Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **13** (2018), No. 4, pp. 397-409
DOI: 10.21915/BIMAS.2018403

ON PRIMITIVE AXIAL ALGEBRAS OF JORDAN TYPE

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Dedicated to Professor Robert L. Griess, Jr. on the occasion of his 71st birthday

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Abstract

In this note we give an overview of our knowledge regarding primitive axial algebras of Jordan type half and connections between 3-transposition groups and Matsuo algebras. We also show that primitive axial algebras of Jordan type η admit a Frobenius form, for any η.

1. Introduction

The purpose of this note is threefold. In §2 we give an overview of our knowledge regarding *primitive axial algebras of Jordan type half*. This is taken from [2]. In fact we focus in §2 on one of the main results in [2] which characterizes Jordan algebras of Clifford type amongst primitive axial algebras of Jordan type half. The primitive axial algebras of Jordan type $\eta \neq \frac{1}{2}$ are reviewed (amongst other things) by Jon Hall in another paper of this volume. In §3, we complete, for the case $\eta = \frac{1}{2}$, a result connecting 3-transposition groups and Matsuo algebras, established in [1, Theorem 6.3] for $\eta \neq \frac{1}{2}$. In §4 we show that any primitive axial algebra of Jordan type η (any η) admits a Frobenius form.

Received May 9, 2017 and in revised form September 14, 2017.

AMS Subject Classification: Primary: 17A99; Secondary: 17C99, 17B69.

Key words and phrases: Axial algebra, 3-transposition, Jordan algebra, Frobenius form.

We start by recalling a few definitions. We do not give the historical background as it can be best found in the introduction to [1].

All algebras A in this note are *commutative, non-associative* over a field $\mathbb{F}$ of *characteristic not* 2.

For $a \in A$ the *adjoint operator* ad_a is *multiplication by* a, so

$$\mathrm{ad}_a \colon A \to A,\ x \mapsto xa.$$

An *axis* in A is, by definition, a *semisimple idempotent*, i.e., an idempotent whose minimal ad-polynomial has few distinct linear factors; where the minimal ad-polynomial is the minimal polynomial of the linear operator ad_a (we are *not* assuming that A is finite dimensional, however, we are assuming that ad_a has a minimal polynomial).

Axial algebras, introduced recently by Hall, Rehren and Shpectorov ([1]), are, by definition, algebras generated by axes. When certain *fusion rules*, i.e. multiplication rules, between the eigenspaces corresponding to an axis, are imposed the structure of axial algebras remains interesting yet it is more rigid.

Given an element $a \in A$ and a scalar $\lambda \in \mathbb{F}$, the λ-eigenspace of ad_a is denoted $A_\lambda(a)$, so:

$$A_\lambda(a) := \{x \in A \mid xa = \lambda x\}.$$

(We allow $A_\lambda(a) = 0$.)

Axial algebras of Jordan type η, where $\eta \notin \{0, 1\}$ is fixed, are algebras generated by a set of axes $\mathcal{A}$ such that for each $a \in \mathcal{A}$:

(1) The minimal ad-polynomial of a divides $(x-1)x(x-\eta)$.

(2) The *fusion rules* imitate the *Peirce multiplication rules* in Jordan algebras. These fusion rules are:

$$A_1(a)A_1(a) \subseteq A_1(a) \quad \text{and} \quad A_0(a)A_0(a) \subseteq A_0(a),$$

$$A_1(a)A_0(a) = \{0\},$$

$$(A_0(a) + A_1(a))A_\eta(a) \subseteq A_\eta(a), \quad \text{and} \quad A_\eta(a)^2 \subseteq A_0(a) + A_1(a).$$

In particular, if we set

$$A_+(a) = A_1(a) \oplus A_0(a) \quad \text{and} \quad A_-(a) = A_\eta(a).$$

then

$$A_\delta(a)A_\epsilon(a) \subseteq A_{\delta\epsilon}(a)\,,$$

for $\delta, \epsilon \in \{+, -\}$.

Thus, for example, Jordan algebras are axial algebras of Jordan type $\frac{1}{2}$, provided that they are generated by idempotents.

An axis $a \in A$ is *absolutely primitive* if $A_1(a) = \mathbb{F}a$ (this is stronger than the usual notion of primitivity). We call an absolutely primitive axis a satisfying (1), (2) above an *η-axis.*

A *primitive axial algebra of Jordan type* η is an algebra generated by η-axes. For $\eta \neq \frac{1}{2}$, primitive axial algebras of Jordan type η were thoroughly analyzed by Hall, Rehren, and Shpectorov in [1]. The case $\eta = \frac{1}{2}$, is much less understood and is of a different nature. This case is the focus of [2] and of §§2,3 of this note.

Given an η-axis $a \in A$, recall that

$$A = \overbrace{A_1(a) \oplus A_0(a)}^{A_+(a)} \oplus \overbrace{A_\eta(a)}^{A_-(a)}\,.$$

The map $\tau(a)\colon A \to A$ defined by $x^{\tau(a)} = x_+ - x_-$, where $x = x_+ + x_- \in A_+(a) + A_-(a)$, is an automorphism of A of order 1 or 2. It is called *the Miyamoto involution corresponding to a.*

1.1. Jordan algebras of Clifford type

A Jordan algebra of Clifford type $J(V, B)$ consists of the following information:

(1) A vector space V over $\mathbb{F}$ together with a symmetric bilinear form B on V. The corresponding quadratic form is denoted $q(v) = B(v, v)$.

(2) The Jordan algebra $J(V, B)$ is $\mathbb{F}\mathbb{1} \oplus V$ with multiplication defined by

$$\mathbb{1} \text{ is the identity and } v * w = B(v, w)\mathbb{1}, \quad \forall v, w \in V.$$

The algebra $J(V,B)$ comes from the associative *Clifford algebra* $\mathrm{Cl}(V,q)$: it is a sub-Jordan algebra of $\mathrm{Cl}(V,q)^+$, where, as usual, $\mathfrak{A}^+$ denotes the special Jordan algebra that emerges from the associative algebra $\mathfrak{A}$.

Let $J = J(V,B)$. It is easy to check that:

(a) For $u \in V$ and $\alpha \in \mathbb{F}$, the element $\alpha\mathbb{1} + u$ is an idempotent if and only if $\alpha = \frac{1}{2}$ and $q(u) = \frac{1}{4}$.

(b) Assume that $a = \frac{1}{2}\mathbb{1} + u$ is an idempotent in J. Then

  (i) $J_1(a) = \mathbb{F}a$, so a is a $\frac{1}{2}$-axis. (Thus $J(V,B)$ is a primitive axial algebra of Jordan type $\frac{1}{2}$ iff it is generated by idempotents.)

  (ii) $J_0(a) = \mathbb{F}(\mathbb{1} - a)$ (of course $\mathbb{1} - a$ is a $\frac{1}{2}$-axis), and

  (iii) $J_{\frac{1}{2}}(a) = u^{\perp} = J_{\frac{1}{2}}(\mathbb{1} - a)$, where $u^{\perp} = \{v \in V \mid B(u,v) = 0\}$.

(c) It follows that $\tau(a) = \tau(\mathbb{1} - a)$, for any $\frac{1}{2}$-axis a.

The purpose of §2 is to show that property (c) above essentially characterizes Jordan algebras of Clifford type amongst primitive axial algebras of Jordan type $\frac{1}{2}$.

2. Primitive Axial Algebras of Jordan Type Half

Throughout this section A is a primitive axial algebra of Jordan type η, generated by a set $\mathcal{A}$ of η-axes.

Let Δ be the graph on the set of all η-axes of A, where distinct a, b form an edge iff $ab \neq 0$. Let also $\Delta_{\mathcal{A}}$ be the full subgraph of Δ on the set $\mathcal{A}$. The purpose of this section is to sketch a proof of the following theorem:

Theorem 2.1. *Assume that $\Delta_{\mathcal{A}}$ is connected and that there are two distinct η-axes $a, b \in A$ such that $\tau(a) = \tau(b)$. Then $\eta = \frac{1}{2}$, $a + b = \mathbb{1}$ is the identity of A, and A is a Jordan algebra of Clifford type.*

In the remainder of this section we will sketch a proof of Theorem 2.1. First we need a theorem that enables us to identify A as a Jordan algebra of Clifford type in the case $\eta = \frac{1}{2}$.

Theorem 2.2. *Let $\eta = \frac{1}{2}$. Assume that A contains two $\frac{1}{2}$-axes $a, b \in \mathcal{A}$ such that $a + b = \mathbb{1}_A$ and such that $v_a v_c \in \mathbb{F}\mathbb{1}_A$, for all $c \in \mathcal{A}$, where $v_c = c - \frac{1}{2}\mathbb{1}_A$. Then A is a Jordan algebra of Clifford type.*

We do not include a proof of Theorem 2.2, see [2, Theorem 5.4].

We will need some information about 2-generated subalgebras of A. This information is taken from [1]. Let $a, b \in \Delta$ with $a \neq b$. We denote by $N_{a,b}$ *the subalgebra generated by* a *and* b. *If* $N_{a,b}$ *contains an identity element, we denote it by* $1_{a,b}$. Note that by [1], 2-*generated subalgebras are at most* 3-*dimensional.*

Lemma 2.3 (Lemma 3.1.2 in [2]). *Let* $a, b \in \Delta$ *with* $a \neq b$. *Then* $N_{a,b}$ *is* 2-*dimensional precisely in the following cases:*

(1) $ab = 0$; *we then denote:* $N_{a,b} = 2B_{a,b}$.
(2) $\eta = -1, ab = -a - b$; *we then denote:* $N_{a,b} = 3C(-1)^{\times}_{a,b}$.
(3) $\eta = \frac{1}{2}, ab = \frac{1}{2}a + \frac{1}{2}b$; *we then denote:* $N_{a,b} = J_{a,b}$.

Furthermore,

(4) *the algebras* $N_{a,b}$ *in cases* (2) *and* (3) *above do not have an identity element.*

The following proposition deals with 2-generated 3-dimensional subalgebras.

Proposition 2.4 (Proposition 4.6 [1]). *Let* $a, b \in \Delta$ *with* $a \neq b$. *Then* $N_{a,b}$ *is* 3-*dimensional precisely when* $ab \neq 0$ *and there exists* $0 \neq \sigma \in N_{a,b}$ *and a scalar* $\varphi = \varphi_{a,b} \in \mathbb{F}$ *such that if we set* $\pi = \pi_{a,b} = (1 - \eta)\varphi - \eta$, *then*

(1) $ab = \sigma + \eta a + \eta b$;
(2) $\sigma v = \pi v$, *for all* $v \in \{a, b, \sigma\}$.

furthermore

(3) $N_{a,b}$ *contains an identity element if and only if* $\pi \neq 0$, *in which case* $1_{a,b} = \frac{1}{\pi}\sigma$.

When $N_{a,b}$ *is* 3-*dimensional we denote:* $N_{a,b} = B(\eta, \varphi)_{a,b}$, *where* $\varphi \in \mathbb{F}$ *is the scalar mentioned above.*

From now on we assume that $\Delta_{\mathcal{A}}$ *is connected.* Note that by [2, Lemma 6.4], $\Delta_{\mathcal{A}}$ is connected iff Δ is connected. Further, we assume that $a, b \in \Delta$ *are distinct with* $\tau(a) = \tau(b)$.

Proposition 2.5 (Proposition 6.5 in [2]). $ab = 0$ *and*

(1) *for any* $c \in \Delta \smallsetminus \{a,b\}$ *exactly one the following holds:*

(i) $ac = bc = 0$.

(ii) $\eta = \frac{1}{2}$, *and for some* $x \in \{a,b\} = \{x,y\}$, *we have* $N_{x,c} = B(\frac{1}{2},0)_{x,c}$ *is 3-dimensional,* $N_{y,c} = J_{y,c}$ *and* $N_{y,c} \subset N_{x,c}$. *Further* $a+b = 1_{x,c}$.

(iii) $\eta = \frac{1}{2}$, $N_{a,c} = N_{b,c}$ *is 3-dimensional and* $a+b = 1_{a,c}$.

(2) *If* d *is an* η*-axis in* A *such that* $\tau(d) = \tau(a)$, *then* $d \in \{a,b\}$.

Proof sketch. By [2, Lemma 3.2.1], for any $c \in \Delta$, we have $ac = 0 \iff c^{\tau(a)} = c$, and since, by definition, $a^{\tau(b)} = a^{\tau(a)} = a$, we see that $ab = 0$.

If $ac = 0$, then, as above $bc = 0$ (and vice versa), so (i) holds. Hence we may assume that $ac \neq 0 \neq bc$.

If $\eta \neq \frac{1}{2}$, then by [1, Proposition 6.5], and since Δ is connected, $a = b$, a contradiction. Thus $\eta = \frac{1}{2}$.

Now consider

$$V := N_{c,c^{\tau(a)}} \subseteq N_{a,c} \cap N_{b,c}.$$

V is either 2 or 3-dimensional. If V is 3-dimensional, then $N_{a,c} = V = N_{b,c}$, and since $ab = 0$, one shows that $a+b = 1_{a,c}$ ([2, Lemma 3.2.5]), so (iii) holds.

So suppose V is 2-dimensional. If both $N_{a,c}$ and $N_{b,c}$ are 2-dimensional, then they both equal to $N_{a,b} = \mathbb{F}a \oplus \mathbb{F}b$. But then $c = a$ or b, a contradiction.

Therefore without loss $N_{a,c}$ is 3-dimensional and V is 2-dimensional. If $V = N_{b,c}$ then (ii) holds: Clearly $N_{b,c} \subset N_{a,c}$ and $a+b = 1_{a,c}$, and then a careful analysis of the situation gives (ii).

The case where both $N_{a,c}$ and $N_{b,c}$ are 3-dimensional and V is 2-dimensional is the hardest case and some precise work is required to get a contradiction. □

Proposition 2.6. $\eta = \frac{1}{2}$ *and*

(1) $xa \neq 0 \neq xb$, *for all* $x \in \Delta \smallsetminus \{a,b\}$;

(2) A *contains an identity element* $\mathbb{1} = a+b$;

(3) *for any* $x \in \Delta$ *such that* $N_{a,x}$ *is 3-dimensional we have* $\mathbb{1} = 1_{a,x}$.

Proof. Let $d(\ ,\ )$ be the distance function on Δ. Let

$$\Delta_1(a) := \{x \in \Delta \mid d(a,x) = 1\}.$$

Since Δ is connected $\Delta_1(a) \neq \emptyset$. Also, by Proposition 2.5(1i), $\Delta_1(a) = \Delta_1(b)$. Let $c \in \Delta_1(a)$. By Proposition 2.5, $\eta = \frac{1}{2}$ and after perhaps interchanging a and b, $N_{a,c}$ is 3-dimensional and $a + b = 1_{a,c}$. Set

$$\mathbb{1} = 1_{a,c} = a + b,$$

then

$$\mathbb{1}c = c, \text{ for all } c \in \Delta_1(a).$$

Let $y \in \Delta \smallsetminus \Delta_1(a)$ be at distance 2 from a in Δ, and let

$$x \in \Delta_1(a) \cap \Delta_1(y).$$

Without loss $N_{a,x}$ is 3-dimensional and $\mathbb{1} = 1_{a,x}$. Now

- $ay = 0 = by \implies \mathbb{1}^{\tau(y)} = (a+b)^{\tau(y)} = a^{\tau(y)} + b^{\tau(y)} = a + b = \mathbb{1}$.
- $\mathbb{1}^{\tau(x)} = \mathbb{1}$ because $\mathbb{1} = 1_{a,x}$.
- $\mathbb{1}y = 0$ so $\mathbb{1}y^{\tau(x)} = 0$.
- $\mathbb{1}x = x$ so $\mathbb{1}x^{\tau(y)} = x^{\tau(y)}$.
- $W := \mathrm{Span}(\{y,\ y^{\tau(x)}\}) \cap \mathrm{Span}(\{x,\ x^{\tau(y)}\}) \neq \{0\}$. Indeed, W is the intersection of two 2-dimensional subspaces of $N_{x,y}$ which is of dimension at most 3.
- $\mathbb{1}$ both annihilates and acts as identity on W, a contradiction.

Hence $\Delta_1(a) = \Delta \smallsetminus \{a, b\}$ and clearly $d(a,b) = 2$ in Δ. But now, as we saw above, $\mathbb{1}c = c$ for all $c \in \Delta$. It follows that $\mathbb{1}$ is the identity of A and (3) holds as well. □

We are now in a position to prove Theorem 2.1.

***Proof of Theorem* 2.1.** We show that the hypotheses of Theorem 2.2 are satisfied. By Proposition 2.6, $\eta = \frac{1}{2}$ and $a + b = \mathbb{1}_A$. Let $c \in \Delta$. Then

$$v_a v_c = (a - \tfrac{1}{2}\mathbb{1})(c - \tfrac{1}{2}\mathbb{1}) = ac - \tfrac{1}{2}a - \tfrac{1}{2}c + \tfrac{1}{4}\mathbb{1} = \sigma_{a,c} + \tfrac{1}{4}\mathbb{1}.$$

Clearly $v_a v_c \in \mathbb{F}\mathbb{1}$ if $c \in \{a, b\}$. Otherwise, by Proposition 2.6(1), $ac \neq 0$. If $N_{a,c}$ is 2-dimensional, then since $ac \neq 0$, $\sigma_{a,c} = 0$, and so $v_a v_c \in \mathbb{F}\mathbb{1}$. If $N_{a,c}$ is 3-dimensional, then by Proposition 2.6(3), $\mathbb{1} = 1_{a,c}$. Furthermore by [1], $\sigma_{a,c} = \pi_{a,c} 1_{a,c} = \pi_{a,c}\mathbb{1}$, for some $\pi_{a,c} \in \mathbb{F}$, and again $v_a v_c \in \mathbb{F}\mathbb{1}$. □

3. 3-transpositions and Matsuo Algebras

Recall that a set of axes $\mathcal{A}$ is closed iff $a^{\tau_b} \in \mathcal{A}$, for all $a, b \in \mathcal{A}$. In this section A is a primitive axial algebra of Jordan type η generated by a closed set of η-axes $\mathcal{A}$, such that $|\mathcal{A}| > 1$.

Let G be a group generated by a normal set of involutions D. Recall that D is called *a set of* 3*-transpositions in* G if $|st| \in \{1, 2, 3\}$, for all $s, t \in D$. The group G is then called a 3*-transposition group.*

Let D be a normal set of 3-transpositions in the group G that generates G. The *Matsuo algebra* associated with the pair (G, D), denoted here $M_\delta(G, D)$, is defined as follows. As a vector space over $\mathbb{F}$ it has the basis D. Multiplication is defined for $x, y \in D$ as follows

$$x \cdot y = \begin{cases} x, & \text{if } y = x \\ 0, & \text{if } |xy| = 2 \\ \delta(x + y - x^y), & \text{if } |xy| = 3. \end{cases}$$

This is extended by linearity to the entire algebra. (Note that we denote multiplication in G by juxtaposition and in $M_\delta(G, D)$ by dot.) By [1, Theorem 6.2], $M_\delta(G, D)$ is a primitive axial algebra of Jordan type 2δ.

The purpose of this section is to prove the following Theorem:

Theorem 3.1. *Suppose that the graph $\Delta_{\mathcal{A}}$ is connected. Let $D := \{\tau_a \mid a \in \mathcal{A}\}$ and $G = \langle D \rangle$. Assume that the map $a \mapsto \tau_a$ on $\mathcal{A}$ is injective and that D is a set of* 3*-transpositions in G. Then A is a quotient of the Matsuo algebra $M_{\frac{\eta}{2}}(G, D)$.*

Remark 3.2. Theorem 3.1 was proved in [1, Theorem 6.3] for $\eta \neq \frac{1}{2}$. The proof for $\eta = \frac{1}{2}$ needed a correction, in view of [2]. Note that the summand $\oplus_{i \in I}\mathbb{F}$ does not appear in Theorem 3.1 since we are assuming that $\Delta_{\mathcal{A}}$ is connected. We also mention that for $\eta \neq \frac{1}{2}$, the map on $\mathcal{A}$ defined by $a \mapsto \tau_a$ is always injective, by [2, Proposition 6.5], and since $\Delta_{\mathcal{A}}$ is connected.

We included a proof of Theorem 3.1 for all η for completeness.

Lemma 3.3. $ab = \frac{\eta}{2}a + \varphi_{a,b}b - \frac{\eta}{2}a^{\tau_b}$, *for all* $a, b \in \mathcal{A}$.

Proof. Clearly this holds when $a = b$ (since, by definition, $\varphi_{a,a} = 1$, and $a^{\tau_a} = a$), so assume $a \neq b$. Suppose first that $N_{a,b}$ is 2-dimensional. We use [2, Lemma 3.1.2]. If $N_{a,b} = 2B_{a,b}$, then $ab = 0, \varphi_{a,b} = 0$, and $a^{\tau_b} = a$ (see also [2, Lemma 3.2.1]), so the claim holds.

Suppose next that $N_{a,b} = 3C(-1)^{\times}_{a,b}$. Then $\eta = -1, ab = -a - b, \varphi_{a,b} = -\frac{1}{2}$ and $a^{\tau_b} = -a - b$ (see also [2, Lemma 3.1.8]), so the claim holds.

Assume that $N_{a,b} = J_{a,b}$. Then $\eta = \frac{1}{2}, ab = \frac{1}{2}a + \frac{1}{2}b, \varphi_{a,b} = 1$ and $a^{\tau_b} = 2b - a$ (see also [2, Lemma 3.1.9]), so again the claim holds.

We may assume that $N_{a,b}$ is 3-dimensional. Set $\varphi := \varphi_{a,b}$. By [2, Theorem 3.1.3(6)], $a^{\tau(b)} = -\frac{2}{\eta}\sigma - \frac{2(\eta-\varphi)}{\eta}b - a$. Also, $\sigma = ab - \eta a - \eta b$. Hence we get

$$\begin{aligned} &\tfrac{2}{\eta}\sigma = -a - \tfrac{2(\eta-\varphi)}{\eta}b - a^{\tau_b} \\ \Longleftrightarrow\ &\sigma = -\tfrac{\eta}{2}a - (\eta-\varphi)b - \tfrac{\eta}{2}a^{\tau_b} \\ \Longleftrightarrow\ &ab = \tfrac{\eta}{2}a + \varphi b - \tfrac{\eta}{2}a^{\tau_b}. \end{aligned}$$ □

Corollary 3.4 (See Corollary 1.2 in [1]). *A is spanned over* $\mathbb{F}$ *by* $\mathcal{A}$.

Proof. This is immediate from Lemma 3.3 and the definition of a closed set of axes. □

Lemma 3.5. *Suppose that*

$$\text{the map } a \mapsto \tau_a \text{ on } \mathcal{A} \text{ is injective.} \qquad (*)$$

Let $a, b \in \mathcal{A}$ *be distinct. Then*

(1) *if* $(\tau_a\tau_b)^2 = 1$, *then* $ab = 0$.

(2) *if* $(\tau_a\tau_b)^3 = 1$, *then* $\varphi_{a,b} = \frac{\eta}{2}$.

Proof. (1): By [2, Lemmas 3.2.7(2) and 3.1.6(2)] and by $(*)$, $N_{a,b} = 2B_{a,b}$, so (1) holds (see also [2, Lemma 3.1.2(1a)]).

(2): If $\eta \neq \frac{1}{2}$, then (2) follows from [1, Proposition 4.8]. So suppose $\eta = \frac{1}{2}$. By [2, Lemma 3.2.7(1) and Corollary 3.3.2] and by $(*)$, we get $\varphi_{a,b} = \frac{1}{4}$. □

We can now prove Theorem 3.1.

***Proof of Theorem* 3.1.** Set $M := M_{\frac{\eta}{2}}(G, D)$. We claim that the map

$$f\colon M \to A : \ \tau_a \mapsto a,$$

extended by linearity is a surjective algebra homomorphism. Note that f is well defined since the map $a \mapsto \tau_a$ is injective on $\mathcal{A}$.

Now f is surjective by Corollary 3.4. Next we need to check that

$$f(\tau_a \cdot \tau_b) = ab, \text{ for all } a, b \in \mathcal{A}. \tag{$*$}$$

If $a = b$, then $\tau_a \cdot \tau_b = \tau_a$, and $ab = a$, so $(*)$ holds.

If $|\tau_a\tau_b| = 2$, then $\tau_a \cdot \tau_b = 0$, while by Lemma 3.5(1), $ab = 0$, so $(*)$ holds in this case as well.

Finally assume that $|\tau_a\tau_b| = 3$. Then

$$\tau_a \cdot \tau_b = \tfrac{\eta}{2}(\tau_a + \tau_b - \tau_a^{\tau_b}) = \tfrac{\eta}{2}(\tau_a + \tau_b - \tau_{a^{\tau_b}}),$$

where the last equality follows from the standard fact that $\tau_a^{\tau_b} = \tau_{a^{\tau_b}}$. Thus $f(\tau_a \cdot \tau_b) = \frac{\eta}{2}(a + b - a^{\tau_b})$. However, by Lemma 3.5(2) and Lemma 3.3, $ab = \frac{\eta}{2}(a + b - a^{\tau_b})$, so $(*)$ holds in this case as well and the proof of the theorem is complete. □

4. The Existence of a Frobenius Form

Recall that a non-zero bilinear form $(\cdot\,,\cdot)$ on an algebra A is called *Frobenius* if the form associates with the algebra product, that is,

$$(xy, z) = (x, yz)$$

for all $x, y, z \in A$.

For primitive axial algebras of Jordan type η, we specialize the concept of Frobenius form further by asking that the condition $(e, e) = 1$ be satisfied for each η-axis e.

The purpose of this section is to prove the following theorem:

Theorem 4.1. *Let A be a primitive axial algebra of Jordan type η. Then A admits a Frobenius form $(\cdot\,,\,\cdot)$. Furthermore, $(e\,,\,u) = \varphi_e(u)$, for any η-axis $e \in A$ and any $u \in A$.*

The proof of Theorem 4.1 depends on two properties of primitive axial algebras of Jordan type. The first is Corollary 3.4. The second is proven in [1] (Lemma 4.2 below).

For an η-axis $e \in A$, let φ_e be the projection function with respect to e. That is, for $u \in A$, we have that $u = \varphi_e(u)e + u_0 + u_\eta$, where u_0 and u_η are eigenvectors of the adjoint linear transformation ad_e for the eigenvalues 0 and η, respectively.

Lemma 4.2 (Lemma 4.4 in [1])**.** *For a primitive axial algebra A of Jordan type and for any η-axes $e, f \in A$, we have $\varphi_e(f) = \varphi_f(e)$.*

Note that by [1] the constant $\varphi_{a,b}$, that we used earlier for η-axes a, b, is the same as $\varphi_a(b)$.

***Proof of Theorem* 4.1.** We start by defining the bilinear form $(\cdot\,,\,\cdot)$ on A. Using Corollary 3.4 we can select a basis $\mathcal{B}$ of A consisting of η-axes, and we let

$$(a\,,\,b) = \varphi_a(b),\ \text{ for all } a, b \in \mathcal{B}.$$

Extending by linearity we get the bilinear form $(\cdot\,,\,\cdot)$. Note that Lemma 4.2 implies that $(\cdot\,,\,\cdot)$ is symmetric.

Lemma 4.3. (1) $(e\,,\,u) = \varphi_e(u)$, *for all η-axes $e \in A$ and all $u \in A$;*
(2) $(e\,,\,e) = 1$, *for all η-axes $e \in A$;*
(3) $(\cdot\,,\,\cdot)$ *is invariant under automorphisms of A.*

Proof. (1&2): Let e be an η-axis and suppose that

$$\varphi_e(b) = (e, b),\ \text{ for all } b \in \mathcal{B}. \tag{$*$}$$

Since φ_e is linear,

$$\varphi_e(u) \ = \ \varphi_e\Big(\sum_{b\in\mathcal{B}} \alpha_b b\Big) = \sum_{b\in\mathcal{B}} \alpha_b \varphi_e(b)$$

$$= \sum_{b\in\mathcal{B}} \alpha_b(e,b) = (e, \sum_{b\in\mathcal{B}} \alpha_b b) = (e,u),$$

and (1) holds for e. Now if $e = a \in \mathcal{B}$, then $(*)$ holds by definition, so (1) holds for a. Suppose $e \notin \mathcal{B}$. Let $b \in \mathcal{B}$. Then $\varphi_e(b) = \varphi_b(e)$, by Lemma 4.2, and $\varphi_b(e) = (b,e)$, as (1) holds for b. Finally, since $(\cdot\,,\,\cdot)$ is symmetric $(b,e) = (e,b)$, so $\varphi_e(b) = (e,b)$, and $(*)$ holds for any η-axis e. This shows that (1) holds.

In particular, for every η-axis $e \in A$, we have that $(e,e) = 1$, since, clearly, $\varphi_e(e) = 1$. Thus (2) holds.

(3): Let $\psi \in \mathrm{Aut}(A)$, if $u = \varphi_e(u)e + u_0 + u_\eta$ is the decomposition of $u \in A$ with respect to the η-axis e, then $u^\psi = \varphi_e(u)e^\psi + u_0^\psi + u_\eta^\psi$ is the decomposition of u^ψ with respect to the η-axis e^ψ. Hence $\varphi_{e^\psi}(u^\psi) = \varphi_e(u)$, and so $(e^\psi, u^\psi) = (e,u)$. Finally, taking an arbitrary $v \in A$ and decomposing it with respect to the basis $\mathcal{B}$ as $v = \sum_{b\in\mathcal{B}} \alpha_b b$, we get that $(v^\psi, u^\psi) = (\sum_{b\in\mathcal{B}} \alpha_b b^\psi, u^\psi) = \sum_{b\in\mathcal{B}} \alpha_b (b^\psi, u^\psi) = \sum_{b\in\mathcal{B}} \alpha_b(b,u) = (\sum_{b\in\mathcal{B}} \alpha_b b, u) = (v,u)$. So indeed, $(\cdot\,,\,\cdot)$ is invariant under the automorphisms of A. □

Lemma 4.4. *For every η-axis $e \in A$, different eigenspaces of* ad_e *are orthogonal with respect to $(\cdot\,,\,\cdot)$.*

Proof. Clearly, if $u \in A_0(e) + A_\eta(e)$ then $(e,u) = \varphi_e(u) = 0$. Hence $A_1(e) = \mathbb{F}e$ is orthogonal to both $A_0(e)$ and $A_\eta(e)$. It remains to show that these two are also orthogonal to each other. Let $u \in A_0(e)$ and $v \in A_\eta(e)$, the fact that $(\cdot\,,\,\cdot)$ is invariant under τ_e gives us $(u,v) = (u^{\tau_e}, v^{\tau_e}) = (u,-v) = -(u,v)$. Clearly, this means that $(u,v) = 0$. □

We are now ready to complete the proof that $(\cdot\,,\,\cdot)$ associates with the algebra product. Note that the identity

$$(x,yz) = (xy,z)$$

that we need to prove is linear in x, y, and z. In particular, since A is spanned by η-axes, we may assume that y is an η-axis. Furthermore, since A decomposes as the sum of the eigenspaces of ad_y, we may assume that x and z are eigenvectors of ad_y, say, for the eigenvalues μ and ν. We have two cases:

If $\mu = \nu$ then

$$(x, yz) = (x, \nu z) = \nu(x, z) = \mu(x, z) = (\mu x, z) = (yx, z) = (xy, z).$$

If $\mu \neq \nu$ then

$$(x, yz) = \nu(x, z) = 0 = \mu(x, z) = (xy, z),$$

since $A_\mu(y)$ and $A_\nu(y)$ are orthogonal to each other. Thus, in both cases we have the desired equality $(x, yz) = (xy, z)$, proving that the form $(\cdot\,,\,\cdot)$ is Frobenius. Also, by Lemma 4.3(1), the second part of Theorem 4.1 holds. □

References

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2. J. I. Hall, Y. Segev and S. Shpectorov, Miyamoto involutions in axial algebras of Jordan type half, to appear in *Israel J. Math.* (`https://arxiv.org/abs/1610.01307`)