvers.

5. Extension to Case of Three Solid Tori

Although the results of our paper Chang, et al. [16] form a good introduction of the issue of deconvolution for the Pompeiu problem into the Heisenberg group setting, these results are incomplete and leave room for further investigation of a number of important issues. This is not surprising, as the situation for the Pompeiu problem in the Heisenberg group setting is much more complicated and not as much is known as in the Euclidean setting. However, we establish the above cases for the Pompeiu problem with a ball and a sphere, as well as the case of two balls. The other issue of extending a given convolution from Euclidean space to the Heisenberg group setting is treated in [16], as described below. This also relates to the issue of extending the Hörmander result for strongly coprime distributions, as we describe below. Although the two results from [16] given above both deal with cases in which a pair of deconvolvers are formed for a given pair $(f*T_1, f*T_2)$, it is not difficult to visualize how this procedure would extend to the case of $(f*T_1, \ldots, f*T_n)$. Here we treat one specific example for three convolution equations $(f*T_1, f*T_2, f*T_3)$, and we produce three deconvolving sequences $\{\nu_{1,j}\}, \{\nu_{2,j}\}, \{\nu_{3,j}\}$, which allow deconvolution and recovery of f. This example is an extension of the three squares result of [9] to a higher dimensional analogue involving instead three solid tori. We furthermore observe below in Section 6 how this result will also allow us to carry over the approach to extension from the Bessel ray developed in [16]. In order to state the three squares theorem of [9] we first state the integral conditions and the relevant definitions.

$$\int_{\tau(\mathcal{P})} f(x)dx = 0 \qquad \text{for all } \tau \in \mathcal{J} \text{ and } P \in \mathcal{P} \tag{5.1}$$

where $\mathcal{J}$ is the group of all translations of $\mathbf{R}^2$. The three squares result of Berenstein and Taylor states the following.

Theorem 5.1 ([9], Theorem 1)**.** *Suppose $\mathcal{P}$ is a finite family of squares with sides parallel to the coordinate axes. Then $f \equiv 0$ is the only continuous solution of equation* (5.1) *if and only if $\mathcal{P}$ contains three squares of side length a_1, a_2, a_3 such that $a_1/a_2, a_2/a_3, a_3/a_1$ are irrational.*

The case where rotations are used and the squares are no longer parallel to the axes is discussed later in this section. The methods applied to yield the Heisenberg deconvolution results of [3] depend upon application of the Gelfand transform for a set that is radial or polyradial. For this reason we generalize the square as a product of intervals to a product of disks, or a solid torus. The injectivity of the Pompeiu transform in $\mathbf{H}^n$ for $(n+1)$ solid tori has been established in [2, 3], subject to the appropriate numerical criteria on the radii. Here we show that the deconvolution result of [16] for two balls of appropriate radii described above will also extend to the case of three solid tori of appropriate radii. Thinking of these as Cartesian products of disks in $\mathbf{C}^1$, this gives one extension of the above result for three squares from $\mathbf{R}^2$ to $\mathbf{C}^2 \subset \mathbf{H}^2$. The result of [2, 3] implies that when the appropriate condition for radii is met, these three solid tori will jointly possess the Pompeiu property, and the associated Pompeiu transform will be injective. This is the transform associated to the integrals

$$\int_{|z_1|,|z_2|\le r_1} L_{\mathbf{g}} f(\mathbf{z},0)\,d\mu_{r_1}(\mathbf{z}) \qquad \text{for all } \mathbf{g} \in \mathbf{H}^2$$
$$\int_{|z_1|,|z_2|\le r_2} L_{\mathbf{g}} f(\mathbf{z},0)\,d\mu_{r_2}(\mathbf{z}) \qquad \text{for all } \mathbf{g} \in \mathbf{H}^2$$
$$\int_{|z_1|,|z_2|\le r_3} L_{\mathbf{g}} f(\mathbf{z},0)\,d\mu_{r_3}(\mathbf{z}) \qquad \text{for all } \mathbf{g} \in \mathbf{H}^2,$$

also expressible as convolution equations

$$((f * T_{r_1})(\mathbf{g}), (f * T_{r_2})(\mathbf{g}), (f * T_{r_3})(\mathbf{g}))$$

for the distributions T_j, for $j = 1, 2, 3$, defined by

$$\langle T_j, \phi\rangle = \int_{|z_1|,|z_2|\le r_j} \phi(\mathbf{z},0)\,d\mu_{r_j}(\mathbf{z}).$$

Our next result then yields the existence of sequences of deconvolving distributions $\{\nu_{1,j}\}, \{\nu_{2,j}\}$, and $\{\nu_{3,j}\}$

Theorem 5.2. *We assume that* r_1, r_2, *and* r_3 *satisfy the conditions*

1. $\left(\frac{r_1}{r_2}, \frac{r_2}{r_3}, \frac{r_3}{r_1}\right) \notin \mathcal{Q}(J_1) = \left\{\frac{\gamma x}{\gamma y} : J_1(x) = J_1(y) = 0, \gamma \in \mathbf{R}^*\right\}$
2. $\left(\frac{r_1}{r_2}, \frac{r_2}{r_3}, \frac{r_3}{r_1}\right)^2 \notin \mathcal{Q}(\Psi_k^{(0)}) = \left\{\frac{\gamma x}{\gamma y} : \Psi_k^{(0)}(x) = \Psi_k^{(0)}(y) = 0, \gamma \in \mathbf{R}^*\right\}$, *for all* $k \in \mathbf{Z}_+$.

Then $\widetilde{T_1}, \widetilde{T_2}$, *and* $\widetilde{T_3}$ *do not have any common zeros. Consider the sequence of compact sets* $\{K_j\} \subset \mathcal{H}_b$ *given below, which forms a compact exhaustion of the Heisenberg brush* $\mathcal{H}_b$. *Then there exist sequences of functions* $\{\nu_{1,j}\}, \{\nu_{2,j}\}$, *and* $\{\nu_{3,j}\}$ *with the property that*

$$\widetilde{T}_{r_1}\widetilde{\nu}_{1,j} + \widetilde{T}_{r_2}\widetilde{\nu}_{2,j} + \widetilde{T}_{r_3}\widetilde{\nu}_{3,j} \equiv 1 \qquad \text{on } K_j$$

and

$$\widetilde{T}_{r_1}\widetilde{\nu}_{1,j} + \widetilde{T}_{r_2}\widetilde{\nu}_{2,j} + \widetilde{T}_{r_3}\widetilde{\nu}_{3,j} \equiv 0 \qquad \text{on } V_j^c$$

In particular, note how these sequences of deconvolvers yield a limiting process to recover f, as described in [16], as follows

$$\begin{aligned}&(f * T_{r_1} * \nu_{1,j} + f * T_{r_2} * \nu_{2,j} + f * T_{r_3} * \nu_{3,j})\widetilde{\ }|_{K_j}\\ &\quad = (f * (T_{r_1} * \nu_{1,j} + T_{r_2} * \nu_{2,j} + T_{r_3} * \nu_{3,j}))\widetilde{\ }|_{K_j} = \widetilde{f}|_{K_j}\end{aligned}$$

and

$$\begin{aligned}&(f * T_{r_1} * \nu_{1,j} + f * T_{r_2} * \nu_{2,j} + f * T_{r_3} * \nu_{3,j})\widetilde{\ }|_{V_j^c}\\ &\quad = (f * (T_{r_1} * \nu_{1,j} + T_{r_2} * \nu_{2,j} + T_{r_3} * \nu_{3,j}))\widetilde{\ }|_{V_j^c} = 0.\end{aligned}$$

Thus the deconvolvers yield a sequence $\{f_j\}$ given by $f_j = f * T_{r_1} * \nu_{1,j} + f * T_{r_2} * \nu_{2,j} + f * T_{r_3} * \nu_{3,j}$, and clearly $f_j \to f$ as $j \to \infty$

We give a brief illustration of how this result is an extension of the proof established for the deconvolution results of [16, Theorem 1 and 2]. It is primarily an extension of the methods established there to the Heisenberg brush $\mathcal{H}_b$ and to additional distributions.

Proof. Our proof begins with the existence of the appropriate neighborhoods separating the zero sets of the relevant Gelfand transforms. We begin with the Bessel zeros and expand outward to include the Laguerre zeros. Consider

$$\widetilde{T}_{r_1}(0;\rho), \qquad \widetilde{T}_{r_2}(0;\rho), \quad \text{and} \quad \widetilde{T}_{r_3}(0;\rho).$$

For $j = 1, 2, 3$, we have that

$$\widetilde{T}_{r_j}(0;\rho) = \frac{J_1(r_j\rho_1)}{(r_j\rho_1)} \frac{J_1(r_j\rho_2)}{(r_j\rho_2)}.$$

Since the Bessel functions have isolated and simple zeros, the zero sets for these transforms extend from isolated simple zeros by the Cartesian product structure. Thus the conditions on the radii guarantee we do not have simultaneous vanishing of all the distributions for any $p \in \mathcal{H}_b$. Again let $V_j = \{\text{zeros of } \widetilde{T}_{r_j}\}$ and let $U_j = V_j \cap \mathcal{H}_\rho$ be the Bessel zeros. The procedure established for constructing the deconvolving sequences $\{\nu_{1,j}\}$, $\{\nu_{2,j}\}$, and $\{\nu_{3,j}\}$ first requires a system of neighborhoods $\{C_i\}$, $\{C_i^{'}\}$, and $\{C_i^{''}\}$ separating the zeros of $\widetilde{T}_{r_1}(0;\rho)$, $\widetilde{T}_{r_2}(0;\rho)$, and $\widetilde{T}_{r_3}(0;\rho)$. Such neighborhoods can be shown to exist based on an extension of the one-dimensional results using the product structure. Furthermore, these neighborhoods cover the Bessel part of the spectrum$(\mathcal{H}_\rho)_b$. Also utilize the auxiliary neighborhood systems $\{B_i\}$, $\{B_i^{'}\}$, and $\{B_i^{''}\}$, slightly smaller, and $\{V_i\}$, $\{V_i^{'}\}$, and $\{V_i^{''}\}$, slightly larger, as in [16, Theorems 1 and 2] to facilitate extension from the central Bessel part to the Laguerre rays $\mathcal{H}_{\mathbf{k},\pm}$. This extension is made using the same idea as in the proofs of the [16] results, but using the product concept. Here we define both $B_{i,j}^a$ and $B_{i,j}^b$ using

$$B_{i,j}^a = \left\{(x,y) : (M_i^-)^2 \le x^2 + y^2 \le (M_i^+)^2 \text{ and } |\frac{y}{x}| \ge 4(j + n/2)\right\},$$

and

$$B_{i,j}^b = \left\{(x,z) : (M_i^-)^2 \le x^2 + z^2 \le (M_i^+)^2 \text{ and } |\frac{z}{x}| \ge 4(j + n/2)\right\},$$

where M_i^+ and M_i^- are the appropriate upper and lower bounds. Here we choose both a j_i^a and j_i^b to be such that, for each $j \ge j_i^a$ exactly one of the Laguerre zeros on the ray $(\mathcal{H}_{j,\pm})_b$ is inside of $B_{i,j}^a \cap (\mathcal{H}_{j,\pm})_b$, and likewise for j_i^b with respect to $B_{i,j}^b$. Then letting $j_i = \max\{j_i^a, j_i^b\}$, we can let

$$B_{i,j_i} = \Big\{(x,y,z) : (M_i^-)^2 \le x^2 + y^2 \le (M_i^+)^2, (M_i^-)^2 \le x^2 + z^2 \le (M_i^+)^2, \left|\frac{y}{x}\right| \ge 4(j_i + n/2), \text{ and } \left|\frac{z}{x}\right| \ge 4(j_i + n/2)\Big\}$$

These same j_i are carried over to form similar neighborhood systems for $\{C_{i,j_i}\}$, $\{C'_{i,j'_i}\}$, $\{C''_{i,j''_i}\}$,

We then cover the rest of the Laguerre spectrum of $\mathcal{H}_b$ with neighborhoods $\{D_j\}$, $\{D'_j\}$, and $\{D''_j\}$, also separating the zeros within these Laguerre rays.

From here the proof proceeds identically to the cases of [16, Theorem 1 and 2]. The "local identities" for each of these compact neighborhoods are formed. Likewise form the appropriate "local identities" for the additional neighborhoods for the Laguerre rays, also separating the zeros there. These are then patched together using the construction $\rho = \rho_1 + \rho_2 - \rho_1 * \rho_2$ to produce local identities on the K_j and which avoid the appropriate zero sets. This finally yields $\{\rho_{1,j}\}$, $\{\rho_{2,j}\}$, and $\{\rho_{3,j}\}$, such that $(\widetilde{\rho}_{1,j} + \widetilde{\rho}_{2,j} + \widetilde{\rho}_{3,j})\rceil_{K_j} \equiv 1$ and $(\widetilde{\rho}_{1,j} + \widetilde{\rho}_{2,j} + \widetilde{\rho}_{3,j})\rceil_{V_j^c} \equiv 0$ and also such that each $\widetilde{\rho}_{i,j}$ is supported away from the zeros of $\widetilde{T}_{r_i}(0;\rho)$ in $K_j \cap (\mathcal{H}_\rho)_b$.

We then invert each of $\widetilde{T}_{r_i}$ on the appropriate sets, using the construction from [16, Theorem 1 and 2]. Here $\phi_j \in C^\infty$ satisfies $\phi_j(t) = \frac{1}{t}$ for $|t| \geq M_j$ and $\phi_j(t) = 0$ for $|t| \leq M_j/2$, for appropriately chosen M_j. The formation of the $\phi_j \circ T_{r_1}$, $\phi_j \circ T_{r_2}$, and $\phi_j \circ T_{r_3}$ on the appropriate sets completes the inversion. The deconvolving sequences $\{\nu_{1,j}\}$, $\{\nu_{2,j}\}$, and $\{\nu_{3,j}\}$, given by $\nu_{1,j} = \rho_{1,j} * (\phi_j \circ T_{r_1})$, $\nu_{2,j} = \rho_{2,j} * (\phi_j \circ T_{r_2})$, and $\nu_{3,j} = \rho_{3,j} * (\phi_j \circ T_{r_3})$, will then have the desired properties. □

The article [16] also considers this result from the perspective of the Weyl calculus, in which the Bessel and Laguerre parts of the spectrum are unified in an operator-valued Bessel function. This perspective is useful because this unification of conditions emphasizes the close relation between the Euclidean and Heisenberg settings, as observed in [18]. We mention that a similar expression of Theorem 5.2 is also possible, in which the conditions for the radii become the following:

$$ker\left\{j_1\left(r_i\sqrt{|\lambda|(P_1^2+Q_1^2)}\right)\right\} \cap ker\left\{j_1\left(r_j\sqrt{|\lambda|(P_2^2+Q_2^2)}\right)\right\} = \{0\}$$
$$\text{for } i \neq j \text{ and } i,j \in \{1,2,3\}\ .$$

We return to the significance of this perspective later, particularly in relating the results in Euclidean and Heisenberg settings.

Rotations play an important role in the Pompeiu problem, and particularly in Euclidean space a significant part of the research addresses issues including rotations. Much of what has been done in the Heisenberg groups setting depends on radial Gelfand transform and addresses radial sets. However there have been some attempts to include rotations, including [14, 15, 19], and we are also considering this issue in the current work. The standard use of rotations in the Pompeiu problem allows the reduction to one set when a continuum of rotations are used, beginning with the work of [11] and including some results in the Heisenberg setting, such as [14, 15]. However, in certain cases a finite number of rotations of a given set have been used, such as the result of [10, 7] extending the three squares theorem to three rotations of a given square. This particular result has the advantage of no longer requiring exceptional sets for the lengths of the sides of the squares, provided the rotations are properly selected. We restate this theorem for rotations of three squares from [10, 7], where S is any square with sides parallel to the coordinate axes.

Theorem 5.3 ([10, 7])**.** *Let $f \in L^1_{loc} \cap C(\mathbf{R}^2)$, and $\theta_1, \theta_2 \in (0, \pi/2)$. Consider integral conditions for three rotations of a square*

$$\int_S f(\mathbf{x}-\mathbf{y})d\mu_S(\mathbf{x}) = 0 \qquad \textit{for all } \mathbf{y} \in \mathbf{R}^2,$$

$$\int_{\theta_1 \cdot S} f(\mathbf{x}-\mathbf{y})d\mu_{\theta_1,S}(\mathbf{x}) = 0 \quad \textit{for all } \mathbf{y} \in \mathbf{R}^2,$$

and

$$\int_{\theta_2 \cdot S} f(\mathbf{x}-\mathbf{y})d\mu_{\theta_2,S}(\mathbf{x}) = 0 \quad \textit{for all } \mathbf{y} \in \mathbf{R}^2,$$

Assume θ_1 and θ_2 satisfy the following condition:

For all $k \in \mathbf{Z} \setminus \{0\}, y \in \mathbf{R}$, we have

either 1. $(\cos\theta_1 k - \sin\theta_1 y) \notin (\mathbf{Z} \setminus \{0\})$ *and* $(\sin\theta_1 k + \cos\theta_1 y) \notin (\mathbf{Z} \setminus \{0\})$

or 2. $(\cos\theta_2 k - \sin\theta_2 y) \notin (\mathbf{Z} \setminus \{0\})$ *and* $(\sin\theta_2 k + \cos\theta_2 y) \notin (\mathbf{Z} \setminus \{0\})$.

Then we may conclude $f \equiv 0$.

Regarding the condition given in the theorem, it is stated in general form to apply to a wide range of possibilities for θ_1, θ_2. For instance, the case of $\theta_1 = \pi/6$ and $\theta_2 = \pi/4$ clearly satisfies the condition, based on properties of irrational numbers, while the case of $\theta_1 = \pi/6$ and $\theta_2 = \pi/3$ clearly does not satisfy these conditions. It is an interesting but subtle aspect of the Pompeiu problem that certain groups of three rotations will meet the condition and thus possess the Pompeiu property, while others will not.

Although our above result for three complex tori is a direct extension of the three squares result, there are added complexities in extending to a result for rotations of the complex tori. The main issue in going from a Cartesian product of two intervals in $\mathbf{R}^2$ to a Cartesian product of two disks in $\mathbf{C}^2$, the corresponding Fourier transforms go from sine functions, whose zeros are periodically distributed, to Bessel functions (of the first kind) J_1 whose zeros are asymptotic to a sine function. The problem becomes considerably harder from this irregularity in the distribution of the zeros. Although these zero sets are well known and thoroughly described, this added irregularity in the distribution introduces a higher level of difficulty to the problem. There is a subtle point with exact location of the zeros and avoiding overlapping, and the problem is more complex since the distance between consecutive zeros is not fixed. However, we can state a corresponding result as follows, first for injectivity of the Pompeiu transform for three rotations of solid tori in $\mathbf{C}^2$, where the solid torus $S_r = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1| \le r, |z_2| \le r\}$.

Proposition 5.4. *Let $f \in L^1_{loc} \cap C(\mathbf{C}^2)$, and let θ_1 and θ_2 be two rotations from z_1 to z_2. Suppose that f satisfies the integral conditions*

$$\begin{aligned} &\int_{S_r} f(\mathbf{z}-\mathbf{w})d\mu_r(\mathbf{z}) = 0 && \textit{for all } \mathbf{w} \in C^2 \\ &\int_{\theta_1 S_r} f(\mathbf{z}-\mathbf{w})d\mu_{r,\theta_1}(\mathbf{z}) = 0 && \textit{for all } \mathbf{w} \in C^2 \\ &\int_{\theta_2 S_r} f(\mathbf{z}-\mathbf{w})d\mu_{r,\theta_2}(\mathbf{z}) = 0 && \textit{for all } \mathbf{w} \in C^2 \end{aligned}$$

We assume θ_1 and θ_2 satisfy the following condition.

For all $\xi \in \mathbf{C}^n$, one of the transforms $\widehat{T}_r(\xi), \widehat{T}_{r,\theta_1}(\xi), \widehat{T}_{r,\theta_2}(\xi)$ is non-zero.

Then we can conclude that $f \equiv 0$.

The condition for the rotations of this theorem can also be more expressively written in a form comparable to that for Theorem 5.3 as follows:

Assume θ_1 and θ_2 satisfy the following condition.

For all $k \in \Gamma, \xi \in \mathbf{R}$, we have either 1. $(\cos\theta_1 k - \sin\theta_1\xi) \notin \Gamma$ and $(\sin\theta_1 k + \cos\theta_1\xi) \notin \Gamma$,

or 2. $(\cos\theta_2 k - \sin\theta_2\xi) \notin \Gamma$ and $(\sin\theta_2 k + \cos\theta_2\xi) \notin \Gamma$.

Here θ_1 and θ_2 are rotations of the form given in [14] for rotations of complex ellipsoids. Also the set $\Gamma = \{\beta_j : J_1(r\beta_j) = 0\}_{j=1}^{\infty}$ represents the sequence of zeros of the Bessel function J_1 and is required for the reduced condition on the rotations. Note that we left this condition for θ_1 and θ_2 in unsimplified form. In this case the condition for injectivity is fairly complex, and thus we do not have good information regarding which rotations are allowable. Extending this result to $\mathbf{H}^2$ would be essentially the same, with additional conditions on the rotations corresponding to the zero sets for the Laguerre part of the spectrum.

Proposition 5.5. *Let $f \in L^\infty \cap C(\mathbf{H}^2)$, and let θ_1 and θ_2 be two rotations from z_1 to z_2. Suppose that f satisfies the integral conditions*

$$\int_{S_r} L_{\mathbf{g}} f(\mathbf{z},0)\,d\mu_r(\mathbf{z}) = 0 \qquad \textit{for all } \mathbf{g} \in \mathbf{H}^2,$$

$$\int_{\theta_1 S_r} L_{\mathbf{g}} f(\mathbf{z},0)\,d\mu_{r,\theta_1}(\mathbf{z}) = 0 \qquad \textit{for all } \mathbf{g} \in \mathbf{H}^2,$$

$$\int_{\theta_2 S_r} L_{\mathbf{g}} f(\mathbf{z},0)\,d\mu_{r,\theta_2}(\mathbf{z}) = 0 \qquad \textit{for all } \mathbf{g} \in \mathbf{H}^2.$$

We assume θ_1 and θ_2 satisfy the following conditions.

1. *For all $\rho \in (\mathbf{R}_+)^2$, one of the transforms $\widetilde{T}_r(0;\rho), \widetilde{T}_{r,\theta_1}(0;\rho), \widetilde{T}_{r,\theta_2}(0;\rho)$ is non-zero.*
2. *For all $(\lambda,\mathbf{k}) \in \mathbf{R}^* \times (\mathbf{Z}_+)^2$ one of the transforms $\widetilde{T}_r(\lambda,\mathbf{k})$, $\widetilde{T}_{r,\theta_1}(\lambda,\mathbf{k})$, $\widetilde{T}_{r,\theta_2}(\lambda,\mathbf{k})$ is non-zero.*

Then we can conclude that $f \equiv 0$.

As a brief outline of the proofs for these two results, we mention that the conditions given above are those needed to ensure the Fourier transforms of the appropriate distributions, or Gelfand transform for the Heisenberg setting, do not all vanish for any given point. From there the standard methods are used to yield the results. Note that both of these results include conditions for the rotations of the solid torus, that these must avoid a certain exceptional set related to the zeros of the Bessel function J_1, and in the second case there is the additional condition relating to zeros of $\Psi_{\mathbf{k}}^{(0)}$ from the Laguerre part of the spectrum. It would be possible to state the above rotation of three squares result of [10, 7] in the same format, involving zeros of the sine function. However, in that case it is easy to reduce the condition to a simpler form, and to find rotations that provide injectivity, based on periodic distribution of zeros and on properties of irrational numbers arising from the rotations. However, the same question is not as easy when extended to solid tori. We leave this as an open question to simplify the condition to more fully describe the conditions for which rotations lead to injectivity. Also we would like to demonstrate examples of specific cases of rotations for which the condition provides injectivity. These results are fairly direct extensions of the well known results for three squares and for three rotations of a given square, stated primarily for the purpose of comparison. The result for three solid tori provides an additional example, in the realm of polyradial distributions rather than radial, to which we can apply the deconvolution procedure established above and in [16]. We finally mention that we can easily prove, based on established methods, that the Pompeiu property does hold in each of these cases if a continuum of rotations is provided. This applies to rotations of a solid torus in $\mathbf{C}^n$ with a translation group provided by either $\mathbf{C}^n$ or $\mathbf{H}^n$.

6. Extending Deconvolution from the Bessel Ray

The work we have done for the deconvolution problem in the Heisenberg group setting has dealt with cases of radial distributions or polyradial distributions. In each of these cases there is a central Bessel ray, or products of rays for polyradial, where the Gelfand transform corresponds to the Fourier-Bessel transform in the associated Euclidean space $\mathbf{C}^n$. We consider the issue of expanding from a given convolution on the Euclidean space to $\mathbf{C}^n$ to a deconvolution for the Heisenberg group $\mathbf{H}^n$. In addition we are concerned with the related issue of extending the Hörmander strongly coprime condition. In the case of $T_1, \ldots, T_m$ radial distributions on $\mathbf{C}^n$ satisfying the Hörmander strongly coprime condition, the existence of compactly supported distributions $\nu_1, \ldots, \nu_m$ such that

$$\widehat{T}_1(\xi)\widehat{\nu}_1(\xi) + \cdots + \widehat{T}_m(\xi)\widehat{\nu}_m(\xi) \equiv 1 \quad \xi \in \mathbf{C}^n,$$

which can also be expressed in terms of the Gelfand transform on the central Bessel ray as

$$\widetilde{T}_1(0;\rho)\widetilde{\nu}_1(0;\rho) + \cdots + \widetilde{T}_m(0;\rho)\widetilde{\nu}_m(0;\rho) \equiv 1 \quad \rho \in \mathbf{R}_+.$$

Our goal is to extend the Euclidean deconvolution from the central Bessel ray $\mathcal{H}_\rho$ to the entire Heisenberg fan. We must find compactly supported distributions $\mu_1, \ldots, \mu_m$ satisfying

$$\widetilde{T}_1(p)\widetilde{\mu}_1(p) + \cdots + \widetilde{T}_m(p)\widetilde{\mu}_m(p) \equiv 1 \quad p \in \mathcal{H}.$$

and such that $\widetilde{\mu}_j(0;\rho) = \widetilde{\nu}_j(0;\rho) = \widehat{\nu}_j(\xi)$ for $j \in \{1, \ldots, m\}$, where $|\xi| = \rho$.

We first recall the result from [16] for extension from Bessel ray for two radial distributions.

Theorem 6.1 ([16], Theorem 5)**.** *Consider S_r and T_r radial distributions satisfying the Hörmander strongly coprime condition on $\mathbf{C}^n$, i.e. such that there exist ν_1 and ν_2, radial, compactly supported distributions satisfying $\widehat{T}_r(\xi)\widehat{\nu}_1(\xi) + \widehat{S}_r(\xi)\widehat{\nu}_2(\xi) \equiv 1$ for all $\xi \in \mathbf{C}^n$. Also assume that for all $(\lambda, k) \in \mathbf{R}^* \times \mathbf{Z}_+$ either $\widetilde{S}_r(\lambda, k) \neq 0$ or $\widetilde{T}_r(\lambda, k) \neq 0$. There there exist μ_1, μ_2 such that*

$$\widetilde{S}_r(p)\widetilde{\mu}_1(p) + \widetilde{T}_r(p)\widetilde{\mu}_2(p) \equiv 1 \qquad \text{for all } p \in \mathcal{H},$$

and such that $\widetilde{\mu}_1(0;p) = \widehat{\nu}_1(\xi)$ and $\widetilde{\mu}_1(0;p) = \widehat{\nu}_1(\xi)$ for all $\rho \in \mathbf{R}_+$ and for all $\xi \in \mathbf{C}^n$, where $|\xi| = \rho$.

This result provides the desired extension for cases including ball and sphere of a fixed radius and the two balls of appropriate radii mentioned above. Furthermore it applies to many other cases and related problems and can be easily extended. A direct corollary of the methods used to prove the above method allows extension to m radial, compactly supported distributions, $T_1, \ldots, T_m$ satisfying the Hörmander strongly coprime condition provided they do not all vanish for any point $(\lambda, k) \in \mathbf{R}^* \times \mathbf{Z}_+$. It is similarly possible to extend to the case of polyradial distributions satisfying the Hörmander strongly coprime condition and not all vanishing for any point $(\lambda, \mathbf{k}) \in \mathbf{R}^* \times (\mathbf{Z}_+)^n$.

Corollary 6.2. *Let T_1, ..., T_m be radial distributions satisfying the Hörmander strongly coprime condition on* $\mathbf{C}^n$, *i.e. such that there exist* $\nu_1, \ldots, \nu_m$, *radial, compactly supported distributions satisfying* $\widehat{T}_1(\xi)\widehat{\nu}_1(\xi) + \cdots + \widehat{T}_m(\xi)\widehat{\nu}_m(\xi) \equiv 1$ *for all* $\xi \in \mathbf{C}^n$. *Also assume that for all* $(\lambda, k) \in \mathbf{R}^* \times \mathbf{Z}_+$ *there exists* $j \in \{1, \ldots, m\}$ *such that* $\widetilde{T}_j(\lambda, k) \neq 0$. *There there exist* $\mu_1, \ldots, \mu_m$ *such that*

$$\widetilde{T}_1(p)\widetilde{\mu}_1(p) + \cdots + \widetilde{T}_m(p)\widetilde{\mu}_m(p) \equiv 1 \qquad \text{for all } p \in \mathcal{H},$$

and such that $\widetilde{\mu}_j(0; \rho) = \widehat{\nu}_j(\xi)$ *for* $j \in \{1, \ldots, m\}$ *and for all* $\rho \in \mathbf{R}_+$ *and for all* $\xi \in \mathbf{C}^n$, *where* $|\xi| = \rho$.

Corollary 6.3. *Consider* $T_1, \ldots, T_m$ *be polyradial distributions satisfying the Hörmander strongly coprime condition on* $\mathbf{C}^n$, *such that there exist* $\nu_1, \ldots, \nu_m$, *polyradial, compactly supported distributions satisfying* $\widehat{T}_1(\xi)\widehat{\nu}_1(\xi) + \cdots + \widehat{T}_m(\xi)\widehat{\nu}_m(\xi) \equiv 1$ *for all* $\xi \in \mathbf{C}^n$. *Also assume that for all* $(\lambda, \mathbf{k}) \in \mathbf{R}^* \times (\mathbf{Z}_+)^n$ *there exists* $j \in \{1, \ldots, m\}$ *such that* $\widetilde{T}_j(\lambda, \mathbf{k}) \neq 0$. *There there exist* $\mu_1, \ldots, \mu_m$ *such that*

$$\widetilde{T}_1(p)\widetilde{\mu}_1(p) + \cdots + \widetilde{T}_m(p)\widetilde{\mu}_m(p) \equiv 1 \qquad \text{for all } p \in \mathcal{H},$$

and such that $\widetilde{\mu}_j(0; \rho) = \widehat{\nu}_j(\xi)$ *for* $j \in \{1, \ldots, m\}$ *and for all* $\rho \in (\mathbf{R}_+)^n$ *and for all* $\xi \in \mathbf{C}^n$, *where* $|\xi_j| = \rho_j$.

In this study of deconvolution in the Heisenberg group setting $\mathbf{H}^n$, it turns out to be very useful to make use of what is known from $\mathbf{C}^n$. We have also considered the related problem of extending a given deconvolution for $\mathbf{C}^n$ to work on all of $\mathbf{H}^n$. All of this turns out to be relevant when considering a potential extension of the strongly coprime condition into the Heisenberg setting, as we discuss in the next section.

7. Extension of the Strongly Coprime Condition

This ability to extend an existing convolution for $\mathbf{C}^n$ to the Heisenberg setting $\mathbf{H}^n$ by expanding from the central Bessel ray corresponds directly to the desired extension of the Hörmander strongly coprime condition to the Heisenberg setting. This strongly coprime condition is one of the primary results in the deconvolution problem, and we are working on finding an appropriate extension. The above results outline a method of extension for a given (radial) deconvolution from the central Bessel ray $\mathcal{H}_\rho$, representing Euclidean space, to the entire Heisenberg fan $\mathcal{H}$ representing the Heisenberg group setting. Based on these results it appears the desired extension should be possible. Although we have not yet proven this result in full generality, such an extension has been demonstrated at least for specialized cases where the convolvers and deconvolvers are radial or polyradial. The method of this extension depended upon the collection of sets used in separating the zero sets as the "platform" on which the deconvolving sequences were constructed. Furthermore these "platforms" facilitate interpolation between the given Euclidean deconvolution on the central Bessel ray $\mathcal{H}_\rho$ and the patching together of local inverses $\frac{1}{\tilde{T}_1}, \ldots, \frac{1}{\tilde{T}_n}$ on the Laguerre rays $\mathcal{H}_{k,\pm}$ away from the central ray, where the zeros are locally finite. In effect this blends together the method of applying local inverses, as in [16, Theorem 1], with extant deconvolution structure on the central Bessel ray $\mathcal{H}_\rho$ which derives from the Hörmander strongly coprime condition.

Thus the extension of the Hörmander strongly coprime condition for the radial case appears to be based on two main criteria. The first condition is to satisfy the Euclidean version of the strongly coprime condition on the central Bessel ray $\mathcal{H}_\rho$ and thereby establish the existence of compactly supported deconvolvers on this ray. Then the second condition would be avoidance of common zeros on the Laguerre rays combined with the existence of the appropriate collection of sets to separate the zeros on these rays. Using the procedure for extension given above and in [16, Theorem 5], this procedure provides a means to extend the deconvolution from the central Bessel ray to the entire spectrum of the Heisenberg fan, $\mathcal{H}$. The situation for the polyradial case is much the same. Thus extension from the central Bessel ray $\mathcal{H}_\rho$ to the rest of the Heisenberg fan $\mathcal{H}$ is possible because of the "platforms" used to separate the zeros, the locally finite nature of the zeros, and the ability to piece together the local inverses $\frac{1}{\tilde{T}_1}, \ldots, \frac{1}{\tilde{T}_n}$ away from their zero sets. It is thus a direct extension of the deconvolution method developed in [16], the relation between the Bessel and Laguerre parts of the spectrum of the Heisenberg fan $\mathcal{H}$, and the subspace topology of this Heisenberg fan $\mathcal{H}$ and its relation to the Gelfand transform.

For this next fundamental result in deconvolution of finding a suitable version of the Hörmander strongly coprime condition that extends to the Heisenberg group setting, $\mathbf{H}^n$, we are very close to establishing a general result. We have connected this problem to the issue of extending a given deconvolution from the central Bessel ray $\mathcal{H}_\rho$ for which we have the result [16, Theorem 5] also extended to greater generality in Corollary 6.2 and Corollary 6.3. Based on the issues of the structure of zero sets, the topology of the Heisenberg fan and brush $\mathcal{H}$ and $\mathcal{H}_b$, and its relation to the Gelfand transform, this is where the essential issue will be. We are continuing our efforts to resolve the remaining points, and we expect to attain a more general result in the sequel.

In our ongoing work on this issue we first intend to establish a general result for the case of radial distributions, as well as the case of polyradial distributions, as these are the cases to which the Gelfand transform applies. More generally a larger goal would be to move beyond the rotation invariant distributions of the radial or polyradial cases, however this would require a new approach and a new set of tools as the Gelfand transform would not apply to such cases. Our current work in the extension of a given deconvolution from the complex space $\mathbf{C}^n$ to the Heisenberg group $\mathbf{H}^n$ has outlined a means to provide an extension to the Heisenberg fan or brush, particularly in the radial and polyradial cases. Although these appear close to yielding an extension of the Hörmander strongly coprime condition, there is still more work to do. And particularly for the more general case of compactly supported distributions not assumed to be radial or polyradial, we are just at the beginning of this interesting problem.

8. Further Questions and Issues

As mentioned above, there are still a number of important points to study related to the Pompeiu transform in the Heisenberg group setting and particularly for the related problem of its inversion through deconvolution. One of the most important of these was just treated in the last section, related to both the extension of a given deconvolution from Euclidean space to the Heisenberg setting and the related extension of the Hörmander strongly coprime condition to a suitable version for the Heisenberg setting. We appear to be very close to this result, at least for the radial and polyradial cases. Recall that the Pompeiu results in the Heisenberg setting are directly analogous to those from Euclidean space within the context of the group Fourier transform and the Weyl calculus, as first demonstrated in [2, 3] for the cases of balls and solid tori. In [17] this type of result was extended to more general distributions including moments, and it was observed in the paper [18] that many results should carry over to the Heisenberg setting, and particularly that deconvolution should be able to work. At this point, we have a similar conjecture for the extension of a version of the Hörmander result for strongly coprime distributions and for extension from complex space $\mathbf{C}^n$ to the Heisenberg group setting, $\mathbf{H}^n$. Based on our current work on this problem, at least the case for radial and polyradial distributions should carry over based on the approach of extending from the central Bessel ray, as outlined above. Although the formulation for the Weyl calculus and ability to give results we need are based on special formulations for the radial and polyradial cases, it is also important to consider extensions to the more general cases. Some of the relevant issues are mentioned briefly below. We are continuing to investigate one other topic related to deconvolution which we originally considered in [16], the relation of distribution of the zero sets to the convergence of the deconvolving sequences. Although it was easy to demonstrate convergence to a compactly supported distribution in the case where there was a uniform separation in the zero sets, this issue had more subtleties in the case where the zero sets coalesced. Within this question we must also consider the issue of N-well approximation of the ratio of radii $\frac{r_i}{r_j}$ by the appropriate zero set, which is directly related to the strongly coprime condition. See [16] for more information on this issue. It turns out the space of convergence for the deconcolving sequences is directly related to some very interesting arithmetic issues related to location of the zeros for $\widetilde{T}_1, \ldots, \widetilde{T}_m$. We are continuing to explore these issues in relation to convergence and in relation to extension of the strongly coprime condition.

Some of the new results of this paper involved an extension of the three squares theorem of [9] for $\mathbf{R}^2$ to a higher dimensional analogue of three solid tori for $\mathbf{C}^2$ and $\mathbf{H}^2$. This Pompeiu result for three tori allowed us to extend the Heisenberg deconvolution method beyond a pair of distributions. The same approach used for three tori in $\mathbf{H}^2$ will generalize to $(n+1)$ tori in $\mathbf{H}^n$. These results also generalized to allow extension of a given deconvolution for such distributions from $\mathbf{C}^n$ to $\mathbf{H}^n$. This gives an additional example of a specific case where the strongly coprime condition for Euclidean space can partially extend to the Heisenberg setting, implying existence of deconvolving sequences in the Heisenberg setting provided additional conditions on the Laguerre zeros are satisfied. The version of the three squares theorem using three appropriate rotations of a given square was also addressed for three rotations of a solid torus in $\mathbf{C}^2$ or $\mathbf{H}^2$. We raised the issue of finding a simplification of the condition for the rotations, which is much more complex due to the nature of the zero sets. Ideally would like to simplify the condition and be able to say something describing which finite sets of rotations will work. More fully describing these cases will also be an important part of extending from Pompeiu results to deconvolution. This is an important issue to consider, as we would like to address the issue of rotations within our work on deconvolution in the Heisenberg setting. Such work may form important first steps related to the issue of rotation and the important issues discussed below.

Many other important topics remain for deconvolution in the Heisenberg setting, and even for the Pompeiu problem much remains to be explored in this setting. This includes several topics in the Pompeiu problem which have only recently been addressed in the Heisenberg setting, such as sets of higher codimension [15] or sets of the same codimension as the ambient space [19]. In each of these areas there are additional issues to be explored for the Pompeiu setting, plus more topics related to the extension to deconvolution. The issue of rotation plays a central role in the Pompeiu problem, and we have begun consideration of this issue, including [14] and some of the results in Section 5. In addition we have some current work dealing with certain cases of non-radial sets in the direction of the work of [11]. Much more work remains to more fully extend the work on this aspect of the Pompeiu problem into the Heisenberg setting. Furthermore, in order to carry over the work on the Pompeiu problem with rotations to the problem of deconvolution it is good to be able to reduce to a finite number of such rotations, as discussed in Section 5. Both of these topics will be important ongoing areas of investigation related to rotations and the Pompeiu problem.

We mention that both the Pompeiu problem and its extension to the problem of deconvolution have local forms. For instance, in the Euclidean setting the work of [4] on the local Pompeiu problem was extended in [6] to include inversion of the Pompeiu transform. We have begun work on the local problem for the Heisenberg group setting, and this is a very interesting and important problem. We expect interesting developments in this direction in future work. As a final point, the local versions of these issues open the problem to new perspectives and approaches, for instance the work on the problem of deconvolution from the perspective of frames and irregular sampling [12, 13, 25, 26], including a local version of the three squares theroem [22]. This new perspective provides an interesting alternative approach to deconvolution, and in particular has provided the ability to prove local results. It is also worthwhile to consider how such approaches may extend to the Heisenberg setting.

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