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A NOTE ON THE SECOND MAIN THEOREM FOR HOLOMORPHIC CURVES INTO ALGEBRAIC VARIETIES

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Abstract

In this short note, we extend the Second Main Theorem established by Min Ru [20] to holomorphic curves into algebraic varieties intersecting numerically equivalent ample divisors.

1. Introduction

In recent years, there has been some significant progress in the study of both qualitative and quantitative aspects of the geometric and arithmetic properties of the complement of divisors in an algebraic projective variety. For recent qualitative results, where the divisors are not necessarily linearly equivalent, see [3], [5], [15], [13], and [11]; for recent quantitative results, see [4], [6],[7], [8], [19] and [20]. The qualitative results began from the Little Picard theorem in the geometric (complex analysis) side, and Siegel's theorem in the arithmetic side, while the quantitative aspect started from Nevanlinna's Second Main Theorem for meromorphic functions (as well as H. Cartan's theorem in the higher dimensions), and Roth's theorem in Diophantine approximation (as well Schmidt's subspace theorem in the higher dimensions). The quantitative results, in the spirit of Nevanlinna-Roth-Cartan-Schmidt, extend and strengthen the qualitative results.

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The above mentioned progress was essentially initiated by the breakthrough method introduced by Corvaja and Zannier [3], where they used Schmidt's subspace theorem to give a new proof of the classical result of Siegel on integral points on affine curves. In their Annals paper [5], they applied the method to study integral points on the complement of divisors in the surface, where the divisors are not necessarily linear equivalent. Later, Aaron Levin [13] significantly improved their results and obtained the sharp result in the surface case, as well as extended the results to higher dimensions. However, all results obtained are of qualitative-type. One of the main results in [5](see Theorem 1 in [5]) is stated as follows: *Let X be a geometrically irreducible nonsingular algebraic surface and $D_1, \dots, D_q$ be distinct irreducible divisors located in general position on X, both defined over a number field. Assume that there exist positive integers $n_1, \dots, n_q$ such that $(n_iD_i).(n_jD_j)$ is a positive constant (i.e independent of i, j for all pairs $1 \leq i, j \leq q$). If $q \geq 4$, then the S-integral points of $X \backslash \{D_1, \dots, D_q\}$ is degenerate, i.e. there is a curve on X containing all the S-integral points in $X(k)$.* In that paper they made a further remark (see the last three lines on Page 706, [5]) that one may prove that *the condition that $(n_iD_i).(n_jD_j)$ is constant amounts to the n_jD_j, $1 \leq j \leq q$, being numerically equivalent.* This is indeed an easy consequence of the Hodge Index Theorem, as shown in this manuscript (see Corollary 2.4). Nevertheless, it gives a strong motivation to study Schmidt's subspace theorem and the Second Main-type Theorem in Nevanlinna theory for numerically equivalent divisors.

On the other hand, on the quantitative side, J.H. Evertse and R. Ferretti [6], [7], by using a different method, established a Schmidt's subspace-type Theorem for the complement of divisors in an arbitrary projective variety $X \subset \mathbb{P}^N$ where the divisors are coming from hypersurfaces in $\mathbb{P}^N$. By a slight reformulation, one actually only needs to assume that the divisors are linearly equivalent on X to an ample divisor. The discussion above thus naturally leads to the question whether the result still holds for divisors which are only numerically equivalent. Such result in arithmetic part was just established by Aaron Levin in his recent preprint [14]. We note that the extension of Evertse and Ferretti's result to numerically equivalent divisors immediately implies the result of Corvaja and Zannier in [5] mentioned earlier, using Corollary 2.4. Moreover, it indeed gives a quantitative extension of their result. The counterpart of Evertse and Ferretti's result (see also [4]) in Nevanlinna theory is due to Min Ru [20] (see also [19]), where he proved a Second Main Theorem for algebraically non-degenerate holomorphic map $f : \mathbb{C} \to X$ intersecting D, where $X \subset \mathbb{P}^N$ is an arbitrary smooth projective variety, and D is the union of divisors coming from hypersurfaces in $\mathbb{P}^N$. The purpose of this short note is to extend Min Ru's result to numerically equivalent divisors, following the argument of Levin. We note that the counterpart of Corvaja and Zannier ([5]) in Nevanlinna theory is due to Liu-Ru [15], and the result obtained in this paper again gives a quantitative extension of Liu-Ru's result.

It is also worth noting that, as being mentioned in Corvaja and Zannier's paper ([5], paragraph spanning pages 708-709), Theorem 1 in [5] intersects the results due to Vojta on semi-abelian varieties (see [23]). Indeed, Corollary 0.3 in [23] generalizes the Corvaja-Zannier result [5] to all dimensions (with 4 in the surface case replaced by dim $X+2$ in general). This requires a slight bit of extra work (e.g., the Picard number ρ in Vojta's result needs to be replaced by the (free) rank of the subgroup of the Neron-Severi group generated by the irreducible components of D; one must also use that numerical equivalence and algebraic equivalence agree up to torsion). Similarly, one has the corresponding analytic qualitative result (using Bloch's conjecture and its generalizations). We also refer the readers to the related paper [17] of Noguchi and Winkelmann, where some further results along these lines are discussed. Thus, the main result of this paper more generally gives a quantitative generalization of these results (in all dimensions).

2. Some Background Material

In this section, we briefly recall some definitions and facts, especially the definition of the Weil and height (characteristic) functions that will be used throughout the paper. Let X be a smooth complex projective variety and $L \to X$ be a positive line bundle. Denote by $\|\cdot\|$ a hermitian fiber metric in L and by ω its Chern form. Let $f : \mathbb{C} \to X$ be a holomorphic map. We define

$$T_{f,L}(r) = \int_1^r \frac{dt}{t} \int_{|z|<t} f^*\omega,$$

and call it the *characteristic* (or *height*) function of f with respect to L. It is independent of, up to bounded term, the choices of the metric on L. The definition can be extended to arbitrary line bundle. Indeed, since any line bundle L can be written as $L = L_1 \otimes L_2^{-1}$ with L_1, L_2 are both positive, we define $T_{f,L}(r) = T_{f,L_1}(r) - T_{f,L_2}(r)$. For an effective divisor D on X, we define

$$T_{f,D}(r) := T_{f,\mathcal{O}(D)}(r). \tag{2.1}$$

If $X = \mathbb{P}^n(\mathbb{C})$ and $L = \mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)$, then we simply write $T_{f,\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)}(r)$ as $T_f(r)$. The characteristic (or height) function $T_{f,L}(r)$ (or $T_{f,D}(r)$) satisfies the following properties (see [22]):

(a) **Additivity**: If L_1 and L_2 are two line bundles on X, then

$$T_{f,L_1\otimes L_2}(r) = T_{f,L_1}(r) + T_{f,L,_2}(r) + O(1).$$

(b) **Funtoriality**: If $\phi : X \to X'$ is a morphism and if L is a line bundle on X', then

$$T_{f,\phi^* L}(r) = T_{\phi\circ f,L}(r) + O(1).$$

(c) **Base locus**: If the image of f is not contained in the base locus of L, then $T_{f,L}(r)$ is bounded from below.

(d) **Globally generated line bundles**: If L is a line bundle over X, and is generated by its global sections, then $T_{f,L}(r)$ is bounded from below.

Let $D = (s)$ be a divisor on X with $s \in H^0(X, L)$, where $H^0(X, L)$ is the set of holomorphic sections of L. Assume that image of f is not contained in the support of D. The **proximity function** for f relative to D is the function

$$m_f(r, D) = \int_0^{2\pi} \log \frac{1}{\|s(f(re^{i\theta}))\|} \frac{d\theta}{2\pi}.$$

The counting function is defined as

$$N_f(r, D) = \int_1^r \frac{n_f(t, D)}{t} dt$$

where $n_f(t, D)$ is the number of zeros of f^*s inside $\{|z| < t\}$, counting multiplicities.

By the First Main Theorem, we have

$$T_{f,D}(r) = m_f(r, D) + N_f(r, D) + O(1).$$

Below, we give an alternative definition of the proximity function $m_f(r, D)$ using the notion of Weil-function.

Definition 2.1. Let D be a Cartier divisor on X. A local *Weil function* for D is a function $\lambda_D : (X \backslash \mathrm{Supp} D) \to \mathbb{R}$ such that for all $x \in X$ there is an open neighborhood U of x in X, a nonzero rational function f on X with $D|_U = (f)$, and a continuous function $\alpha : U \to \mathbb{R}$ such that

$$\lambda_D(x) = -\log |f(x)| + \alpha(x)$$

for all $x \in (U \backslash \mathrm{Supp}\ D)$.

Note that a continuous (fiber) metric $\|\cdot\|$ on the line sheaf $\mathcal{O}_X(D)$ determines a Weil function for D given by $\lambda_D(x) = -\log \|s(x)\|$ where s is the rational section of $\mathcal{O}_X(D)$ such that $D = (s)$. An example of Weil function for the hyperplanes $H = \{a_0 x_0 + \cdots + a_n x_n = 0\}$ is given by

$$\lambda_H(x) = \log \frac{\max_{0 \le i \le n} |x_i| \max_{0 \le i \le n} |a_i|}{|a_0 x_0 + \cdots + a_n x_n|}.$$

We define, for any holomorphic map $f : \mathbb{C} \to X$ whose image is not contained in the support of D,

$$m_f(r, D) = \int_0^{2\pi} \lambda_D(f(re^{i\theta})) \frac{d\theta}{2\pi}.$$

This definition agrees, up to a bound term, with the definition given earlier.

Weil functions λ_D satisfy analogues of properties which the height functions carry (see (a)-(d) above) for all $P \in X$ where the relevant Weil-functions are defined (see [22]):

(a) **Additivity**: If λ_1 and λ_2 are Weil functions for Cartier divisors D_1 and D_2 on X, respectively, then $\lambda_1 + \lambda_2$ extends uniquely to a Weil function for $D_1 + D_2$.

(b) **Functoriality**: If λ is a Weil function for a Cartier divisor D on X, and if $\phi : X' \to X$ is a morphism such that $\phi(X') \not\subset \mathrm{Supp}D$, then $x \mapsto \lambda(\phi(x))$ is a Weil function for the Cartier divisor ϕ^*D on X'.

(c) **Normalization**: If $X = \mathbb{P}^n$, and if $D = \{x_0 = 0\} \subset X$ is the hyperplane at infinity, then the function

$$\lambda_D([x_0 : \cdots : x_n]) := \log \frac{\max\{|x_0|, \ldots, |x_n|\}}{|x_0|}$$

is a Weil function for D.

(d) **Uniqueness**: If both λ_1 and λ_2 are Weil functions for a Cartier divisor D on X, then $\lambda_1 = \lambda_2 + O(1)$.

(e) **Boundedness from below**: If D is an effective divisor and λ is a Weil function for D, then λ is bounded from below.

(f) **Principal divisors**: If D is a principal divisor (f), then $-\log|f|$ is a Weil function for D.

Hence we have

Proposition 2.2. *Let $f : \mathbb{C} \to X$ be a holomorphic map. The proximity function and counting function of f have the following properties.*

(a) **Additivity**: *If D_1 and D_2 are two divisors on X, then*

$$m_f(r, D_1 + D_2) = m_f(r, D_1) + m_f(r, D_2) + O(1)$$

$$N_f(r, D_1 + D_2) = N_f(r, D_1) + N_f(r, D_2) + O(1).$$

(b) **Funtoriality**: *If $\phi : X \to X'$ is a morphism and D' is a divisor on X' whose support does not contain the image of $\phi \circ f$, then*

$$m_f(r, \phi^*D') = m_{\phi\circ f}(r, D') + O(1) \quad \textit{and} \quad N_f(r, \phi^*D') = N_{\phi\circ f}(r, D') + O(1).$$

(c) **Effective divisors**: *If D is effective, then $m_f(r, D)$ and $N_f(r, D)$ are bounded from below.*

In each of the above cases, the implied constants in $O(1)$ depends on the varieties, divisors, and morphisms, but not on f and r.

We also recall some notations and results in algebraic geometry. Let X be a smooth projective variety. Two divisors D_1 and D_2 are said to be *linearly equivalent on* X, denoted by $D_1 \sim D_2$, if $D_1 - D_2 = (f)$ for some meromorphic function f on X. This is the same as saying there is a sheaf isomorphism $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2), 1 \mapsto f$. Two divisors D_1 and D_2 are said to be *numerically equivalent on* X, denoted by $D_1 \equiv D_2$, if $D_1.C = D_2.C$ for all irreducible curves C on X. Obviously, linearly equivalence implies numerical equivalence. Recall that the intersection numbers are defined as follows: According to the result of Kleiman, let F be a coherent sheaf and $L_1, \ldots, L_t$ be t line bundles over X, then $\chi(X, L_1^{n_1} \otimes \cdots \otimes L_t^{n_t} \otimes F)$ is a numerical polynomial in $n_1, \ldots, n_t$ of total degree $\dim(supp(F))$ (i.e. a polynomial with rational coefficients which assumes integer values whenever $n_1, \ldots, n_t$ are integer). Let $D_1, \ldots, D_t$ be effective divisors on X and $L_i = \mathcal{O}_X(D_i)$. Let Y be a closed subscheme of X of dimension t. Then the *intersection number* of $D_1, \ldots, D_t$ with Y, denoted by $(D_1 \cdots D_t \cdot Y)$ is defined as the coefficient of the monomial $n_1 \cdots n_t$ in $\chi(X, L_1^{n_1} \otimes \cdots \otimes L_t^{n_t} \otimes F)$.

We need the following result (see [1], page 120).

Theorem 2.3 (Hodge Index Theorem)**.** *Let X be a smooth complex projective surface. Let $h \in H^{1,1}_{\mathbb{R}}(X)$ with $h^2 > 0$. Then the cup product form is negative definite on $h^{\perp} \subset H^{1,1}_{\mathbb{R}}(X)$.*

It gives the following corollary (compare with (2.15) Corollary in [1], page 120).

Corollary 2.4. *Let X be a non-singular complex projective surface. Let D_1, D_2 be two distinct effective divisors. Assume that $D_1.D_2 = D_1^2 = D_2^2 > 0$. Then D_1 and D_2 are numerically equivalent.*

Proof. Let $h = [D_1]$. Then $h^2 = D_1^2 > 0$. Moreover, $D_1.(D_1 - D_2) = D_1^2 - D_1.D_2 = 0$ and $(D_1 - D_2)^2 = D_1^2 - 2D_1.D_2 + D_2^2 = 0$. So the above Hodge Index Theorem implies that $[D_1 - D_2] = 0 \in H^{1,1}_{\mathbb{R}}(X)$ which means that D_1 and D_2 are numerically equivalent. $\square$

3. The Second Main Theorem for Numerically Equivalent Ample Divisors

We first recall the result of Ru [20] on the Second Main Theorem for holomorphic curves into projective varieties. Let X be a smooth complex projective variety of dimension $n \geq 1$. Let $D_1, \ldots, D_q$ be effective divisors on X with $q > n$. $D_1, \ldots, D_q$ are said to be *in general position* (on X) if for any subset of $n+1$ elements $\{i_0, \ldots, i_n\} \subset \{1, \ldots, q\}$,

$$\mathrm{supp} D_{i_0} \cap \cdots \cap \mathrm{supp} D_{i_n} = \emptyset,$$

where $\mathrm{supp}(D)$ means the support of the divisor D. A map $f : \mathbb{C} \to X$ is said to be *algebraically non-degenerate* if the image of f is not contained in any proper subvarieties of X. The result of Ru [20] is stated as follows.

Theorem A (Ru's Second Main Theorem)**.** *Let $X \subset \mathbb{P}^N(\mathbb{C})$ be a smooth complex projective variety of dimension $n \geq 1$. Let $D_1, \ldots, D_q$ be hypersurfaces in $\mathbb{P}^N(\mathbb{C})$ of degree d_j, located in general position on X. Let $f : \mathbb{C} \to X$ be an algebraically non-degenerate holomorphic map. Then, for every $\epsilon > 0$,*

$$\sum_{j=1}^{q} d_j^{-1} m_f(r, D_j) \leq (n + 1 + \epsilon) T_f(r) \ \|_E,$$

where "$\|_E$" means the inequality holds for all $r \in (0, +\infty)$ except for a possible set E with finite Lebesgue measure.

We first give a slight reformulation of the above result.

Theorem B (Ru's Second Main Theorem, reformulated)**.** *Let X be a smooth complex projective variety of dimension $n \geq 1$. Let $D_1, \ldots, D_q$ be effective divisors on X, located in general position. Suppose that there exists an ample divisor A on X and positive integers d_j such that $D_j \sim d_j A$ (i.e. D_j is linearly equivalent to $d_j A$) for $j = 1, \ldots, q$. Let $f : \mathbb{C} \to X$ be an algebraically non-degenerate holomorphic map. Then, for every $\epsilon > 0$,*

$$\sum_{j=1}^{q} d_j^{-1} m_f(r, D_j) \leq (n + 1 + \epsilon) T_{f,A}(r) \ \|_E.$$

Proof. Let N be a positive integer such that NA is very ample and N is divisible by d_j for $j = 1, \ldots, q$. Let $\phi : X \to \mathbb{P}^m(\mathbb{C})$ be the canonical embedding of X into $\mathbb{P}^m(\mathbb{C})$ associated to NA, where $m = \dim H^0(X, \mathcal{O}_X(NA)) - 1$. Then $\frac{N}{d_j} D_j = \phi^* H_j$ for some hyperplanes H_j in $\mathbb{P}^m(\mathbb{C})$. From the assumption that $D_1, \ldots, D_q$ are in general position on X, $H_1, \ldots, H_q$ are in general position on $X \subset \mathbb{P}^m(\mathbb{C})$ (or more precisely on the image of X under ϕ). Moreover from the functoriality and additivity of Weil functions, for $P \in X \backslash \mathrm{Supp} D_j$, we have

$$\lambda_{H_j}(\phi(P)) = \frac{N}{d_j} \lambda_{D_j}(r) + O(1),$$

so

$$m_{\phi \circ f}(r, H_j) = \frac{N}{d_j} m_f(r, D_j) + O(1).$$

Also, from the functoriality of height (characteristic) functions, we have

$$N T_{f,A}(r) = T_{f,NA}(r) = T_{\phi \circ f}(r) + O(1),$$

where $T_{\phi \circ f}(r) := T_{\phi \circ f, \mathcal{O}_{\mathbb{P}^m}(1)}(r)$. Applying Theorem A to the map $\phi \circ f$ and the hyperplanes H_j for $j = 1, \ldots, q$, we have

$$\sum_{j=1}^{q} m_{\phi \circ f}(r, H_j) \leq (n + 1 + \epsilon) T_{\phi \circ f}(r) \ \|_E.$$

The result then follows by substituting the identities above(we note that here the exceptional set E might change, nevertheless it is still of finite Lebesgue measure). $\square$

The main result of this short note is the following result,which says that Theorem B above remains true if we replace linear equivalence by numerical equivalence.

Main Theorem. *Let X be a smooth complex projective variety of dimension $n \geq 1$. Let $D_1, \ldots, D_q$ be effective divisors on X, located in general position. Suppose that there exists an ample divisor A on X and positive integers d_j such that $D_j \equiv d_j A$ for $j = 1, \ldots, q$. Let $f : \mathbb{C} \to X$ be an algebraically non-degenerate holomorphic map. Then, for every $\epsilon > 0$,*

$$\sum_{j=1}^{q} d_j^{-1} m_f(r, D_j) \leq (n + 1 + \epsilon) T_{f,A}(r) \ \|_E.$$

To prove the theorem, the following result in algebraic geometry, due to Matsusaka [16] (see also [12]), plays an important role.

Theorem 3.1 (Matsusaka)**.** *Let A be an ample Cartier divisor on a projective variety X. Then there exists a positive integer N_0 such that for all $N \geq N_0$, and any Cartier divisor D with $D \equiv NA$, D is very ample.*

Lemma 3.2 (See also (d) of Proposition 1.2.9 in [22])**.** *Let A be an ample Cartier divisor on a projective variety X. Let $f : \mathbb{C} \to X$ be a holomorphic map. Then, for any $\epsilon > 0$ and any effective divisor D with $D \equiv A$,*

$$T_{f,D}(r) \leq (1+\epsilon)T_{f,A}(r) + O(1),$$

where $O(1)$ is a constant which is independent of f and r.

Proof. Let N_0 be the integer in Theorem 3.1 for the given ample divisor A. Then $NA - (N - N_0)D$ is very ample for any $N \geq N_0$. Thus, by the additivity property of the height (characteristics function) functions,

$$T_{f,NA}(r) - T_{f,(N-N_0)D}(r) = T_{f,NA-(N-N_0)D}(r) \geq O(1).$$

That is

$$(N - N_0)T_{f,D}(r) \leq NT_{f,A}(r) + O(1).$$

With N being taken as $N = \frac{(1+\epsilon)N_0}{\epsilon}$, it gives the desired result. ☐

We are now ready to prove our main result.

Proof of the Main Theorem. By replacing D_j with $\frac{d}{d_j}D_j$ with $d = \mathrm{lcm}\{d_1, \ldots, d_q\}$, A by dA, and using the additivity of Weil functions and heights (up to bounded functions), we see that it suffices to prove the case where we can assume that $d_1 = d_2 = \cdots = d_q = 1$, i.e. $D_j \equiv A$ for $j = 1, \ldots, q$. For the given $\epsilon > 0$, let N_0 be the integer in Theorem 3.1 for our given A. Take N with

$$N_0 < \frac{\epsilon}{4q}N.$$

By the choice of N_0, we have that $NA - (N - N_0)D_j$ is very ample for $j = 1, \ldots, q$. Since the divisors $D_1, \ldots, D_q$ are in general position and $NA - (N - N_0)D_j$ is very ample for all j, there exist effective divisors E_j such that $(N - N_0)D_j + E_j$ is linearly equivalent to NA for all $1 \leq j \leq q$, and the divisors $(N-N_0)D_1+E_1,\dots,(N-N_0)D_q+E_q$ are in general position. Applying Theorem B to the linearly equivalent divisors $(N-N_0)D_j+E_j$ (which are all linearly equivalent to NA), $j=1,\dots,q$, we get

$$\sum_{j=1}^{q} m_f(r,(N-N_0)D_j+E_j) \le \Big(n+1+\frac{\epsilon}{2}\Big)\, T_{f,NA}(r) \quad \|_E.$$

Using additivity and that the Weil functions λ_{E_j} are bounded from below outside of the support of E_j and $T_{f,NA}(r)=NT_{f,A}(r)$, we obtain

$$\sum_{j=1}^{q}\Big(1-\frac{N_0}{N}\Big) m_f(r,D_j) \;\le\; \Big(n+1+\frac{\epsilon}{2}\Big)\, T_{f,A}(r) \quad \|_E,$$

i.e.

$$\sum_{j=1}^{q} m_f(r,D_j) \;\le\; \frac{N_0}{N}\sum_{j=1}^{q} m_f(r,D_j)+\Big(n+1+\frac{\epsilon}{2}\Big)\, T_{f,A}(r) \quad \|_E.$$

Note that in the above inequality, the exceptional set E might change, nevertheless it is still of finite Lebesgue measure. On the other hand, by Lemma 3.2 with $\epsilon=1$ and the First Main Theorem, we get

$$m_f(r,D_j) \le T_{f,D_j}(r)+O(1) \le 2T_{f,A}(r)+O(1).$$

Thus, by the choice of N that $N_0<\frac{\epsilon}{4q}N$, we obtain

$$\sum_{j=1}^{q} m_f(r,D_j) \le \frac{2qN_0}{N}T_{f,A}(r)+\Big(n+1+\frac{\epsilon}{2}\Big)\, T_{f,A}(r) \le (n+1+\epsilon)T_{f,A}(r) \quad \|_E.$$

This finishes the proof of the Main Theorem.

Corollary 3.3. *Let X be a complex smooth projective surface and $D_1,\dots,D_q$ be distinct irreducible ample divisors located in general position on X (i.e. no three of them share a common point). Assume that there exist positive integers $n_1,\dots,n_q$ such that $(n_iD_i).(n_jD_j)$ is a positive constant (i.e independent of i,j for all pairs $1\le i,j\le q$). Let $f:\mathbb{C}\to X$ be an algebraically non-degenerate holomorphic map. Then, for every $\epsilon>0$,*

$$\sum_{j=1}^{q} n_j m_f(r,D_j) \le (3+\epsilon)\Big(\frac{1}{q}\sum_{j=1}^{q} n_j T_{D_j,f}(r)\Big) \quad \|_E.$$

In particular, with the same assumptions about the divisors $D_1, \ldots, D_q$, if $q \geq 4$, then every holomorphic map $f : \mathbb{C} \to X \backslash \cup_{j=1}^q D_j$ must be algebraically degenerate.

Proof. From Corollary 2.4, we know that $n_j D_j, 1 \leq j \leq q$, are numerically equivalent. Therefore applying the Main Theorem to the divisors $n_j D_j$, together with the additivity property of Weil functions and heights (up to bounded functions), gives

$$\sum_{j=1}^q n_j m_f(r, D_j) \leq (3+\epsilon)\Big(\frac{1}{q}\sum_{j=1}^q n_j T_{f,D_j}(r)\Big) \ \|_E.$$

Now assume that $f : \mathbb{C} \to X \backslash \cup_{j=1}^q D_j$ and that f is algebraically non-degenerate. Since $n_j D_j$ and D_j share the same support and the image of f omits the support of D_j, we have $N_f(r, nD_j) = 0$, thus from the First Main Theorem,

$$m_f(r, n_j D_j) = T_{f, n_j D_j}(r) + O(1).$$

Thus, we get

$$\begin{aligned} \sum_{j=1}^q n_j T_{f,D_j}(r) + O(1) &= \sum_{j=1}^q n_j m_f(r, D_j) \\ &\leq \frac{3+\epsilon}{q}\Big(\sum_{j=1}^q n_j T_{f,D_j}(r)\Big) \ \|_E, \end{aligned}$$

which is a contradiction when $q \geq 4$. □

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