Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **8** (2013), No. 4, pp. 491-503

A MINIMIZATION PROBLEM ASSOCIATED WITH THE CHERN-SIMONS MODEL WITH DOUBLE VORTEX POINTS ON A TORUS

CHANG-SHOU LIN[1,a] AND SHUSEN YAN[2,b]

[1]Taida Institute of Mathematical Sciences, Center for Advanced Study, National Taiwan University, Taipei 106, Taiwan.
[a]E-mail: cslin@math.ntu.edu.tw
[2]Department of Mathematics, The University of New England, Armidale, NSW 2351, Australia
[b]E-mail: syan@turing.une.edu.au

Abstract

In this paper, we will study the following minimization problem

$$\inf\Big\{\frac{1}{2}\int_\Omega |Du|^2 - 8\pi\ln\int_\Omega e^{u+u_0} :\ u\in H\Big\},$$

where $\big\{u\in H^1_{loc}(\mathbb{R}^2):\ u$ is doubly periodic in Ω, and $\int_\Omega u=0\big\}$, $u_0(x)=-4\pi G(x,p_1)-4\pi G(x,p_2)$ and $G(x,p)$ is the Green function of $-\Delta$ in Ω with singularity at p subject to the periodic boundary condition. We will introduce a quantity $D(p)$ for $p\in\Omega$ and prove that if $D(p)>0$ at a maximum point of u_0, then the above problem has a minimizer.

1. Introduction

Let e_1 and e_2 be two linear independent vectors in $\mathbb{R}^2$, and let

$$\Omega=\big\{x\in\mathbb{R}^2 : x=t_1e_1+t_2e_2,\ \ \text{for some}\ \ t_1,\,t_2\in[0,1]\big\}.$$

Suppose p_1 and p_2 are two points in Ω. Here, p_1 and p_2 may coincide. Let

$$u_0(x)=-4\pi G(x,p_1)-4\pi G(x,p_2), \tag{1.1}$$

Received June 06, 2013 and in revised form August 29, 2013.
AMS Subject Classification: 35J60, 58E11.
Key words and phrases: Concentration, mean field equation, Chern-Simons model.

where $G(x,p_j)$ is the Green function of $-\Delta$ in Ω with singularity at p_j, subject to the periodic boundary condition. That is, $G(x,p_j)$ satisfies

$$-\Delta G(x,p_j)=\delta_{p_j}-\frac{1}{|\Omega|},$$

where $\delta_{p_j}(x)$ is the Dirac measure at $p_j\in\Omega$, and $|\Omega|$ is the measure of Ω.

Define

$$I(u) \;=\; \frac{1}{2}\int_\Omega |Du|^2-8\pi\ln\int_\Omega e^{u+u_0}, \tag{1.2}$$

and

$$\begin{aligned} H \;=\; \big\{u\in H^1_{loc}(\mathbb{R}^2):\; & u \text{ is doubly periodic with periodic cell } \Omega,\\ & \text{and } \int_\Omega u=0\big\}. \end{aligned}\tag{1.3}$$

In this paper, we consider the following minimization problem:

$$\inf\big\{I(u):\; u\in H\big\}. \tag{1.4}$$

By a Moser-Trudinger type inequality, $I(u)$ is bounded from below in H. But it is not coercive in H. It turns out that the existence of a minimizer for (1.4) is a very delicate problem.

It is well-known now that the existence of a minimizer of (1.4) is related to the existence of bubbling solution for the following problem:

$$\begin{cases}\Delta u+\frac{1}{\varepsilon^2}e^{u+u_0}(1-e^{u+u_0})=\frac{8\pi}{|\Omega|}, & \text{in } \Omega,\\ u \;\text{ is doubly periodic in } \partial\Omega\end{cases} \tag{1.5}$$

where and ε is the Chern-Simons constant. In fact, Nolasco and Tarantello [13] proved the following result:

Theorem A. *There is an $\varepsilon_0>0$, such that for any $\varepsilon\in(0,\varepsilon_0]$, (1.5) has a solution u_ε, which can be decomposed to*

$$u_\varepsilon=w_\varepsilon+c_\varepsilon,\qquad \int_\Omega w_\varepsilon=0,$$

for some constant c_ε satisfying $c_\varepsilon\to-\infty$ as $\varepsilon\to0$. And up to a subsequence, one of the following is true

(i) *if* (1.4) *is achieved, then* $w_\varepsilon \to w$ *in* $C^q(\Omega)$ *for any* $q \geq 0$ *as* $\varepsilon \to 0$, *and* w *is a minimizer of* (1.4)*;*

(ii) *if* (1.4) *is not achieved, then there exists a* $p_0 \in \Omega$, *satisfying* $u_0(p_0) = \max_{x\in\Omega} u_0(x)$ *and*

$$\frac{e^{w_\varepsilon + u_0}}{\int_\Omega e^{w_\varepsilon + u_0}} \to \delta_{p_0},$$

in the sense of measure.

Concerning the minimization problem (1.4), the following result was proved in [5]:

Theorem B. *Suppose that* Ω *is a rectangle and* $p_1 = p_2$. *Then* (1.4) *is not achieved.*

More result on this problem can be found in [9].

Note that generally it is very difficult to check whether (1.4) is achieved or not. The aim of this paper is to give another condition, which will guarantee the existence of a minimizer for (1.4).

For any $p \in \Omega$, we define the following quantity

$$D(p) = \lim_{r\to 0} \int_{\Omega\setminus B_r(p)} \frac{e^{8\pi(\gamma(y,p)-\gamma(p,p))+u_0(y)-u_0(p)} - 1}{|y-p|^4}\, dy - \int_{\mathbb{R}^2\setminus\Omega} \frac{1}{|y-p|^2}\, dy,$$

where $\gamma(y,p)$ is the regular part of the Green function $G(y,p)$. Noting that $\Delta u_0 + 8\pi\Delta\gamma(y,p) = 0$, we find that $D(p)$ is well defined.

The main result of this paper is the following:

Theorem 1.1. *If there exists a maximum point* p *of* u_0 *with* $D(p) > 0$, *then* (1.4) *is achieved.*

The quantity $D(p)$ was first introduced in [2]. In the last two decades, existence of bubbling solutions have been studied extensively for Chern-Simons model [1, 6, 7, 10, 11, 12] and the related mean field equation [3, 4, 8]. When the number of the vortex points is bigger than two, the Cherm-Simons model always has bubbling solutions. The case for two vortex points is very delicate. If $D(p) < 0$ at a maximum point of u_0, (1.5) has a bubbling solution. See [11]. Here, we show that if $D(p) > 0$ at a maximum point of u_0, (1.4) is achieved. This result complements to the result in [11].

In the rest of this section, let us explain the proof of Theorem 1.1.

In [13], Nolasco and Tarantello constructed a sequence of solution u_ε for (1.5), satisfying

(i) $u_\varepsilon = w_\varepsilon + c_\varepsilon$ with $\int_\Omega w_\varepsilon = 0$ and

$$e^{c_\varepsilon} = \frac{16\pi\varepsilon^2}{\int_\Omega e^{u_0+w_\varepsilon}\Big(1+\sqrt{1-32\pi\varepsilon^2\frac{\int_\Omega e^{2(u_0+w_\varepsilon)}}{\big(\int_\Omega e^{u_0+w_\varepsilon}\big)^2}}\Big)}. \tag{1.6}$$

(ii) $I(w_\varepsilon) \to \inf_{u\in H} I(u)$ as $\varepsilon \to 0$.

This construction shows that (1.4) has special minimization sequence, which is related to the solution of the Chern-Simons model (1.5). Nolasco and Tarantello [13] proved that the sequence w_ε is either convergent strongly in H to a minimizer of $I(u)$, or blow-up at a maximum point p of the function u_0. In the blow-up case, Lemma 4.12 in [13] shows

$$I(w_\varepsilon) \to -8\pi\big(4\pi\gamma(p,p) + u_0(p) + \ln\pi + 1\big),$$

as $\varepsilon \to 0$. Thus, to prove that (1.4) is achieved, we only need to show that

$$\inf_{u\in H} I(u) < -8\pi\big(4\pi\gamma(p,p) + u_0(p) + \ln\pi + 1\big). \tag{1.7}$$

So, in the next section, we will construct a suitable function $w \in H$, such that

$$I(w) < -8\pi\big(4\pi\gamma(p,p) + u_0(p) + \ln\pi + 1\big),$$

which will imply (1.7).

2. The Proof of the Main Result

Without loss of generality, in this paper, we assume $|\Omega| = 1$.

Let $p \in \Omega$ be a maximum point of u_0. Denote

$$V_{p,\mu}(y) = \ln\frac{8\mu^2}{(1+\mu^2|y-p|^2)^2}, \quad \mu > 0.$$

Then, $u = V_{p,\mu}$ is a solution of

$$\begin{cases} -\Delta u = e^u, & \text{in } \mathbb{R}^2; \\ \int_{\mathbb{R}^2} e^u = 8\pi. \end{cases}$$

Here, we always assume that the constant $\mu > 0$ is large.

In this section, we will construct a function $w_\mu \in H$ such that

$$I(w_\mu) < -8\pi\big(4\pi\gamma(p,p) + u_0(p) + \ln\pi + 1\big). \tag{2.1}$$

Firstly, we define w_μ^* as follows.

$$w_\mu^*(y) = \begin{cases} V_{p,\mu}(y) + 8\pi\big(\gamma(y,p) - \gamma(p,p)\big)\big(1 - \frac{1}{\theta\mu^2}\big), & y \in B_d(p); \\ V_{0,\mu}(d) + 8\pi\big(G(y,p) - \frac{1}{2\pi}\ln\frac{1}{d} - \gamma(p,p)\big)\big(1 - \frac{1}{\theta\mu^2}\big), & y \in \Omega \setminus B_d(p), \end{cases} \tag{2.2}$$

where $\theta > 0$ is a fixed small constant, and the constant d is chosen to make $w_\mu^* \in C^1(\Omega)$. Thus, d satisfies

$$\mu V'(\mu d) = -\frac{4}{d}\big(1 - \frac{1}{\theta\mu^2}\big),$$

where $V(y) = V_{0,1}(y)$, which gives

$$\frac{(\mu d)^2}{1 + (\mu d)^2} = 1 - \frac{1}{\theta\mu^2}.$$

So, we obtain

$$d^2 = \theta + O\big(\frac{1}{\mu^2}\big). \tag{2.3}$$

Then, we take $L_\mu = -\int_\Omega w_\mu^*(y)\,dy$, and

$$w_\mu(y) = w_\mu^*(y) + L_\mu. \tag{2.4}$$

So $\int_\Omega w_\mu(y)\,dy = 0$. We will prove in this section that if $\mu > 0$ is large enough, (2.1) holds. We have

Proposition 2.2. *We have*

$$I(w_\mu)=-8\pi(4\pi\gamma(p,p)+\max_{x\in\Omega}u_0(x)+\ln\pi+1)-\frac{8D(p)}{\mu^3}+O\Big(\frac{\ln\mu}{\mu^4}\Big).$$

Proof. We have

$$\begin{aligned}I(w_\mu)&=\frac{1}{2}\int_\Omega|Dw_\mu|^2-8\pi\ln\int_\Omega e^{w_\mu^*+L_\mu+u_0(p)+u_0(y)-u_0(p)}\\&=\frac{1}{2}\int_\Omega|Dw_\mu|^2-8\pi L_\mu-8\pi u_0(p)-8\pi\ln\int_\Omega e^{w_\mu^*+u_0(y)-u_0(p)}.\end{aligned}\tag{2.5}$$

We estimate $\frac{\partial I(w_\mu)}{\partial\mu}$ first:

$$\frac{\partial I(w_\mu)}{\partial\mu}=-\int_\Omega\Delta w_\mu\frac{\partial w_\mu}{\partial\mu}-8\pi\frac{\partial L_\mu}{\partial\mu}-\frac{8\pi\int_\Omega e^{w_\mu^*+u_0(y)-u_0(p)}\frac{\partial w_\mu^*}{\partial\mu}}{\int_\Omega e^{w_\mu^*+u_0(y)-u_0(p)}}.\tag{2.6}$$

Noting that $\frac{\partial w_\mu}{\partial\mu}=\frac{\partial w_\mu^*}{\partial\mu}+\frac{\partial L_\mu}{\partial\mu}$ and $\int_\Omega\Delta w_\mu=0$, we find

$$\begin{aligned}-\int_\Omega\Delta w_\mu\frac{\partial w_\mu}{\partial\mu}&=-\int_\Omega\Delta w_\mu\frac{\partial w_\mu^*}{\partial\mu}-\frac{\partial L_\mu}{\partial\mu}\int_\Omega\Delta w_\mu\\&=-\int_\Omega\Delta w_\mu\frac{\partial w_\mu^*}{\partial\mu}\\&=\int_\Omega\big(1_{B_d(p)}e^{V_{p,\mu}}-8\pi\big(1-\frac{1}{\theta\mu^2}\big)\big)\frac{\partial w_\mu^*}{\partial\mu}.\end{aligned}\tag{2.7}$$

Since $L_\mu=-\int_\Omega w_\mu^*$, we find

$$-\int_\Omega\frac{\partial w_\mu^*}{\partial\mu}=\frac{\partial L_\mu}{\partial\mu}.\tag{2.8}$$

So, we obtain from (2.7) and (2.8)

$$-\int_\Omega\Delta w_\mu\frac{\partial w_\mu}{\partial\mu}-8\pi\frac{\partial L_\mu}{\partial\mu}=\int_\Omega\big(1_{B_d(p)}e^{V_{p,\mu}}+\frac{8\pi}{\theta\mu^2})\big)\frac{\partial w_\mu^*}{\partial\mu}.\tag{2.9}$$

Write

$$w_\mu^*=V_{p.\mu}+\xi_\mu,$$

where

$$\xi_\mu=\begin{cases}8\pi(\gamma(y,p)-\gamma(p,p))\big(1-\frac{1}{\theta\mu^2}\big), & \text{in } B_d(p),\\ V_\mu(d)-V_{p.\mu}+8\pi(G(y,p)-\frac{1}{2\pi}\ln\frac{1}{d}-\gamma(p,p))\big(1-\frac{1}{\theta\mu^2}\big), & \text{in } \Omega\setminus B_d(p).\end{cases}$$

Then

$$\frac{\partial\xi_\mu}{\partial\mu}=\begin{cases}8\pi(\gamma(y,p)-\gamma(p,p))\frac{2}{\theta\mu^3}, & \text{in } B_d(p),\\ \frac{\partial}{\partial\mu}\big(V_\mu(d)-V_{p,\mu}\big)+8\pi(G(y,p)-\frac{1}{2\pi}\ln\frac{1}{d}-\gamma(p,p))\frac{2}{\theta\mu^3} & \text{in } \Omega\setminus B_d(p).\end{cases}$$

Using the above estimate, we find

$$\begin{aligned}\int_\Omega 1_{B_d(p)}e^{V_{p,\mu}}\frac{\partial\xi_\mu}{\partial\mu}&=\frac{16\pi}{\theta\mu^3}\int_{B_d(p)}e^{V_{p,\mu}}(\gamma(y,p)-\gamma(p,p))\\ &= O\Big(\frac{1}{\mu^3}\int_{B_d(p)}e^{V_{p,\mu}}|y-p|^2\Big)=O\big(\frac{\ln\mu}{\mu^5}\big).\end{aligned}\tag{2.10}$$

and

$$\begin{aligned}\frac{1}{\theta\mu^2}\int_\Omega\frac{\partial\xi_\mu}{\partial\mu}&=\frac{1}{\theta\mu^2}\int_{\Omega\setminus B_d(p)}\frac{\partial}{\partial\mu}\big(V_\mu(d)-V_{p,\mu}\big)+O\big(\frac{1}{\mu^5}\big)\\ &= \frac{1}{\theta\mu^2}\int_{\Omega\setminus B_d(p)}\Big(\frac{4\mu|y-p|^2}{1+\mu^2|y-p|^2}-\frac{4\mu d^2}{1+\mu^2d^2}\Big)+O\big(\frac{1}{\mu^5}\big)\\ &= \frac{1}{\theta\mu^2}\int_{\Omega\setminus B_d(p)}\Big[\frac{4}{\mu}\Big(1+O\big(\frac{1}{\mu^2|y-p|^2}\big)\Big)-\frac{4}{\mu}\Big(1+O\big(\frac{1}{\mu^2d^2}\big)\Big)\Big]+O\big(\frac{1}{\mu^5}\big)\\ &= O\big(\frac{1}{\mu^5}\big).\end{aligned}\tag{2.11}$$

Combining (2.9), (2.10), and (2.11), we are led to

$$-\int_\Omega\Delta w_\mu\frac{\partial w_\mu}{\partial\mu}-8\pi\frac{\partial L_\mu}{\partial\mu}=\int_\Omega\big(1_{B_d(p)}e^{V_{p,\mu}}+\frac{8\pi}{\theta\mu^2})\big)\frac{\partial V_{p,\mu}}{\partial\mu}+O\big(\frac{\ln\mu}{\mu^5}\big).\tag{2.12}$$

On the other hand, by (2.3), we have

$$\begin{aligned}&\int_\Omega\big(1_{B_d(p)}e^{V_{p,\mu}}+\frac{8\pi}{\theta\mu^2}\big)\frac{\partial V_{p,\mu}}{\partial\mu}\\ &= \int_{B_d(p)}e^{V_{p,\mu}}\frac{\partial V_{p,\mu}}{\partial\mu}+\frac{8\pi}{\theta\mu^2}\big(-\frac{2}{\mu}+\int_\Omega\frac{4}{\mu(1+\mu^2|y-p|^2)}\big)\\ &= \int_{B_d(x)}e^{V_{x,\mu}}\frac{\partial V_{x,\mu}}{\partial\mu}-\frac{16\pi}{\theta\mu^3}+O\big(\frac{\ln\mu}{\mu^5}\big)\end{aligned}$$

$$
\begin{aligned}
&= \int_{B_d(x)} e^{V_{x,\mu}} \frac{\partial V_{x,\mu}}{\partial \mu} - \frac{16\pi}{d^2\mu^3} + O\big(\frac{\ln \mu}{\mu^5}\big) \\
&= \int_{B_d(x)} e^{V_{x,\mu}} \frac{\partial V_{x,\mu}}{\partial \mu} - \frac{16}{\mu^3} \int_{\mathbb{R}^2 \setminus B_d(x)} \frac{dy}{|y-x|^4} + O\big(\frac{\ln \mu}{\mu^5}\big) \\
&= \int_{B_d(x)} e^{V_{x,\mu}} \frac{\partial V_{x,\mu}}{\partial \mu} + \int_{\mathbb{R}^2 \setminus B_d(x)} e^{V_{x,\mu}} \frac{\partial V_{x,\mu}}{\partial \mu} + O\big(\frac{\ln \mu}{\mu^5}\big) \\
&= \int_{\mathbb{R}^2} e^{V_{x,\mu}} \frac{\partial V_{x,\mu}}{\partial \mu} + O\big(\frac{\ln \mu}{\mu^5}\big) = O\big(\frac{\ln \mu}{\mu^5}\big),
\end{aligned}
\tag{2.13}
$$

which, together with (2.12), gives

$$
-\int_\Omega \Delta w_\mu \frac{\partial w_\mu}{\partial \mu} - 8\pi \frac{\partial L_\mu}{\partial \mu} = O\big(\frac{\ln \mu}{\mu^5}\big). \tag{2.14}
$$

Inserting (2.14) into (2.6), we find

$$
\frac{\partial I(w_\mu)}{\partial \mu} = -\frac{8\pi \int_\Omega e^{w_\mu^* + u_0(y) - u_0(p)} \frac{\partial w_\mu^*}{\partial \mu}}{\int_\Omega e^{w_\mu^* + u_0(y) - u_0(p)}} + O\big(\frac{\ln \mu}{\mu^5}\big). \tag{2.15}
$$

Let

$$
f(y,p) = 8\pi\big(\gamma(y,p) - \gamma(p,p)\big) + u_0(y) - u_0(p). \tag{2.16}
$$

It is easy to check that

$$
\begin{aligned}
\int_\Omega e^{w_\mu^* + u_0(y) - u_0(p)} \frac{\partial \xi_\mu}{\partial \mu} &= \frac{16\pi}{\theta\mu^3} \int_{B_d(p)} e^{V_{p,\mu} + f(y,p)} (\gamma(y,p) - \gamma(p,p)) + O\big(\frac{1}{\mu^5}\big) \\
&= O\big(\frac{\ln \mu}{\mu^5}\big),
\end{aligned}
\tag{2.17}
$$

and

$$
\int_\Omega e^{w_\mu^* + u_0(y) - u_0(p)} = 8\pi + O\big(\frac{\ln \mu}{\mu^2}\big). \tag{2.18}
$$

From (2.15), (2.17) and (2.18), we obtain

$$
\frac{\partial I(w_\mu)}{\partial \mu} = -\frac{8\pi \int_\Omega e^{w_\mu^* + u_0(y) - u_0(p)} \frac{\partial V_{p,\mu}}{\partial \mu}}{\int_\Omega e^{w_\mu^* + u_0(y) - u_0(p)}} + O\big(\frac{\ln \mu}{\mu^5}\big). \tag{2.19}
$$

Let

$$
f_\mu(y,p) = 8\pi\big(\gamma(y,p) - \gamma(p,p)\big)\big(1 - \frac{1}{\theta\mu^2}\big) + u_0(y) - u_0(p). \tag{2.20}
$$

Then

$$\begin{aligned}
&\int_\Omega e^{w_\mu^*+u_0(y)-u_0(p)}\frac{\partial V_{p,\mu}}{\partial\mu}\\
&=\int_{B_d(p)} e^{V_{p,\mu}+f_\mu(y,p)}\frac{\partial V_{p,\mu}}{\partial\mu}+\int_{\Omega\setminus B_d(p)} e^{w_\mu^*+u_0(y)-u_0(p)}\frac{\partial V_{p,\mu}}{\partial\mu}\\
&=\int_{B_d(p)} e^{V_{p,\mu}+f_\mu(y,p)}\frac{\partial V_{p,\mu}}{\partial\mu}+\frac{8\mu^2}{(1+\mu^2d^2)^2}\int_{\Omega\setminus B_d(p)}\frac{d^{4(1-\frac{1}{\theta\mu^2})}}{|y-p|^{4(1-\frac{1}{\theta\mu^2})}}e^{f_\mu(y,p)}\frac{\partial V_{p,\mu}}{\partial\mu}\\
&=\int_{B_d(p)} e^{V_{p,\mu}+f_\mu(y,p)}\frac{\partial V_{p,\mu}}{\partial\mu}+\frac{8}{\mu^2}\int_{\Omega\setminus B_d(p)}\frac{1}{|y-p|^4}e^{f_\mu(y,p)}\frac{\partial V_{p,\mu}}{\partial\mu}+O\big(\frac{1}{\mu^5}\big)\\
&=\int_\Omega e^{V_{p,\mu}+f_\mu(y,p)}\frac{\partial V_{p,\mu}}{\partial\mu}+O\big(\frac{1}{\mu^5}\big)\\
&=\int_\Omega e^{V_{p,\mu}}\frac{\partial V_{p,\mu}}{\partial\mu}+\int_\Omega e^{V_{p,\mu}}\big(e^{f_\mu(y,p)}-1\big)\frac{\partial V_{p,\mu}}{\partial\mu}+O\big(\frac{1}{\mu^5}\big). \qquad (2.21)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_\Omega e^{V_{p,\mu}}\frac{\partial V_{p,\mu}}{\partial\mu} &= \frac{\partial}{\partial\mu}\int_\Omega e^{V_{p,\mu}}\\
&= -\frac{\partial}{\partial\mu}\int_{\mathbb{R}^2\setminus\Omega} e^{V_{p,\mu}}=\frac{16}{\mu^3}\int_{\mathbb{R}^2\setminus\Omega}\frac{1}{|y-p|^4}\,dy+O\big(\frac{1}{\mu^5}\big). \qquad (2.22)
\end{aligned}$$

Noting that

$$e^{f_\mu(y,p)}-1=e^{f(y,p)}-1+O\big(\frac{1}{\mu^2}\big),$$

we find

$$\int_\Omega e^{V_{p,\mu}}\big(e^{f_\mu(y,p)}-1\big)\frac{\partial V_{p,\mu}}{\partial\mu}=\int_\Omega e^{V_{p,\mu}}\big(e^{f(y,p)}-1\big)\frac{\partial V_{p,\mu}}{\partial\mu}+O\big(\frac{1}{\mu^5}\big). \qquad (2.23)$$

Combining (2.21), (2.22) and (2.23), we are led to

$$\begin{aligned}
&\int_\Omega e^{w_\mu^*+u_0(y)-u_0(p)}\frac{\partial V_{p,\mu}}{\partial\mu}\\
&= \frac{16}{\mu^3}\int_{\mathbb{R}^2\setminus\Omega}\frac{1}{|y-p|^4}\,dy+\int_\Omega e^{V_{p,\mu}}\big(e^{f(y,p)}-1\big)\frac{\partial V_{p,\mu}}{\partial\mu}+O\big(\frac{\ln\mu}{\mu^5}\big)\\
&= \frac{16}{\mu^3}\int_{\mathbb{R}^2\setminus\Omega}\frac{1}{|y-p|^4}\,dy-\int_\Omega\frac{16}{\mu^3|y-p|^4}\big(e^{f(y,p)}-1\big)+O\big(\frac{\ln\mu}{\mu^5}\big)\\
&= -\frac{16D(p)}{\mu^3}+O\big(\frac{\ln\mu}{\mu^5}\big). \qquad (2.24)
\end{aligned}$$

From (2.24) and (2.18), we obtain

$$-\frac{8\pi\int_\Omega e^{w_\mu^*+u_0(y)-u_0(p)}\frac{\partial w_\mu^*}{\partial\mu}}{\int_\Omega e^{w_\mu^*+u_0(y)-u_0(p)}}=\frac{16D(p)}{\mu^3}+O\big(\frac{\ln\mu}{\mu^5}\big). \tag{2.25}$$

Combining (2.19) and (2.25), we are led to

$$\frac{\partial I(w_\mu)}{\partial\mu}=\frac{16D(p)}{\mu^3}+O\big(\frac{\ln\mu}{\mu^5}\big). \tag{2.26}$$

We are now ready to estimate $I(w_\mu)$. From (2.26), we find

$$\lim_{t\to+\infty}I(w_t)-I(w_\mu)=\frac{8D(p)}{\mu^2}+O\big(\frac{\ln\mu}{\mu^4}\big), \tag{2.27}$$

which gives

$$I(w_\mu)=\lim_{t\to+\infty}I(w_t)-\frac{8D(p)}{\mu^2}+O\big(\frac{\ln\mu}{\mu^4}\big). \tag{2.28}$$

Finally, we estimate $\lim_{t\to+\infty}I(w_t)$. By (2.5),

$$I(w_t)=-\frac12\int_\Omega w_t\Delta w_t-8\pi\ln\int_\Omega e^{w_t^*+u_0(y)-u_0(p)}-8\pi L_t-8\pi u_0(p) \tag{2.29}$$

But from $\int_\Omega w_t=0$,

$$-\frac12\int_\Omega w_t\Delta w_t=\frac12\int_{B_d(p)}e^{V_{p,t}}w_t+4\pi(1-\frac{1}{\theta t^2})\int_\Omega w_t=\frac12\int_{B_d(p)}e^{V_{p,t}}w_t. \tag{2.30}$$

It is easy to check

$$\int_\Omega e^{w_t^*+u_0(y)-u_0(p)}\to 8\pi,\quad \text{as } t\to+\infty, \tag{2.31}$$

and

$$\begin{aligned}&\frac12\int_{B_d(p)}e^{V_{p,t}}w_t-8\pi L_t\\ &=\frac12\big(\ln(8t^2)+L_t\big)\int_{B_d(p)}e^{V_{p,t}}-8\pi L_t+\frac12\int_{B_d(p)}e^{V_{p,t}}\ln\frac{1}{(1+t^2|y-p|^2)^2}\\ &\quad+\frac12\int_{B_d(p)}e^{V_{p,t}}8\pi\big(\gamma(y,p)-\gamma(p,p)\big)\big(1-\frac{1}{\theta t^2}\big)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\big(\ln(8t^2)+L_t\big)\big(8\pi+O\big(\frac{1}{t^2}\big)\big) - 8\pi L_t+\int_{\mathbb{R}^2}\frac{4}{(1+|y|^2)^2}\ln\frac{1}{(1+|y|^2)^2}+o(1)\\
&= \frac{1}{2}\big(\ln(8t^2) - L_t\big)8\pi - 8\pi + o(1) + O\big(\frac{|L_t|}{t^2}\big), \qquad (2.32)
\end{aligned}
$$

since

$$
\begin{aligned}
\int_{\mathbb{R}^2}\frac{4}{(1+|y|^2)^2}\ln\frac{1}{(1+|y|^2)^2} &= 8\pi\int_0^{+\infty}\frac{1}{(1+t)^2}\ln\frac{1}{1+t}\,dt\\
&= 8\pi\int_0^{+\infty}\frac{1}{(1+t)^2}\,dt = 8\pi.
\end{aligned}
$$

Moreover, from $\int_\Omega G(y,p)\,dy = 0$, we get

$$
\int_\Omega \gamma(y,p)\,dy = -\frac{1}{2\pi}\int_\Omega \ln\frac{1}{|y-p|}\,dy.
$$

As a result,

$$
\begin{aligned}
-L_t=\int_\Omega w_t^* &= \int_{B_d(p)}\big(V_{p,t} - 4\big(1-\frac{1}{\theta t^2}\big)\ln\frac{1}{|y-p|}\big)\,dy\\
&\quad+\int_{\Omega\setminus B_d(p)}\big(V_t(d)-4\big(1-\frac{1}{\theta t^2}\big)\ln\frac{1}{d}\big)dy-\gamma(p,p)8\pi\big(1-\frac{1}{\theta t^2}\big)\\
&= \pi d^2\ln(8t^{-2}) + \int_{B_d(p)}\ln\frac{|y-p|^4}{(\frac{1}{t^2}+|y-p|^2)^2}\,dy\\
&\quad+(1-\pi d^2)\Big(\ln\frac{8}{t^2}+\ln\frac{d^4}{(d^2+\frac{1}{t^2})^2}\Big) - 8\pi\gamma(p,p)+o(1)\\
&= \ln(8t^{-2}) - 8\pi\gamma(p,p) + o(1). \qquad (2.33)
\end{aligned}
$$

From (2.32) and (2.33), we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_{B_d(p)} e^{V_{p,t}}w_t - 8\pi L_t\\
&= \frac{1}{2}\Big(\ln(8t^2)+\ln(8t^{-2}) - 8\pi\gamma(p,p)+o(1)\Big)8\pi - 8\pi + o(1) + O\big(\frac{\ln t}{t^2}\big)\\
&= 8\pi\big(\ln 8 - 4\pi\gamma(p,p) - 1\big) + o(1). \qquad (2.34)
\end{aligned}
$$

Combining (2.29), (2.30), (2.31) and (2.34), we obtain

$$
\begin{aligned}
I(w_t) &= 8\pi\big(\ln 8 - 4\pi\gamma(p,p) - 1\big) - 8\pi u_0(p) - 8\pi\ln(8\pi) + o(1)\\
&= -8\pi\big(4\pi\gamma(p,p)+u_0(p)+\ln\pi+1\big)+o(1),
\end{aligned}
$$

which gives

$$\lim_{t\to+\infty} I(w_t) = -8\pi\big(4\pi\gamma(p,p) + u_0(p) + \ln\pi + 1\big). \qquad \square$$

Proof of Theorem 1.1. If $D(p) > 0$, it follows from Proposition 2.2,

$$\inf_{u\in H} I(u) \le I(w_\mu) < -8\pi\big(4\pi\gamma(p,p) + u_0(p) + \ln\pi + 1\big),$$

if $\mu > 0$ is large enough. Thus the minimizing sequence constructed in [13] converges strongly in H and (1.4) is achieved. $\square$

Acknowledgment

The second author is partially supported by ARC.

References

1. H. Chan, C.-C. Fu and C.-S. Lin, Non-topological multi-vortex solutions to the self-dual Chern-Simons-Higgs equation, *Comm. Math. Phys.*, **231**(2002), no. 2, 189-221.
2. S.-Y. A. Chang, C.-C. Chen and C.-S. Lin, Extremal functions for a mean field equation in two dimension. Lectures on partial differential equations, 61–93, *New Stud. Adv. Math.*, 2, Int. Press, Somerville, MA, 2003.
3. C.-C. Chen and C.-S. Lin, Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces, *Comm. Pure Appl. Math.*, **55**(2002), 728-771.
4. C.-C. Chen and C.-S. Lin, Topological degree for a mean field equation on Riemann surfaces, *Comm. Pure Appl. Math.*, **56**(2003), 1667-1727.
5. C.-C. Chen, C.-S. Lin and G. Wang, Concentration phenomena of two-vortex solutions in a Chern-Simons model, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5), **3**(2004), no. 2, 367-397.
6. K. Choe, Asymptotic behavior of condensate solutions in the Chern-Simons-Higgs theory, *J. Math. Phy.*, **48**(2007), 103501.
7. K. Choe and N. Kim, Blow-up solutions of the self-dual Chern-Simons-Higgs vertex equation, *Ann. I.H. Poinaré Anal., Non Linéaire*, **25**(2008), 313-338.
8. P. Esposito, M. Grossi and A. Pistoia, On the existence of blowing-up solutions for a mean field equation, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 22(2005), 227-257.
9. C.-S. Lin and C.-L. Wang, Elliptic functions, Green functions and the mean field equation on tori, *Ann. Math.* (2), **172**(2010), no. 2, 911-954.

10. C.-S. Lin abd S. Yan, Bubbling solutions for relativistic Abelian Chern-Simons model on a torus, *Comm. Math. Phy.*, **297**(2010), no. 3, 733-758.

11. C.-S. Lin abd S. Yan, Existence of bubbling solutions for Chern-Simons model on a torus, *Arch. Ration. Mech. Anal.*, **207**(2013), no. 2, 353-392.

12. C.-S. Lin and S. Yan, Bubbling solutions for the SU(3) Chern-Simons model on a torus, *Comm. Pure Appl. Math.*, **66**(2013), 991-1027.

13. M. Nolasco, G. Tarantello, Double vortex condensates in the Chern-Simons-Higgs theory, *Calc. Var. and PDE*, **9**(1999), 31-94.