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ON S^*-QUASINORMAL SUBGROUPS OF PRIME POWER ORDER IN FINITE GROUPS

GUO ZHONG[1,a], BIN XIAO[2,b] AND JUNPEI LUO[2,c]

[1]Faculty of Science and Technology, University of Macau, Macau SAR, China.
[a]E-mail: zhg102003@163.com
[2]Shenzhen Jixiang Primary School, Shenzhen, Guangdong, 518112, P. R. China.
[b]E-mail: 4401833@QQ.com
[c]E-mail: 840343128@QQ.com

Abstract

Let H be a subgroup of a finite group G. We say that H is S^*-quasinormal in G if there is a normal subgroup K of G such that $HK \trianglelefteq G$ and $H \cap K \leq H_{seG}$, where H_{seG} denotes the subgroup of H generated by all those subgroups of H which are S-quasinormally embedded in G. In this paper, we investigate the influence of S^*-quasinormal subgroups on the p-nilpotency of finite groups. Some recent results are extended and generalized.

1. Introduction

Throughout only finite groups are considered. Terminologies and notations employed agree with standard usage, as in Robinson [17].

Recall that two subgroups H and K of a group G are said to be permutable if $HK = KH$. The subgroup H of G is said to be S-quasinormal in G if H permutes with every Sylow subgroups of G, i.e., $HP = PH$ for any Sylow subgroup P of G. This concept was introduced by O.H.Kegel in [12] and has been studied widely by many authors, such as [5, 18]. An interesting question in theory of finite groups is to determine the influence of the embedding properties of members of some distinguished families of subgroups of a group on the structure of the group. Recently, Ballester-Bolinches and Pedraza-Aquilera [4] generalized S-quasinormal subgroups to

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S-quasinormally embedded subgroups. The subgroup K of G is said to be S-quasinormally embedded in G provided every Sylow subgroup of K is a Sylow subgroup of some S-quasinormal subgroup of G. Using this concept, a series of meaningful results on the structure of finite groups have obtained, for example [1-3, 15-16]. As a development, we introduced the following new concept.

Definition 1.1. Let H be a subgroup of a group G. We say that H is S^*-quasinormal in G if there exists some normal subgroup T of G such that $HT \trianglelefteq G$ and $H \cap T \leq H_{seG}$, where H_{seG} is the subgroup generated by all the subgroups of H which are S-quasinormally embedded in G. We call H_{seG} the SE-core of H in G.

It is easy to see that all normal subgroups, quasinormal subgroups, S-quasinormal subgroups and S-quasinormally embedded subgroups are all S^*-quasinormal subgroups. However, the following examples show that the converse is not true in general.

Example 1.2. For any simple non-abelian group G, there always exists a Sylow subgroup which is S-quasinormally embedded in G. So is S^*-quasinormal in G.

Example 1.3. Suppose that $G = S_4$, the symmetric group of degree 4. Take the subgroup $H =< (12) >$. Then H is S^*-quasinormal in G, but not S-quasinormally embedded in G since H and every subgroup of G with order 6 containing H can not permute with every Sylow subgroup of G.

A primary subgroup of a group G is a subgroup of prime power order. The property of primary subgroups has been studied extensively by many scholars in determining the structure of a finite group. For instance, Hall[10] in 1937 proved that G is solvable if and only if every Sylow subgroup of G is complemented in G. Srinivasan[19] in 1980 stated that G is supersolvable if every maximal subgroup of the Sylow subgroups is normal in G. In 2000, Wang[21] proved that G is supersolvable if every maximal subgroup of the Sylow subgroups is c-supplemented in G. In this paper, we continue these work and characterize p-nilpotency of finite groups with the assumption that certain subgroups of prime power order are S^*-quasinormal in G.

2. Preliminaries

For the sake of convenience, we firstly cite some known results in the literature which will be useful in the following.

Lemma 2.1 ([14], Lemma 2.1). *Let H and K be subgroups of a group G.*

(1) *If H is S-quasinormal in G, then H is subnormal in G.*

(2) *If both H and K are S-quasinormal subgroups of G, then both $H \cap K$ and $< H, K >$ are S-quasinormal subgroups of G.*

Lemma 2.2 ([2], Lemma 2.5). *Suppose that a subgroup H of a group G is S-quasinormal embedded in G, P is a Sylow p-subgroup of H. If $H_{seG} = 1$, then P is S-quasinormal in G.*

Lemma 2.3 ([15], Lemma 2.2). *If P is an S-quasinormal p-subgroup of a group G for some prime p, then $N_G(P) \geq O^p(G)$.*

Lemma 2.4 ([7], Lemma 2.6). *Suppose that N is a normal subgroup of a group G and $H \leq K \leq G$. Then:*

(1) $H_{seG} \trianglelefteq H$.

(2) $H_{seG} \leq H_{seK}$.

(3) $H_{seG}N/N \leq (HN/N)_{se(G/N)}$.

(4) *If* $(|N|, |H|) = 1$, *then* $H_{seG}N/N = (HN/N)_{se(G/N)}$.

We often need the following lemma in our proofs.

Lemma 2.5. *Let H be a subgroup of a group G.*

(1) *If H is S^*-quasinormal in G and $H \leq M \leq G$, then H is S^*-quasinormal in M.*

(2) *Let $N \lhd G$ and $N \leq H$. If H is S^*-quasinormal in G, then H/N is S^*-quasinormal in G/N.*

(3) *Let π be a set of primes, H a π-subgroup of G and N a normal π'-subgroup of G. If H is S^*-quasinormal in G, then HN/N is S^*-quasinormal in G/N.*

Proof. By hypothesis, there exists a subgroup K of G such that $HK \trianglelefteq G$ and $H \cap K \leq H_{seG}$.

(1) Then $H(M\cap K)=M\cap HK\trianglelefteq M$ and $H\cap K\leq H_{seG}\leq H_{seM}$ by Lemma 2.4. Hence, H is S^*-quasinormal in M.

(2) We know that $(H/N)(KN/N)\trianglelefteq G/N$. By Lemma 2.4, we get that $H/N\cap KN/N=(H\cap K)N/N\leq H_{seG}N/N\leq(HN/N)_{se(G/N)}=(H/N)_{se(G/N)}$. So H/N is S^*-quasinormal in G/N.

(3) Since $(|G:K|,|N|)=1$, it is easy to see that $N\leq K$ and $(HN/N)\cdot(K/N)\trianglelefteq G/N$. Thus $(HN/N)\cap(K/N)=(H\cap K)N/N\leq H_{seG}N/N=(HN/N)_{se(G/N)}$ by Lemma 2.4. Therefore, HN/N is S^*-quasinormal in G/N.

Lemma 2.6 ([13], Lemma 2.3)**.** *Let G be a group and p a prime number such that $p^{n+1}\nmid|G|$ for some integer $n\geq1$. If $(|G|,(p-1)(p^2-1)\ldots(p^n-1))=1$, then G is p-nilpotent.*

Lemma 2.7 ([23], Lemma 2.7)**.** *Let G be a group. If A is subnormal in G and A is a p-subgroup of G, then $A\leq O_p(G)$.*

Lemma 2.8 ([6], A, Lemma 1.2)**.** *Let U,V and W be subgroups of a group G. The following statements are equivalent.*

(1) $U\cap VW=(U\cap V)(U\cap W)$.

(2) $UV\cap UW=U(V\cap W)$.

3. Main Results

Our first result is to unify and improve the results of [2], [8] and [14] on the p-nilpotency of a group.

Theorem 3.1. *Let G be a group and p be a prime number such that $(|G|,(p-1)(p^2-1)\ldots(p^n-1))=1$ (integer $n\geq1$.) If P is a Sylow p-subgroup in G and every n-maximal subgroup of P is S^*-quasinormal in G, then G is p-nilpotent.*

Proof. Assume that the statement is false and let G be a counterexample of minimal order. We proceed the proof by the following steps.

(1) By Lemma 2.6, $|P|\geq p^{n+1}$, and thus every n-maximal subgroup P_n of P satisfies that $P_n\neq1$.

(2) G is not a simple group.

According to the hypothesis, P_n is S^*-quasinormal in G. By the definition of a S^*-quasinormal subgroup, there is a normal subgroup T of G such that $P_nT \trianglelefteq G$ and $P_n \cap T \leq (P_n)_{seG}$. Suppose that G is simple. If $T = 1$, then $1 \neq P_nT = P_n \trianglelefteq G$, which is a contradiction. If $T = G$, then $1 < P_n \cap T = P_n \leq (P_n)_{seG}$. We can write $(P_n)_{seG} =< U|U$ is a nontrivial S-quasinormally embedded subgroup of G contained in $P_n >$. Let U be an arbitrary S-quasinormally embedded subgroup of G contained in P_n. Then there is an S-quasinormal subgroup K of G such that U is a Sylow p-subgroup of K. Since G is simple, we have $K_G = 1$, By Lemma 2.2, U is S-quasinormal in G. From the arbitrariness of U and Lemma 2.1, P_n is S-quasinormal in G, so $P_n = 1$, in contrary to (1).

(3) G has a unique minimal normal subgroup N such that G/N is p-nilpotent, moreover $\Phi(G)=1$.

From the above we can see that the group G/N satisfies the hypothesis of the theorem which shows that PN/N is a Sylow p-subgroup of G/N. By Lemma 2.6, we may take $|PN/N| \geq p^{n+1}$. Let M_n/N be a n-maximal subgroup of PN/N. Then $M_n = M_n \cap PN = (M_n \cap P)N = P_nN$. Obviously, P_n is a n-maximal subgroup of P. According to the hypothesis, P_n is S^*-quasinormal in G. Therefore, there is a normal subgroup T of G such that $P_nT \trianglelefteq G$ and $P_n \cap T \leq (P_N)_{seG}$. Furthermore, we can see that $TN/N \trianglelefteq G/N, M_n/N \cdot TN/N = P_nN/N \cdot TN/N = P_nTN/N \trianglelefteq G/N$. If $N \cap P_nT = 1$, then $N \cap P_n = N \cap T = 1, N \cap P_nT = (N \cap P_n)(N \cap T)$. If $N \cap P_nT \neq 1$, then $N \leq P_nT$. Since $P_n \cap N = P \cap M_n \cap N = P \cap N$ is a Sylow p-subgroup of N and $|N : N \cap T| = |NT : T| \leq |P_nT : T|, (|N : N \cap P_n|, |N : N \cap T|) = 1, (N \cap P_n)(N \cap T) = N = N \cap P_nT$. By Lemma 2.8, $P_nN \cap TN = (P_n \cap T)N$, and thus $P_nN/N \cap TN/N = (P_nN \cap TN)/N = (P_n \cap T)N/N$. Hence $M_n/N \cap TN/N = P_nN/N \cap TN/N = (P_n \cap T)N/N \leq (P_n)_{seG}N/N \leq (P_nN/N)_{se(G/N)}$ by Lemma 2.4. Thus M_n/N is S^*-quasinormal in G/N. As a result, the factor group G/N satisfies the hypothesis of our theorem. The choice of G yields that G/N is p-nilpotent. As a consequence, the uniqueness of N and $\Phi(G) = 1$ are clear.

(4) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then $G/O_{p'}(G)$ is p-nilpotent according to step (3), so is G, which is contrary to the choice of G.

(5)$O_p(G)=1$.

If $O_p(G)\neq 1$, according to step (3) $N\leq O_p(G)$, there exists a maximal subgroup M of G such that $G=NM$ and $N\cap M=1$. Since $O_p(G)\cap M$ is normalized by N and M, hence by G, the uniqueness of N yields $N=O_p(G)$. Clearly, $P=P\cap NM=N(P\cap M)$. Since $P\cap M<P$, there exists a maximal subgroup P_1 of P which contains $P\cap M$, and hence $P=NP_1$. Pick an n-maximal subgroup P_n of P contained in P_1. It follows by the hypothesis that there is a normal subgroup T of G such that $P_nT\trianglelefteq G$ and $P_n\cap T\leq (P_n)_{seG}$. Let U be a nontrivial S-quasinormally embedded subgroup of G contained in P_n. Then there is an S-quasinormal subgroup K of G such that U is a Sylow p-subgroup of K. If $K_G\neq 1$, then $N\leq K_G\leq K$, so $N\leq U\leq (P_n)_{seG}\leq P_n\leq P_1$. Consequently, $P=NP_1=P_1$, a contradiction. Thus we have $K_G=1$. Furthermore, by Lemma 2.2, U is S-quasinormal in G. From the arbitrariness of U and Lemma 2.1, $(P_n)_{seG}$ is S-quasinormal in G. By Lemmas 2.3 and 2.1, $O^p(G)\leq N_G((P_n)_{seG})$ and $(P_n)_{seG}$ is subnormal in G. By Lemma 2.7, we have $P_n\cap T\leq (P_n)_{seG}\leq O_p(G)=N$, so $P_n\cap T\leq (P_n)_{seG}\leq P_1\cap N$. Furthermore, $P_n\cap T\leq (P_n)_{seG}^G=(P_n)_{seG}^{O^p(G)P}=(P_n)_{seG}^P\leq (P_1\cap N)^P=P_1\cap N\leq N$. It follows that $(P_n)_{seG}^G=P_1\cap N=N$ or $(P_n)_{seG}^G=1$. If $(P_n)_{seG}^G=P_1\cap N=N$, then $N\leq P_1$, a contradiction. If $(P_n)_{seG}^G=1$, then $P_n\cap T=1$ and so $|T_p|\leq p^n$. It follows that T is p-nilpotent by Lemma 2.6. Let $T_{p'}$ be the normal p-complement of T, then $T_{p'}\trianglelefteq G$, we get $T_{p'}=1$ by step (4), and thus T is a normal p-subgroup of G and $T\leq P_nT\leq O_p(G)=N$. If $T\neq 1$, we get $T=P_nT=N$, so $P_n\leq T$, namely, $P_n\cap T=P_n=1$, a contradiction. If $T=1$, then $P_n\trianglelefteq G$, so $N\leq P_n\leq P_1$, a contradiction. Now it is clear that (5) holds.

(6) End of the proof.

If $N\cap P\leq \Phi(P)$, then N is p-nilpotent by Tate′s Theorem([11,IV,4.7]). Therefore, $N_{p'}\trianglelefteq G$. It follows that $N_{p'}\leq O_{p'}(G)=1$. Moreover, N is a p-group, then $N\leq O_p(G)=1$, a contradiction. As a result, there exists a maximal subgroup P_1 of P such that $P=(P\cap N)P_1$. Take a n-maximal subgroup P_n of P contained in P_1. By the hypothesis, there is a normal subgroup T of G such that $P_nT\trianglelefteq G$ and $P_n\cap T\leq (P_n)_{seG}$. Let U be a nontrivial S-quasinormally embedded subgroup of G contained in P_n. Then there is an S-quasinormal subgroup K of G such that U is a Sylow p-subgroup of K. If $K_G\neq 1$, then $N\leq K_G\leq K$, so $U\cap N$ is a Sylow p-subgroup of N. We know $U\cap N \leq P_1\cap N \leq P\cap N$ and $P\cap N$ is a Sylow p-subgroup of N, so $U\cap N = P_1\cap N = P\cap N$. Consequently, $P=(P\cap N)P_1=(P_1\cap N)P_1=P_1$, a contradiction. Hence $K_G=1$, U is S-quasinormal in G by Lemma 2.2. From Lemma 2.1 and the arbitrariness of U, $(P_n)_{seG}$ is S-quasinormal in G, and thus $(P_n)_{seG}$ is subnormal in G by Lemma 2.1. It follows from Lemma 2.7 that $(P_n)_{seG}\leq O_p(G)=1$, so $|T_p|\leq p^n$, therefore T is p-nilpotent by Lemma 2.6. Similarly, we have $T_{p'}=1$ and so $T=1$. It deduce that $P_n \trianglelefteq G, N\leq P_n\leq P_1$, a contradiction. This completes the proof.

Theorem 3.2. *Let G be a group and p be a prime number such that $(|G|,(p-1)(p^2-1)\dots(p^n-1))=1$ for some integer $n\geq 1$. If G has a Sylow p-subgroup P such that every n-maximal subgroup of P not having a p-nilpotent supplement in G is S^*-quasinormal in G, then G is p-nilpotent.*

Proof. Assume that the theorem is not true and G be a counterexample of minimal order. Then we claim that every n-maximal subgroup of P is S^*-quasinormal in G. Otherwise, then P has a n-maximal subgroup P_n by the hypothesis, and P_n has a p-nilpotent supplement T in G. Let H be a minimal non-p-nilpotent subgroup of G containing P. Then H is a minimal non-nilpotent group by ([11, IV,5.4]). Therefore by ([22, Theorem 3.4.11]), we get to know that H has the following properties:

(1) $|H|=p^aq^b$, where p and q are different primes;

(2) $H=H_pH_q$, where H_p is a normal Sylow p-subgroup of H (We may suppose that $H_p=P$ without loss of generality) and H_q is a cyclic Sylow q-subgroup of H;

(3) $H_p/\Phi(H_p)$ is a chief factor of H.

Because $G=P_nT, H=H\cap P_nT=P_n(H\cap T)=P_nL, L=H\cap T$. If $L=H$, then H is contained in T and thus $G=P_nT=T$ is p-nilpotent, a contradiction. As a result, $L<H$ and L is p-nilpotent. Let $L=L_p\times L_q$. Clearly, L_q is a Sylow q-subgroup of H and $L_p=H_p\cap L=H_p\cap H\cap T=H_p\cap T$. Claim that $L_p=1$ and $L_p\nsubseteq \Phi(H_p)=\Phi$. If $L_p=1$, then L is a p'-group, so P_n is a Sylow p-subgroup of H, a contradiction. If $L_p\subseteq\Phi(H_p)$, a contradiction also happens. Next we consider the quotient group H/Φ. Since $L_q\leq N_H(L_p), L_q\Phi/\Phi\leq N_{H/\Phi}(L_p\Phi/\Phi)$. Besides, since H_p/Φ is elementary abelian, $L_p\Phi/\Phi=H_p/\Phi$. Consequently, $L_p\Phi/\Phi\trianglelefteq H/\Phi$. As $L_p\Phi/\Phi\neq 1$ and H_p/Φ is a chief factor of H, $L_p\Phi/\Phi=H_p/\Phi$, so we can get $L_p=H_p$.

This suggests that $L = H$. This contradiction exposes that every n-maximal subgroup of P is S^*-quasinormal in G. It follows from Theorem 3.1 that G is p-nilpotent. This is a final contradiction.

Remark 3.1. The assumption that $(|G|, (p-1)(p^2-1)\dots(p^n-1)) = 1$ in Theorems 3.1-3.2 can not be removed. For instance, we consider $G = A_5$ and $p = 5$. In this case, since every maximal subgroup of Sylow 5-subgroup of G is 1, every maximal subgroup of Sylow 5-subgroup of G is S^*-quasinormal in G. However, G is not 5-nilpotent.

If we remove the hypothesis $(|G|, (p-1)(p^2-1)\dots(p^n-1)) = 1$ in Theorem 3.1, we can prove the following result.

Theorem 3.3. *Let G be a group and P be a Sylow p-subgroup of G for some p a prime of $|G|$. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P not having a p-nilpotent supplement in G is S^*-quasinormal in G, then G is p-nilpotent.*

Proof. It is easy to see that the theorem holds when $p = 2$ by Theorem 3.2. So it suffices to prove the theorem for the case when p is odd. Suppose that the theorem is false and let G be a counterexample of minimal order. We proceed via the following steps. With the same arguments to those used in the proof of Theorem 3.2, we first have the following claim (1).

(1) Every maximal subgroup of P is S^*-quasinormal in G.

(2) $O_{p'}(G) = 1$.

If $L = O_{p'}(G) \neq 1$, then PL/L is a Sylow p-subgroup of G/L. Let T/L be a maximal subgroup of PL/L. Then $T = P_1L$ for some maximal subgroup P_1 of P. It follows from (1) and Lemma 2.5 that P_1L/L is S^*-quasinormal in G/L. Besides, $N_{G/L}(PL/L) = N_G(P)L/L$ (see [22, Lemma 3.6.10]) and therefore it is p-nilpotent. As a result, $G/O_{p'}(G)$ satisfies the hypothesis. It follows that G/L is p-nilpotent and so is G, a contradiction.

(3) If M is a proper subgroup of G containing P, then M is p-nilpotent.

As $N_M(P) \leq N_G(P), N_M(P)$ is p-nilpotent. By (1) and Lemma 2.5, it is easy to see that M satisfies the hypothesis. By the minimality of G, M is p-nilpotent.

(4) $G = PQ$ is soluble and $1 \neq O_p(G) < P$, where Q a Sylow q-subgroup of G with $q \neq p$.

Since G is not p-nilpotent, by a result of Thompson[20, Corollary], there exists a non-trivial characteristic subgroup T of P such that $N_G(T)$ is not p-nilpotent. Choose T such that the order of T is as large as possible. Since $N_G(P)$ is p-nilpotent, we have $N_G(K)$ is p-nilpotent for any characteristic subgroup K of P satisfying $T < K \leq P$. Now, T char $P \lhd N_G(P)$, which gives $T \trianglelefteq N_G(P)$. So $N_G(P) \leq N_G(T)$. According to (3), we have that $N_G(T) = G$ and $T = O_p(G)$. Now, applying the result of Thompson again, we have that $G/O_p(G)$ is p-nilpotent and therefore G is p-solvable. Then for any $q \in \pi(G)$ with $q \neq p$, there exists a Sylow q-subgroup of Q such that PQ is a subgroup of G by [9, Theorem 6.3.5]. If $PQ < G$, then PQ is p-nilpotent by(3), contrary to the choice of G. Therefore, $PQ = G$, as we wished.

(5) There is a unique minimal normal subgroup N in G such that $G = NM$, where M is a maximal subgroup of G, and $N = O_p(G) = C_G(N)$.

Let N be a minimal normal subgroup of G. It follows from (2) and (4) that N is an elementary abelian p-subgroup, and $N \leq O_p(G) < P$. We can see that G/N satisfies the hypothesis. The minimal choice of G yields that G/N is p-nilpotent. As a result, the uniqueness of N and $N \nleq \Phi(G)$ are evident.

(6) $|N| = p$.

Clearly, $P = NM_p$, where M_p is a Sylow p-subgroup of M. Let $M_p \leq P_1$, where P_1 is a maximal subgroup of P. If $P_1 = 1$, then $|N| = |P| = p$. We may assume that $P_1 \neq 1$. Evidently, $N \nleq P_1$. By(1), there is a normal subgroup K of G such that $P_1K \trianglelefteq G$ and $P_1 \cap K \leq (P_1)_{seG}$. Let U be a nontrivial S-quasinormally embedded subgroup of G contained in P_1, and then there is an S-quasinormal subgroup T of G such that U is a Sylow p-subgroup of T. It follows from (5) and $N \nleq P_1$ that we get $N \leq P_1K$ and $K \neq 1$. Therefore, $N \leq K$. If $N \cap P_1 = 1$, then $|N| = p$ from $P = NP_1$. Assume $N \cap P_1 \neq 1$. If $T_G \neq 1$, then $N \leq T_G \leq T$, so $N \leq U \leq P_1$, a contradiction. If $T_G = 1$, then U is S-quasinormal in G by Lemma 2.2, so is $(P_1)_{seG}$ from the arbitrariness of U and Lemma 2.1. As $P_1 \cap K \leq (P_1)_{seG} \leq O_p(G) \cap P_1 = P_1 \cap N \leq P_1 \cap K$,

$P_1 \cap K = P_1 \cap N$. On the other side, $O^p(G) \le N_G((P_1)_{seG})$ by Lemma 2.3. Thus $1 < P_1 \cap N = P_1 \cap K \le (P_1)^G_{seG} = (P_1)^{O^p(G)P}_{seG} = (P_1)^P_{seG} \le (P_1 \cap N)^P = P_1 \cap N \le N$. By the minimal normality of N, $(P_1)^G_{seG} = P_1 \cap N = N$, which is a contradiction. Thus (6) happens.

(7) The final contradiction.

From (6), $Aut(N)$ is a cyclic group of order $p-1$. If $p < q$, then NQ is p-nilpotent by [17, 10.1.9] and (6). Therefore, $Q \le C_G(N) = O_p(G)$, which is a contradiction. Consequently, we may assume that $q < p$. According to (5), $M \cong G/N \cong N_G(N)/C_G(N)$ is isomorphic with some subgroup of $Aut(N)$. Thus Q is a cyclic group. It follows that G is q-nilpotent and so $P \trianglelefteq G$. Moreover, $G = N_G(P)$ is p-nilpotent from the hypothesis. The proof is now completed.

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