Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **15** (2020), No. 1, pp. 63-92
DOI: 10.21915/BIMAS.2020105

THE FLAG MANIFOLD OVER THE SEMIFIELD Z

G. LUSZTIG

Department of Mathematics, M.I.T., Cambridge, MA 02139.
E-mail: gyuri@math.mit.edu

Abstract

Let G be a semisimple group over the complex numbers. We show that the flag manifold $\mathcal{B}$ of G has a version $\mathcal{B}(Z)$ over the tropical semifield Z on which the monoid $G(Z)$ attached to G and Z acts naturally.

0. Introduction

0.1. Let G be a connected semisimple simply connected algebraic group over $\mathbf{C}$ with a fixed pinning (as in [5, 1.1]). In this paper we assume that G is of simply laced type. Let $\mathcal{B}$ be the variety of Borel subgroups of G. In [5, 2.2, 8.8] a submonoid $G_{\geq 0}$ of G and a subset $\mathcal{B}_{\geq 0}$ of $\mathcal{B}$ with an action of $G_{\geq 0}$ (see [5, 8.12]) was defined. (When $G = SL_n$, $G_{\geq 0}$ is the submonoid consisting of the real, totally positive matrices in G.) More generally, for any semifield K, a monoid $\mathfrak{G}(K)$ was defined in [8], so that when $K = \mathbf{R}_{>0}$ we have $\mathfrak{G}(K) = G_{\geq 0}$. (In the case where K is $\mathbf{R}_{>0}$ or the semifield in (i) or (ii) below, a monoid $G(K)$ already appeared in [5, 2.2, 9.10]; it was identified with $\mathfrak{G}(K)$ in [9].)

This paper is concerned with the question of defining the flag manifold $\mathcal{B}(K)$ over a semifield K with an action of the monoid $\mathfrak{G}(K)$ so that in the case where $K = \mathbf{R}_{>0}$ we recover $\mathcal{B}_{\geq 0}$ with its $G_{\geq 0}$-action.

In [9, 4.9], for any semifield K, a definition of the flag manifold $\mathcal{B}(K)$ over K was given (based on ideas of Marsh and Rietsch [10]); but in that definition the lower and upper triangular part of G play an asymmetric role

Received January 18, 2020.
AMS Subject Classification: 20G99.
Key words and phrases: Flag manifold, canonical basis, semifield.
Supported by NSF grant DMS-1855773.

and as a consequence only a part of $\mathfrak{G}(K)$ acts on $\mathcal{B}(K)$ (unlike the case $K = \mathbf{R}_{>0}$ when the entire $\mathfrak{G}(K)$ acts). To get the entire $\mathfrak{G}(K)$ act one needs a conjecture stated in [9, 4.9] which is still open.

In this paper we get around that conjecture and provide an unconditional definition of the flag manifold (denoted by $\mathcal{B}(K)$) over a semifield K with an action of $\mathfrak{G}(K)$ assuming that K is either

(i) the semifield consisting of all rational functions in $\mathbf{R}(x)$ (with x an indeterminate) of the form $x^e f_1/f_2$ where $e \in \mathbf{Z}$ and $f_1 \in \mathbf{R}[x], f_2 \in \mathbf{R}[x]$ have constant term in $\mathbf{R}_{>0}$ (standard sum and product); or

(ii) the semifield $\mathbf{Z}$ in which the sum of a, b is $\min(a, b)$ and the product of a, b is $a + b$.

For K as in (i) we give two definitions of $\mathcal{B}(K)$; one of them is elementary and the other is less so, being based on the theory of canonical bases (the two definitions are shown to be equivalent). For K as in (ii) we only give a definition based on the theory of canonical bases.

A part of our argument involves a construction of an analogue of the finite dimensional irreducible representations of G when G is replaced by the monoid $\mathfrak{G}(K)$ where K is any semifield.

Let W be the Weyl group of G. Now W is naturally a Coxeter group with generators $\{s_i; i \in I\}$ and length function $w \mapsto |w|$. Let $\leq$ be the Chevalley partial order on W.

In §3 we prove the following result which is a $\mathbf{Z}$-analogue of a result (for $\mathbf{R}_{>0}$) in [10].

Theorem 0.2 *The set $\mathcal{B}(\mathbf{Z})$ has a canonical partition into pieces $P_{v,w}(\mathbf{Z})$ indexed by the pairs $v \leq w$ in W. Each such piece $P_{v,w}(\mathbf{Z})$ is in bijection with $\mathbf{Z}^{|w|-|v|}$; in fact, there is an explicit bijection $\mathbf{Z}^{|w|-|v|} \xrightarrow{\sim} P_{v,w}(\mathbf{Z})$ for any reduced expression of w.*

In §3 we also prove a part of a conjecture in [9, 2.4] which attaches to any $v \leq w$ in W a certain subset of a canonical basis, see 3.10.

In §4 we show that our definitions do not depend on the choice of a (very dominant) weight λ.

In §5 we show how some of our results extend to the non-simply laced case.

1. Definition of $\mathcal{B}(\mathbf{Z})$

1.1. In this section we will give the definition of the flag manifold $\mathcal{B}(K)$ when K is as in 0.1(i), (ii).

1.2. We fix some notation on G. Let w_I be the longest element of W. For $w \in W$ let $\mathcal{I}_w$ be the set of all sequences $\mathbf{i} = (i_1, i_2, \dots, i_m)$ in I such that $w = s_{i_1} s_{i_2} \dots s_{i_m}$, $m = |w|$.

The pinning of G consists of two opposed Borel subgroups B^+, B^- with unipotent radicals U^+, U^- and root homomorphisms $x_i : \mathbf{C} \to U^+$, $y_i : \mathbf{C} \to U^-$ indexed by $i \in I$. Let $T = B^+ \cap B^-$, a maximal torus. Let $\mathcal{Y}$ be the group of one parameter subgroups $\mathbf{C}^* \to T$; let $\mathcal{X}$ be the group of characters $T \to \mathbf{C}^*$. Let $\langle , \rangle : \mathcal{Y} \times \mathcal{X} \to \mathbf{Z}$ be the canonical pairing. The simple coroot corresponding to $i \in I$ is denoted again by $i \in \mathcal{Y}$; let $i' \in \mathcal{X}$ be the corresponding simple root. Let $\mathcal{X}^+ = \{\lambda \in \mathcal{X}; \langle i, \lambda \rangle \geq 0 \quad \forall i \in I\}$, $\mathcal{X}^{++} = \{\lambda \in \mathcal{X}; \langle i, \lambda \rangle \geq 1 \quad \forall i \in I\}$. Let $G(\mathbf{R})$ be the subgroup of G generated by $x_i(t), y_i(t)$ with $i \in I, t \in \mathbf{R}$. Let $\mathcal{B}(\mathbf{R})$ be the subset of $\mathcal{B}$ consisting of all $B \in \mathcal{B}$ such that $B = gB^+g^{-1}$ for some $g \in G(\mathbf{R})$. We have $G_{\geq 0} \subset G(\mathbf{R})$, $\mathcal{B}_{\geq 0} \subset \mathcal{B}(\mathbf{R})$. For $i \in I$ we set $\dot{s}_i = y_i(1)x_i(-1)y_i(1) \in G(\mathbf{R})$, an element normalizing T. For $(B, B') \in \mathcal{B} \times \mathcal{B}$ we write $pos(B, B')$ for the relative position of B, B' (an element of W).

1.3. Let K be a semifield. Let $K^! = K \sqcup \{\circ\}$ where $\circ$ is a symbol. We extend the sum and product on K to a sum and product on $K^!$ by defining $\circ + a = a$, $a + \circ = a$, $\circ \times a = \circ, a \times \circ = \circ$ for $a \in K$ and $\circ + \circ = \circ, \circ \times \circ = \circ$. Thus $K^!$ becomes a monoid under addition and a monoid under multiplication. Moreover the distributivity law holds on $K^!$. When K is $\mathbf{R}_{>0}$ we have $K^! = \mathbf{R}_{\geq 0}$ with $\circ = 0$ and the usual sum and product. When K is as in 0.1(i), $K^!$ can be viewed as the subset of $\mathbf{R}(x)$ given by $K \cup \{0\}$ with $\circ = 0$ and the usual sum and product. When K is as in 0.1(ii) we have $0 \in K$ and $\circ \neq 0$.

1.4. Let $V = {}^\lambda V$ be the finite dimensional simple G-module over $\mathbf{C}$ with highest weight $\lambda \in \mathcal{X}^+$. For $\nu \in \mathcal{X}$ let V_ν be the ν-weight space of V with respect to T. Thus V_λ is a line. We fix $\xi^+ = {}^\lambda\xi^+$ in $V_\lambda - 0$. For each $i \in I$ there are well defined linear maps $e_i : V \to V, f_i : V \to V$ such that $x_i(t)\xi = \sum_{n \geq 0} t^n e_i^{(n)} \xi, y_i(t)\xi = \sum_{n \geq 0} t^n f_i^{(n)} \xi$ for $\xi \in V, t \in \mathbf{C}$. Here

$e_i^{(n)} = (n!)^{-1}e_i^n : V \to V, f_i^{(n)} = (n!)^{-1}f_i^n : V \to V$ are zero for $n \gg 0$. For an integer $n < 0$ we set $e_i^{(n)} = 0, f_i^{(n)} = 0$.

Let $\beta = {}^\lambda\beta$ be the canonical basis of V (containing ξ^+) defined in [1]. Let ξ^- be the lowest weight vector in $V - 0$ contained in β. For $b \in \beta$ we have $b \in V_{\nu_b}$ for a well defined $\nu_b \in \mathcal{X}$, said to be the weight of b. By a known property of β (see [1, 10.11] and [2, §3], or alternatively [3, 22.1.7]), for $i \in I, b \in \beta, n \in \mathbf{Z}$ we have

$$e_i^{(n)}b = \sum_{b' \in \beta} c_{b,b',i,n}b', \quad f_i^{(n)}b = \sum_{b' \in \beta} d_{b,b',i,n}b'$$

where

$$c_{b,b',i,n} \in \mathbf{N}, \quad d_{b,b',i,n} \in \mathbf{N}.$$

Hence for $i \in I, b \in \beta, t \in \mathbf{C}$ we have

$$x_i(t)b = \sum_{b' \in \beta, n \in \mathbf{N}} c_{b,b',i,n}t^n b', \quad y_i(t)b = \sum_{b' \in \beta, n \in \mathbf{N}} d_{b,b',i,n}t^n b'.$$

For any $i \in I$ there is a well defined function $z_i : \beta \to \mathbf{Z}$ such that for $b \in \beta$, $t \in \mathbf{C}^*$ we have $i(t)b = t^{z_i(b)}b$.

Let $P = {}^\lambda P$ be the variety of $\mathbf{C}$-lines in V. Let $P^\bullet = {}^\lambda P^\bullet$ be the set of all $L \in P$ such that for some $g \in G$ we have $L = gV_\lambda$. Now $P^\bullet$ is a closed subvariety of P. For any $L \in P^\bullet$ let $G_L = \{g \in G; gL = L\}$; this is a parabolic subgroup of G.

Let $V^\bullet = {}^\lambda V^\bullet = \cup_{L \in P^\bullet} L$, a closed subset of V. For any $\xi \in V, b \in \beta$ we define $\xi_b \in \mathbf{C}$ by $\xi = \sum_{b \in \beta} \xi_b b$. Let $V_{\geq 0} = {}^\lambda V_{\geq 0}$ (resp. $V_{\mathbf{R}}$) be the set of all $\xi \in V$ such that $\xi_b \in \mathbf{R}_{\geq 0}$ (resp. $\xi_b \in \mathbf{R}$) for any $b \in \beta$. We have $V_{\geq 0} \subset V_{\mathbf{R}}$. Note that $V_{\mathbf{R}}$ is stable under the action of $G(\mathbf{R})$ on V. Let $P_{\geq 0} = {}^\lambda P_{\geq 0}$ (resp. $P_{\mathbf{R}}$) be the set of lines $L \in P$ such that $L \cap V_{\geq 0} \neq 0$ (resp. $L \cap V_{\mathbf{R}} \neq 0$.) We have $P_{\geq 0} \subset P_{\mathbf{R}}$.

Let $V^\bullet_{\geq 0} = {}^\lambda V^\bullet_{\geq 0} = V^\bullet \cap V_{\geq 0}$, $P^\bullet_{\geq 0} = {}^\lambda P^\bullet_{\geq 0} = P^\bullet \cap P_{\geq 0}$.

Now let K be a semifield. Let $V(K) = {}^\lambda V(K)$ be the set of formal sums $\xi = \sum_{b \in \beta} \xi_b b, \xi_b \in K^!$. This is a monoid under addition $(\sum_{b \in \beta} \xi_b b) + (\sum_{b \in \beta} \xi'_b b) = \sum_{b \in \beta}(\xi_b + \xi'_b)b$ and we define scalar multiplication $K^! \times V(K) \to V(K)$ by $(k, \sum_{b \in \beta} \xi_b b) \mapsto \sum_{b \in \beta}(k\xi_b)b$.

For $\xi = \sum_{b\in\beta} \xi_b b \in V(K)$ we define $\mathrm{supp}(\xi) = \{b \in \beta; \xi_b \in K\}$.

Let $\mathrm{End}(V(K))$ be the set of maps $\zeta : V(K) \to V(K)$ such that $\zeta(\xi + \xi') = \zeta(\xi) + \zeta(\xi')$ for ξ, ξ' in $V(K)$ and $\zeta(k\xi) = k\zeta(\xi)$ for $\xi \in V(K), k \in K^!$. This is a monoid under composition of maps. Define $\underline{\circ} \in V(K)$ by $\underline{\circ}_b = \circ$ for all $b \in \beta$. The group K (for multiplication in the semifield structure) acts freely (by scalar multiplication) on $V(K) - \underline{\circ}$; let $P(K) = {}^\lambda P(K)$ be the set of orbits of this action.

For $i \in I, n \in \mathbf{Z}$ we define $e_i^{(n)}, f_i^{(n)}$ in $\mathrm{End}(V(K))$ by

$$e_i^{(n)}(b) = \sum_{b'\in\beta} c_{b,b',i,n} b', \quad f_i^{(n)}(b) = \sum_{b'\in\beta} d_{b,b',i,n} b',$$

with $b \in \beta$. Here a natural number N (such as $c_{b,b',i,n}$ or $d_{b,b',i,n}$) is viewed as an element of $K^!$ given by $1+1+\cdots+1$ (N terms, where 1 is the neutral element for the product in K, if $N > 0$) or by $\circ \in K^!$ (if $N = 0$).

For $i \in I, k \in K$ we define $i^k \in \mathrm{End}(V(K))$, $(-i)^k \in \mathrm{End}(V(K))$ by

$$i^k(b) = \sum_{n\in\mathbf{N}} k^n e_i^{(n)} b, \quad (-i)^k(b) = \sum_{n\in\mathbf{N}} k^n f_i^{(n)} b,$$

for any $b \in \beta$. We show:

(a) *The map* $i^k : V(K) \to V(K)$ *is injective. The map* $(-i)^k : V(K) \to V(K)$ *is injective.*

Using a partial order of the weights of V, we can write $V(K)$ as a direct sum of monoids $V(K)_s, s \in \mathbf{Z}$ where $V(K)_s = \{\underline{\circ}\}$ for all but finitely many s and $(-i)^k$ maps any $\xi \in V(K)_s$ to ξ plus an element in the direct sum of $V(K)_{s'}$ with $s' < s$. Then (a) for $(-i)^k$ follows immediatly. A similar proof applies to i^k.

For $i \in I, k \in K$ we define $\underline{i}^k \in \mathrm{End}(V(K))$ by $\underline{i}^k(b) = k^{z_i(b)} b$ for any $b \in \beta$. Let $\mathfrak{G}(K)$ be the monoid associated to G, K by generators and relations in [8, 2.10(i)-(vii)]. (In *loc.cit.* it is assumed that K is as in 0.1(i) or 0.1(ii) but the same definition makes sense for any K.) We have the following result.

Proposition 1.5. *The elements* $i^k, (-i)^k, \underline{i}^k$ *(with* $i\in I, k\in K$*) in* $\mathrm{End}(V(K))$ *satisfy the relations in* [8, 2.10(i)-(vii)] *defining the monoid* $\mathfrak{G}(K)$ *hence they define a monoid homomorphism* $\mathfrak{G}(K) \to \mathrm{End}(V(K))$.

We write the relations in *loc.cit.* (for the semifield $\mathbf{R}_{>0}$) for the endomorphisms $x_i(t), y_i(t), i(t)$ of V with $t \in R_{>0}$. These relations can be expressed as a set of identities satisfied by $c_{b,b',i,n}$, $d_{b,b',i,n}$, $z_i(b)$ and these identities show that the endomorphisms $i^k, (-i)^k, \underline{i}^k$ of $V(K)$ satisfy the relations in *loc.cit.* (for the semifield K). The result follows.

1.6. Consider a homomorphism of semifields $r : K_1 \to K_2$. Now r induces a homomorphism of monoids $\mathfrak{G}_r : \mathfrak{G}(K_1) \to \mathfrak{G}(K_2)$. It also induces a homomorphism of monoids $V_r : V(K_1) \to V(K_2)$ given by $\sum_{b\in\beta} \xi_b b \mapsto \sum_{b\in\beta} r(\xi_b)b$. From the definitions, for $g \in \mathfrak{G}(K_1), \xi \in V(K_1)$, we have $V_r(g\xi) = \mathfrak{G}_r(g)(V_r(\xi))$ where $g\xi$ is given by the $\mathfrak{G}(K_1)$-action on $V(K_1)$ and $\mathfrak{G}_r(g)(V_r(\xi))$ is given by the $\mathfrak{G}(K_2)$-action on $V(K_2)$. Assuming that $r : K_1 \to K_2$ is surjective (so that $\mathfrak{G}_r : \mathfrak{G}(K_1) \to \mathfrak{G}(K_2)$ is surjective) we deduce:

(a) *If E is a subset of $V(K_1)$ which is stable under the $\mathfrak{G}(K_1)$-action on $V(K_1)$, then the subset $V_r(E)$ of $V(K_2)$ is stable under the $\mathfrak{G}(K_2)$-action on $V(K_2)$.*

1.7. In the remainder of this section we assume that $\lambda \in \mathcal{X}^{++}$. Then $L \mapsto G_L$ is an isomorphism $\pi : P^\bullet \xrightarrow{\sim} \mathcal{B}$ and

(a) π restricts to a bijection $\pi_{\geq 0} : P^\bullet_{\geq 0} \xrightarrow{\sim} \mathcal{B}_{\geq 0}$.

See [5, 8.17].

1.8. Let Ω be the set of all open nonempty subsets of $\mathbf{C}$. Let X be an algebraic variety over $\mathbf{C}$. Let X_1 be the set of pairs (U, f_U) where $U \in \Omega$ and $f_U : U \to X$ is a morphism of algebraic varieties. We define an equivalence relation on X_1 in which $(U, f_U), (U', f_{U'})$ are equivalent if $f_U|_{U\cap U'} = f_{U'}|_{U\cap U'}$. Let $\tilde{X}$ be the set of equivalence classes. An element of $\tilde{X}$ is said to be a rational map $f : \mathbf{C} \triangleright X$. For $f \in \tilde{X}$ let Ω_f be the set of all $U \in \Omega$ such that f contains $(U, f_U) \in X_1$ for some f_U; we shall then write $f(t) = f_U(t)$ for $t \in U$. We shall identify any $x \in X$ with the constant map $f_x : \mathbf{C} \to X$ with image $\{x\}$; thus X can be identified with a subset of

$\tilde{X}$. If X' is another algebraic variety over $\mathbf{C}$ then we have $\widetilde{X \times X'} = \tilde{X} \times \tilde{X}'$ canonically. If $F : X \to X'$ is a morphism then there is an induced map $\tilde{F} : \tilde{X} \to \tilde{X}'$; to $f : \mathbf{C} \rhd X$ it attaches $f' : \mathbf{C} \rhd X'$ where for some $U \in \Omega_f$ we have $f'(t) = F(f(t))$ for all $t \in U$. If H is an algebraic group over $\mathbf{C}$ then $\tilde{H}$ is a group with multiplication $\tilde{H} \times \tilde{H} = \widetilde{H \times H} \to \tilde{H}$ induced by the multiplication map $H \times H \to H$. Note that H is a subgroup of $\tilde{H}$. In particular, the group $\tilde{G}$ is defined. Also, the additive group $\tilde{\mathbf{C}}$ and the multiplicative group $\widetilde{\mathbf{C}^*}$ are defined. Also $\tilde{\mathcal{B}}$ is defined.

1.9. Let X be an algebraic variety over $\mathbf{C}$ with a given subset $X_{\geq 0}$. We define a subset $\tilde{X}_{\geq 0}$ of $\tilde{X}$ as follows: $\tilde{X}_{\geq 0}$ is the set of all $f \in \tilde{X}$ such that for some $U \in \Omega_f$ and some $\epsilon \in \mathbf{R}_{>0}$ we have $(0, \epsilon) \subset U$ and $f(t) \in X_{\geq 0}$ for all $t \in (0, \epsilon)$. (In particular, $\tilde{G}_{\geq 0}$ is defined in terms of $G, G_{\geq 0}$ and $\tilde{\mathcal{B}}_{\geq 0}$ is defined in terms of $\mathcal{B}, \mathcal{B}_{\geq 0}$.) If X' is another algebraic variety over $\mathbf{C}$ with a given subset $X'_{\geq 0}$, then $X \times X'$ with its subset $(X \times X')_{\geq 0} = X_{\geq 0} \times X'_{\geq 0}$ gives rise as above to the set $\widetilde{X \times X'}_{\geq 0}$ which can be identified with $\tilde{X}_{\geq 0} \times \tilde{X}'_{\geq 0}$. If $F : X \to X'$ is a morphism such that $F(X_{\geq 0}) \subset X'_{\geq 0}$, then the induced map $\tilde{F} : \tilde{X} \to \tilde{X}'$ carries $\tilde{X}_{\geq 0}$ into $\tilde{X}'_{\geq 0}$ hence it restricts to a map $\tilde{F}_{\geq 0} : \tilde{X}_{\geq 0} \to \tilde{X}'_{\geq 0}$. From the definitions we see that:

(a) *if $\tilde{F}$ is an isomorphism of $\tilde{X}$ onto an open subset of $\tilde{X}'$ and F carries $\tilde{X}_{\geq 0}$ bijectively onto $\tilde{X}'_{\geq 0}$, then the map $\tilde{F}_{\geq 0}$ is a bijection.*

Now the multiplication $G \times G \to G$ carries $G_{\geq 0} \times G_{\geq 0}$ to $G_{\geq 0}$ hence it induces a map $\tilde{G}_{\geq 0} \times \tilde{G}_{\geq 0} \to \tilde{G}_{\geq 0}$ which makes $\tilde{G}_{\geq 0}$ into a monoid; the conjugation action $G \times \mathcal{B} \to \mathcal{B}$ carries $G_{\geq 0} \times \mathcal{B}_{\geq 0}$ to $\mathcal{B}_{\geq 0}$ hence it induces a map $\tilde{G}_{\geq 0} \times \tilde{\mathcal{B}}_{\geq 0} \to \tilde{\mathcal{B}}_{\geq 0}$ which define an action of the monoid $\tilde{G}_{\geq 0}$ on $\tilde{\mathcal{B}}_{\geq 0}$. We define $\tilde{\mathbf{C}}^*{}_{\geq 0}$ in terms of $\mathbf{C}^*$ and its subset $\mathbf{C}^*_{\geq 0} := \mathbf{R}_{>0}$. The multiplication on $\mathbf{C}^*$ preserves $\mathbf{C}^*_{\geq 0}$ hence it induces a map $\tilde{\mathbf{C}}^*{}_{\geq 0} \times \tilde{\mathbf{C}}^*{}_{\geq 0} \to \tilde{\mathbf{C}}^*{}_{\geq 0}$ which makes $\tilde{\mathbf{C}}^*{}_{\geq 0}$ into an abelian group. We define $\tilde{\mathbf{C}}_{\geq 0}$ in terms of $\mathbf{C}$ and its subset $\mathbf{C}_{\geq 0} := \mathbf{R}_{\geq 0}$. The addition on $\mathbf{C}$ preserves $\mathbf{C}_{\geq 0}$ hence it induces a map $\tilde{\mathbf{C}}_{\geq 0} \times \tilde{\mathbf{C}}_{\geq 0} \to \tilde{\mathbf{C}}_{\geq 0}$ which makes $\tilde{\mathbf{C}}_{\geq 0}$ into an abelian monoid. The imbedding $\mathbf{C}^* \subset \mathbf{C}$ induces an imbedding $\tilde{\mathbf{C}}^*{}_{\geq 0} \to \tilde{\mathbf{C}}_{\geq 0}$; the monoid operation on $\tilde{\mathbf{C}}_{\geq 0}$ preserves the subset $\tilde{\mathbf{C}}^*{}_{\geq 0}$ and makes $\tilde{\mathbf{C}}^*{}_{\geq 0}$ into an abelian monoid. This, together with the multiplication on $\tilde{\mathbf{C}}^*{}_{\geq 0}$ makes $\tilde{\mathbf{C}}^*{}_{\geq 0}$ into a semifield. From the definitions we see that this semifield is the same as K in 0.1(i) and that $\tilde{G}_{\geq 0}$ is the monoid associated to G and K in [5, 2.2] (which is the same as $\mathfrak{G}(K)$). We define $\mathcal{B}(K)$ to be $\tilde{\mathcal{B}}_{\geq 0}$ with the action of $\tilde{G}_{\geq 0} = \mathfrak{G}(K)$ described above. This achieves what was stated in 0.1 for K as in 0.1(i).

1.10. In the remainder of this section K will denote the semifield in 0.1(i) and we assume that $\lambda \in \mathcal{X}^{++}$. We associate $\tilde{P}_{\geq 0} = {}^\lambda\tilde{P}_{\geq 0}$ to P and its subset $P_{\geq 0}$ as in 1.9. We associate $\tilde{P}^\bullet_{\geq 0} = {}^\lambda\tilde{P}^\bullet_{\geq 0}$ to $P^\bullet$ and its subset $P^\bullet_{\geq 0}$ as in 1.9. We write $P^\bullet(K) = {}^\lambda P^\bullet(K) = \tilde{P}^\bullet_{\geq 0}$.

We associate $\tilde{V}_{\geq 0} = {}^\lambda\tilde{V}_{\geq 0}$ to V and its subset $V_{\geq 0}$ as in 1.9. We can identify $\tilde{V}_{\geq 0} = V(K)$ (see 1.4). We associate $\tilde{V}^\bullet_{\geq 0} = {}^\lambda\tilde{V}^\bullet_{\geq 0}$ to $V^\bullet$ and its subset $V^\bullet_{\geq 0}$ as in 1.9. We write $V^\bullet(K) = {}^\lambda V^\bullet(K) = \tilde{V}^\bullet_{\geq 0}$. We have $V^\bullet(K) \subset \tilde{V}_{\geq 0}$.

The obvious map $a' : V - 0 \to P$ restricts to a (surjective) map $a'_{\geq 0} : V_{\geq 0} - 0 \to P_{\geq 0}$ and defines a map $\tilde{a}'_{\geq 0} : \tilde{V}_{\geq 0} - 0 \to \tilde{P}_{\geq 0}$. The scalar multiplication $\mathbf{C}^* \times (V - 0) \to V - 0$ carries $\mathbf{C}^*_{\geq 0} \times (V_{\geq 0} - 0)$ to $V_{\geq 0} - 0$ hence it induces a map $\widetilde{\mathbf{C}^*}_{\geq 0} \times (\tilde{V}_{\geq 0} - 0) \to \tilde{V}_{\geq 0} - 0$ which is a (free) action of the group $K = \widetilde{\mathbf{C}^*}_{\geq 0}$ on $\tilde{V}_{\geq 0} - 0 = V(K) - 0$. From the definitions we see that $\tilde{a}'_{\geq 0}$ is surjective and it induces a bijection $(V(K) - 0)/K \xrightarrow{\sim} \tilde{P}_{\geq 0}$. Thus we have $\tilde{P}_{\geq 0} = P(K)$ (notation of 1.4). Note that $P^\bullet(K) \subset P(K)$.

The obvious map $a : V^\bullet - 0 \to P^\bullet$ restricts to a (surjective) map $a_{\geq 0} : V^\bullet_{\geq 0} - 0 \to P^\bullet_{\geq 0}$ and it defines a map $\tilde{a}_{\geq 0} : V^\bullet(K) = \tilde{V}^\bullet_{\geq 0} - 0 \to \tilde{P}^\bullet_{\geq 0} = P^\bullet(K)$. The (free) K-action on $\tilde{V}_{\geq 0} - 0$ considered above restricts to a (free) K-action on $V^\bullet(K) - 0 = \tilde{V}^\bullet_{\geq 0} - 0$. From the definitions we see that $\tilde{a}_{\geq 0}$ is constant on any orbit of this action. We show:

(a) *The map $\tilde{a}_{\geq 0}$ is surjective. It induces a bijection $(V^\bullet(K) - 0)/K \xrightarrow{\sim} P^\bullet(K)$.*

Let $f \in \tilde{P}^\bullet_{\geq 0}$. We can find $U \in \Omega_f$, $\epsilon \in \mathbf{R}_{>0}$ such that $(0, \epsilon) \subset U$ and $f(t) \in P^\bullet_{\geq 0}$ for $t \in (0, \epsilon)$. Using the surjectivity of $a_{\geq 0}$ we see that for $t \in (0, \epsilon)$ we have $f(t) = a(x_t)$ where $t \mapsto x_t$ is a function $(0, \epsilon) \to V^\bullet_{\geq 0} - 0$. We can assume that there exists $B \in \mathcal{B}(\mathbf{R})$ such that $\pi(f(t))$ is opposed to B for all $t \in U$. Let $\mathcal{O} = \{B_1 \in \mathcal{B}; B_1 \text{ opposed to } B\}$; thus we have $\pi(f(t)) \in \mathcal{O}$ for all $t \in U$. Let $B' \in \mathcal{O} \cap \mathcal{B}(\mathbf{R})$ and let $\xi' \in V_{\mathbf{R}} - 0$ be such that $\pi(\mathbf{C}\xi') = B'$. Let U_B be the unipotent radical of B. Then $U_B \to \mathcal{O}$, $u \mapsto uB'u^{-1}$ is an isomorphism. Hence there is a unique morphism $\zeta : \mathcal{O} \to V^\bullet - 0$ such that $\zeta(uB'u^{-1}) = u\xi'$ for any $u \in U_B$. From the definitions we have $\zeta(\mathcal{O} \cap \mathcal{B}(\mathbf{R})) \subset (V_{\mathbf{R}} \cap V^\bullet) - 0$. We define $f' : U \to V^\bullet - 0$ by $f'(t) = \zeta(\pi(f(t)))$. We can view f' as an element of $\tilde{V}^\bullet - 0$ such that $\tilde{a}(f') = f$. Since $\pi(f(t)) \in \mathcal{B}(\mathbf{R})$, we have

$f'(t) \in (V_{\mathbf{R}} \cap V^\bullet) - 0$ for $t \in (0, \epsilon)$. For such t we have $a(f'(t)) = f(t) = a(x_t)$ hence $f'(t) = z_t x_t$ where $t \mapsto z_t$ is a (possibly discontinuous) function $(0, \epsilon) \to \mathbf{R} - 0$. Since $x_t \in V_{\ge 0} - 0$ and $\mathbf{R}_{>0}(V_{\ge 0} - 0) = V_{\ge 0} - 0$, we see that for $t \in (0, \epsilon)$ we have $f'(t) \in (V_{\ge 0} - 0) \cup (-1)(V_{\ge 0} - 0)$. Since $(0, \epsilon)$ is connected and f' is continuous (in the standard topology) we see that $f'(0, \epsilon)$ is contained in one of the connected components of $(V_{\ge 0} - 0) \cup (-1)(V_{\ge 0} - 0)$ that is, in either $V_{\ge 0} - 0$ or in $(-1)(V_{\ge 0} - 0)$. Thus there exists $s \in \{1, -1\}$ such that $sf'(0, \epsilon) \subset V_{\ge 0} - 0$ hence also $sf'(0, \epsilon) \subset V^\bullet_{\ge 0} - 0$. We define $f'' : U \to V^\bullet - 0$ by $f''(t) = sf'(t)$. We can view f'' as an element of $\tilde{V}^\bullet_{\ge 0} - 0$ such that $\tilde{a}_{\ge 0}(f') = f$. This proves that $\tilde{a}_{\ge 0}$ is surjective. The remaining statement of (a) is immediate.

Since $P^\bullet$ and its subset $P^\bullet_{\ge 0}$ can be identified with $\mathcal{B}$ and its subset $\mathcal{B}_{\ge 0}$ (see 1.7(a)), we see that we may identify $P^\bullet(K) = \mathcal{B}(K)$. The action of $\mathfrak{G}(K)$ on $P^\bullet(K)$ induced from that on $V^\bullet(K) - 0$ is the same as the previous action of $\mathfrak{G}(K)$, see [8, 2.13(d)]. This gives a second incarnation of $\mathcal{B}(K)$.

1.11. Let $\mathbf{Z}$ be the semifield in 0.1(ii). Following [5], we define a (surjective) semifield homomorphism $r : K \to \mathbf{Z}$ by $r(x^e f_1/f_2) = e$ (notation of 0.1). Now r induces a surjective map $V_r : V(K) \to V(\mathbf{Z})$ as in 1.6. Let $V^\bullet(\mathbf{Z}) = {}^\lambda V^\bullet(\mathbf{Z}) \subset V(\mathbf{Z})$ be the image under V_r of the subset $V^\bullet(K)$ of $V(K)$. Then $V^\bullet(\mathbf{Z}) - \underline{\circ} = V_r(V^\bullet(K) - 0)$.

The $\mathbf{Z}$-action on $V(\mathbf{Z}) - \underline{\circ}$ in 1.4 leaves $V^\bullet(\mathbf{Z}) - \underline{\circ}$ stable. (We use the K-action on $V^\bullet(K) - 0$.) Let $P^\bullet(\mathbf{Z}) = {}^\lambda P^\bullet(\mathbf{Z})$ be the set of orbits of this action. We have $P^\bullet(\mathbf{Z}) \subset P(\mathbf{Z})$ (notation of 1.4). From 1.6(a) we see that $V^\bullet(\mathbf{Z}) - \underline{\circ}$ is stable under the $\mathfrak{G}(\mathbf{Z})$-action on $V(\mathbf{Z})$ in 1.6. Since the $\mathfrak{G}(\mathbf{Z})$-action commutes with scalar multiplication by $\mathbf{Z}$ it follows that the $\mathfrak{G}(\mathbf{Z})$-action on $V(\mathbf{Z}) - \underline{\circ}$ and $V^\bullet(\mathbf{Z}) - \underline{\circ}$ induces a $\mathfrak{G}(\mathbf{Z})$-action on $P(\mathbf{Z})$ and $P^\bullet(\mathbf{Z})$.

1.12. We set $\mathcal{B}(\mathbf{Z}) = {}^\lambda P^\bullet(\mathbf{Z})$. This achieves what was stated in 0.1 for the semifield $\mathbf{Z}$. This definition of $\mathcal{B}(\mathbf{Z})$ depends on the choice of $\lambda \in \mathcal{X}^{++}$. In §4 we will show that $\mathcal{B}(\mathbf{Z})$ is independent of this choice up to a canonical bijection. (Alternatively, if one wants a definition without such a choice one could take λ such that $\langle i, \lambda \rangle = 1$ for all $i \in I$.)

2. Preparatory Results

2.1. We preserve the setup of 1.4. As shown in [4, 5.3, 4.2], for $w \in W$ and $\mathbf{i} = (i_1, i_2, \dots, i_m) \in \mathcal{I}_w$, the subspace of V generated by the vectors

$$f_{i_1}^{(c_1)} f_{i_2}^{(c_2)} \dots f_{i_m}^{(c_m)} \xi^+$$

for various $c_1, c_2, \dots, c_m$ in $\mathbf{N}$ is independent of $\mathbf{i}$ (we denote it by V^w) and $\beta^w := \beta \cap V^w$ is a basis of it. Let $V'^{\mathbf{i}}$ be the subspace of V generated by the vectors

$$e_{i_m}^{(d_m)} e_{i_{m-1}}^{(d_{m-1})} \dots e_{i_1}^{(d_1)} b_w$$

for various $d_1, d_2, \dots, d_m$ in $\mathbf{N}$, where

$$\begin{aligned} b_w &= \dot{w} \xi^+, \\ \dot{w} &= \dot{s}_{i_1} \dot{s}_{i_2} \dots \dot{s}_{i_m}. \end{aligned}$$

We show:

(a) $$V^w = V'^{\mathbf{i}}.$$

We show that $V^w \subset V'^{\mathbf{i}}$. We argue by induction on $m = |w|$. If $m = 0$, the result is obvious. Assume now that $m \geq 1$. Let $c_1, c_2, \dots, c_m$ be in $\mathbf{N}$. By the induction hypothesis,

(b) $$f_{i_1}^{(c_1)} f_{i_2}^{(c_2)} \dots f_{i_m}^{(c_m)} \xi^+$$

is a linear combination of vectors of form

$$f_{i_1}^{(c_1)} e_{i_m}^{(d_m)} e_{i_{m-1}}^{(d_{m-1})} \dots e_{i_2}^{(d_2)} b_{s_{i_1} w}$$

for various $d_2, \dots, d_m$ in $\mathbf{N}$. Using the known commutation relations between f_{i_1} and e_j we see that (b) is a linear combination of vectors of form

$$e_{i_m}^{(d_m)} e_{i_{m-1}}^{(d_{m-1})} \dots e_{i_2}^{(d_2)} f_{i_1}^{(c_1)} b_{s_{i_1} w}$$

for various $d_2, \dots, d_m$ in $\mathbf{N}$. It is then enough to show that

$$f_{i_1}^{(c_1)} b_{s_{i_1} w} = e_{i_1}^{(d_1)} \dot{s}_{i_1} b_{s_{i_1} w}$$

for some $d_1 \in \mathbf{N}$. This follows from the fact that

(c) $\quad e_{i_1} b_{s_{i_1} w} = 0$ *and* $b_{s_{i_1} w}$ *is in a weight space of* V.

Next we show that $V'^{\mathbf{i}} \subset V^w$. We argue by induction on $m = |w|$. If $m = 0$ the result is obvious. Assume now that $m \geq 1$. Since V^w is stable under the action of $e_i (i \in I)$, it is enough to show that $b_w \in V^w$. By the induction hypothesis, $b_{s_{i_1} w} \in V^{s_{i_1} w}$. Using (c), we see that for some $c_1 \in \mathbf{N}$ we have

$$b_w = \dot{s}_{i_1} b_{s_{i_1} w} = f_{i_1}^{(c_1)} b_{s_{i_1} w} \in f_{i_1}^{(c_1)} V^{s_{i_1} w} \subset V^w.$$

This completes the proof of (a).

From [3, 28.1.4] one can deduce that $b_w \in \beta$. From (a) we see that $b_w \in V^w$. It follows that

(d) $$b_w \in \beta^w.$$

2.2. For $v \leq w$ in W we set

$$\mathcal{B}_{v,w} = \{B \in \mathcal{B}, pos(B^+, B) = w, pos(B^-, B) = w_I v\}$$

(a locally closed subvariety of $\mathcal{B}$) and

$$(\mathcal{B}_{v,w})_{\geq 0} = \mathcal{B}_{\geq 0} \cap \mathcal{B}_{v,w}.$$

We have $\mathcal{B} = \sqcup_{v \leq w \text{ in } W} \mathcal{B}_{v,w}$, $\mathcal{B}_{\geq 0} = \sqcup_{v \leq w \text{ in } W} (\mathcal{B}_{v,w})_{\geq 0}$.

2.3. Recall that there is a unique isomorphism $\phi : G \to G$ such that $\phi(x_i(t)) = y_i(t)$, $\phi(y_i(t)) = x_i(t)$ for all $i \in I, t \in \mathbf{C}$ and $\phi(g) = g^{-1}$ for all $g \in T$. This carries Borel subgroups to Borel subgroups hence induces an isomorphism $\phi : \mathcal{B} \to \mathcal{B}$ such that $\phi(B^+) = B^-$, $\phi(B^-) = B^+$. For $i \in I$ we have $\phi(\dot{s}_i) = \dot{s}_i^{-1}$. Hence ϕ induces the identity map on W. For $v \leq w$ in W we have $ww_I \leq vw_I$; moreover,

(a) ϕ *defines an isomorphism* $\mathcal{B}_{ww_I, vw_I} \xrightarrow{\sim} \mathcal{B}_{v,w}$.

(See [9, 1.4(a)] From the definition we have

(b) $\phi(G_{\geq 0}) = G_{\geq 0}$.

From [5, 8.7] it follows that

(c) $\phi(\mathcal{B}_{\geq 0}) = \mathcal{B}_{\geq 0}$.

From (a), (c) we deduce:

(d) *ϕ defines a bijection* $(\mathcal{B}_{ww_I,vw_I})_{\geq 0} \xrightarrow{\sim} (\mathcal{B}_{v,w})_{\geq 0}$.

By [2, §3] there is a unique linear isomorphism $\phi : V \to V$ such that $\phi(g\xi) = \phi(g)\phi(\xi)$ for all $g \in G$, $\xi \in V$ and such that $\phi(\xi^+) = \xi^-$; we have $\phi(\beta) = \beta$ and $\phi^2(\xi) = \xi$ for all $\xi \in V$.

2.4. Assume now that $\lambda \in \mathcal{X}^{++}$. Let $B \in \mathcal{B}_{v,w}$ and let $L \in P^\bullet$ be such that $\pi(L) = B$. Let $\xi \in L - 0$, $b \in \beta$. We show:

(a) $\xi_b \neq 0 \implies b \in \beta^w \cap \phi(\beta^{vw_I})$.

We have $B = gB^+g^{-1}$ for some $g \in B^+\dot{w}B^+$. Then $\xi = cg\xi^+$ for some $c \in \mathbf{C}^*$. We write $g = g'\dot{w}g''$ with $g' \in U^+, g'' \in B^+$. We have $\xi = c'g'\dot{w}\xi^+ = c'g'b_w$ where $c' \in \mathbf{C}^*$. By 2.1(d) we have $b_w \in \beta^w$. Moreover, V^w is stable by the action of U^+; we see that $\xi \in V^w$. Since $\xi_b \neq 0$ we have $b \in \beta^w$. Let $B' = \phi(B)$. We have $B' \in \mathcal{B}_{ww_I,vw_I}$ (see 2.3(a)). Let $L' = \phi(L) \in P^\bullet$ and let $\xi' = \phi(\xi) \in L' - 0$, $b' = \phi(b) \in \beta$. We have $\xi'_{b'} \neq 0$. Applying the first part of the proof with B, L, ξ, v, w, b replaced by B', L', ξ', v', w', b' we obtain $b' \in \beta^{vw_I}$. Hence $b \in \phi(\beta^{vw_I})$. Thus, $b \in \beta^w \cap \phi(\beta^{vw_I})$, as required.

2.5. We return to the setup of 1.4. For $i \in I$ we set

$$V^{e_i} = \{\xi \in V; e_i(\xi) = 0\} = \Big\{\xi \in V; \sum_{b\in\beta} \xi_b c_{b,b',i,1} = 0 \text{ for all } b' \in \beta\Big\},$$
$$V^{f_i} = \{\xi \in V; f_i(\xi) = 0\} = \Big\{\xi \in V; \sum_{b\in\beta} \xi_b d_{b,b',i,1} = 0 \text{ for all } b' \in \beta\Big\}.$$

If $\xi \in V_{\geq 0}$, the condition that $\sum_{b\in\beta} \xi_b c_{b,b',i,1} = 0$ is equivalent to the condition that $\xi_b c_{b,b',i,1} = 0$ for any b, b' in β. Thus we have

$$V_{\geq 0} \cap V^{e_i} = \Big\{\xi \in V_{\geq 0}; \xi = \sum_{b\in\beta^{e_i}} \xi_b b\Big\}$$

where $\beta^{e_i} = \{b \in \beta; c_{b,b',i,1} = 0 \text{ for any } b' \in \beta\}$. Similarly, we have

$$V_{\geq 0} \cap V^{f_i} = \Big\{\xi \in V_{\geq 0}; \xi = \sum_{b\in\beta^{f_i}} \xi_b b\Big\}$$

where $\beta^{f_i} = \{b \in \beta; d_{b,b',i,1} = 0 \text{ for any } b' \in \beta\}$.

Now the action of $\dot{s}_i$ on V defines an isomorphism $\mathcal{T}_i : V^{e_i} \to V^{f_i}$. If $b \in \beta^{e_i}$ we have $\mathcal{T}_i(b) = f_i^{(\langle i,\nu_b \rangle)} b = \sum_{b' \in \beta} d_{b,b',i,\langle i,\nu_b \rangle} b'$; in particular, we have $\mathcal{T}_i(b) \in V_{\geq 0} \cap V^{f_i}$. Thus $\mathcal{T}_i$ restricts to a map $\mathcal{T}'_i : V_{\geq 0} \cap V^{e_i} \to V_{\geq 0} \cap V^{f_i}$. Similarly the action of $\dot{s}_i^{-1}$ restricts to a map $\mathcal{T}''_i : V_{\geq 0} \cap V^{f_i} \to V_{\geq 0} \cap V^{e_i}$. This is clearly the inverse of $\mathcal{T}'_i$.

2.6. Now let K be a semifield. Let

$$V(K)^{e_i} = \Big\{ \sum_{b \in \beta} \xi_b b; \xi_b \in K^! \text{ if } b \in \beta^{e_i}, \xi_b = \circ \text{ if } b \in \beta - \beta^{e_i} \Big\},$$
$$V(K)^{f_i} = \Big\{ \sum_{b \in \beta} \xi_b b; \xi_b \in K^! \text{ if } b \in \beta^{f_i}, \xi_b = \circ \text{ if } b \in \beta - \beta^{f_i} \Big\}.$$

We define $\mathcal{T}_{i,K} : V(K) \to V(K)$ by

$$\sum_{b \in \beta} \xi_b b \mapsto \sum_{b' \in \beta} (\sum_{b \in \beta} d_{b,b',i,\langle i,\nu_b \rangle} \xi_b) b'$$

(notation of 1.4). From the results in 2.5 one can deduce that

(a) $\mathcal{T}_{i,K}$ *restricts to a bijection* $\mathcal{T}'_{i,K} : V(K)^{e_i} \xrightarrow{\sim} V(K)^{f_i}$.

2.7. Let K be a semifield. We define an involution $\phi : V(K) \to V(K)$ by $\phi(\sum_{b \in \beta} \xi_b b) = \sum_{b \in \beta} \xi_{\phi(b)} b$. (Here $\xi_b \in K^!$; we use that $\phi(\beta) = \beta$.) This restricts to an involution $V(K) - \underline{\circ} \to V(K) - \underline{\circ}$ which induces an involution $P(K) \to P(K)$ denoted again by ϕ.

3. Parametrizations

3.1. In this section K denotes the semifield in 0.1(i). For $v \leq w$ in W we define $\mathcal{B}_{v,w}(K) = \widetilde{\mathcal{B}_{v,w}}_{\geq 0}$ as in 1.9 in terms of $\mathcal{B}_{v,w}$ and its subset $(\mathcal{B}_{v,w})_{\geq 0}$. We have

$$\mathcal{B}(K) = \sqcup_{v \leq w \text{ in } W} \mathcal{B}_{v,w}(K).$$

3.2. We preserve the setup of 1.4. We now fix $v \leq w$ in W and $\mathbf{i} = (i_1, i_2, \ldots, i_m) \in \mathcal{I}_w$. According to [10], there is a unique sequence $q_1, q_2, \ldots,$

q_m with $q_k \in \{s_{i_k}, 1\}$ for $k \in [1,m]$, $q_1q_2 \dots q_m = v$ and such that $q_1 \le q_1q_2 \le \dots \le q_1q_2\dots q_m$ and $q_1 \le q_1s_{i_2}, q_1q_2 \le q_1q_2s_{i_3}, \dots, q_1q_2\dots q_{m-1} \le q_1q_2\dots q_{m-1}s_{i_m}$. Let $[1,m]' = \{k \in [1,m]; q_k = 1\}$, $[1,m]'' = \{k \in [1,m]; q_k = s_{i_k}\}$. Let A be the set of maps $h : [1,m]' \to \mathbf{C}^*$; this is naturally an algebraic variety over $\mathbf{C}$. Let $A_{\ge 0}$ be the subset of A consisting of maps $h : [1,m]' \to \mathbf{R}_{>0}$. Following [10], we define a morphism $\sigma : A \to G$ by $h \mapsto g(h)_1 g(h)_2 \dots g(h)_m$ where

(a) *$g(h)_k = y_{i_k}(h(k))$ if $k \in [1,m]'$ and $g(h)_k = \dot{s}_{i_k}$ if $k \in [1,m]''$.*

We show:

(b) *If $h \in A_{\ge 0}$, then $\sigma(h)\xi^+ \in V^w$, so that $\sigma(h)$ is a linear combination of vectors $b \in \beta^w$. Moreover, $(\sigma(h)\xi^+)_{b_w} \ne 0$.*

From the properties of Bruhat decomposition, for any $h \in A_{\ge 0}$ we have $\sigma(h) \in B^+\dot{w}B^+$, so that $\sigma(h)\xi^+ = cu\dot{w}\xi^+ = cub_w$ where $c \in \mathbf{C}^+$, $u \in U^+$. Since $b_w \in V^w$ and V^w is stable under the action of U^+, it follows that $cu\dot{w}\xi^+ \in V^w$. More precisely, $ub_w = b_w$ plus a linear combination of elements $b \in \beta$ of weight other than that of b_w. This proves (b).

We show:

(c) *Let $h \in A_{\ge 0}$. Assume that $i \in I$ is such that $|s_iw| > |w|$ and that $b \in \beta$ is such that $(\sigma(h)\xi^+)_b \ne 0$. Then $\nu_b \ne \nu_{b_w} + i'$.*

Since $|s_iw| > |w|$ we have $e_ib_w = 0$. We write $\sigma(h)x^+ = cub_w$ with c, u as in the proof of (b). Now ub_w is a linear combination of vectors of the form $e_{j_1}e_{j_2}\dots e_{j_k}b_w$ with $j_t \in I$. Such a vector is in a weight space $V(\nu)$ with $\nu = \nu_{b_w} + j'_1 + j'_2 + \dots + j'_k$. If $j'_1 + j'_2 + \dots + j'_k = i'$ then $k = 1$ and $j_1 = i$. But in this case we have $e_{j_1}e_{j_2}\dots e_{j_k}b_w = e_ib_w = 0$. The result follows.

3.3. Let $h \in A_{\ge 0}$. Let $k \in [1,m]''$. The following result appears in the proof of [10, 11.9].

(a) *We have $(g(h)_{k+1}g(h)_{k+2}\dots g(h)_m)^{-1}x_{i_k}(a)g(h)_{k+1}g(h)_{k+2}\dots g(h)_m \in U^+$.*

From (a) it follows that for $\xi \in V$ we have

$$e_{i_k}(g(h)_{k+1}g(h)_{k+2}\dots g(h)_m\xi) = g(h)_{k+1}g(h)_{k+2}\dots g(h)_m(e'\xi)$$

where $e' : V \to V$ is a linear combination of products of one or more factors $e_j, j \in I$. When $\xi = \xi^+$ we have $e'\xi = 0$ hence $e_{i_k}(g(h)_{k+1}g(h)_{k+2}\dots g(h)_m\xi^+) = 0$. We can write uniquely

$$g(h)_{k+1}g(h)_{k+2}\dots g(h)_m\xi^+ = \sum_{\nu\in\mathcal{X}}(g(h)_{k+1}g(h)_{k+2}\dots g(h)_m\xi^+)_\nu$$

with $(g(h)_{k+1}g(h)_{k+2}\dots g(h)_m\xi^+)_\nu \in V_\nu$. We have

$$\sum_{\nu\in\mathcal{X}} e_{i_k}((g(h)_{k+1}g(h)_{k+2}\dots g(h)_m\xi^+)_\nu) = 0.$$

Since the elements $e_{i_k}((g(h)_{k+1}g(h)_{k+2}\dots g(h)_m\xi^+)_\nu)$ (for various $\nu \in \mathcal{X}$) are in distinct weight spaces, it follows that $e_{i_k}((g(h)_{k+1}g(h)_{k+2}\dots g(h)_m\xi^+)_\nu) = 0$ for any $\nu \in \mathcal{X}$. If $\xi \in V_\nu$ satisfies $e_{i_k}\xi = 0$, then

(b) $\dot{s}_{i_k}\xi = f_{i_k}^{(\langle i_k,\nu\rangle)}\xi.$

(If $\langle i_k, \nu\rangle < 0$ then $\xi = 0$ so that both sides of (b) are 0.) We deduce

(c)
$$\begin{aligned}&g(h)_k((g(h)_{k+1}g(h)_{k+2}\dots g(h)_m\xi^+)_\nu)\\ &\quad= f_{i_k}^{(\langle i_k,\nu\rangle)}((g(h)_{k+1}g(h)_{k+2}\dots g(h)_m\xi^+)_\nu)\end{aligned}$$

for any $\nu \in \mathcal{X}$.

3.4. Let $h \in A_{\ge 0}$. For any $k \in [1, m]$ we set $[k, m]' = [k, m] \cap [1, m]'$, $[k, m]'' = [k, m] \cap [1, m]''$. Let $\mathcal{E}_{\ge k}$ be the set of all maps $\chi : [k, m]' \to \mathbf{N}$. (If $[k, m]' = \emptyset$, $\mathcal{E}_{\ge k}$ consists of a single element.) For $\chi \in \mathcal{E}_{\ge k}$ and $k' \in [k, m]$ let $\chi_{\ge k'}$ be the restriction of χ to $[k', m]'$.

We now define an integer $c(k, \chi)$ for any $k \in [1, m]''$ and any $\chi \in \mathcal{E}_{\ge k}$ by descending induction on k. We can assume that $c(k', \chi')$ is defined for any $k' \in [k + 1, m]''$ and any $\chi' \in \mathcal{E}_{\ge k'}$. We set $c_{k,\chi} = \langle i_k, \nu\rangle$ where

(a)
$$\nu = \lambda - \sum_{\kappa\in[k+1,m]'}\chi(\kappa)i'_\kappa - \sum_{\kappa\in[k+1,m]'';c(\kappa,\chi_{\ge\kappa})\ge 0} c(\kappa, \chi_{\ge\kappa})i'_k \in \mathcal{X}.$$

This completes the inductive definition of the integers $c(k, \chi)$.

Next we define for any $k \in [1,m]$ and any $\chi \in \mathcal{E}_{\geq k}$ an element $\mathcal{J}_{k,\chi} \in V$ by

$$\mathcal{J}_{k,\chi} = g(h)^{\chi}_{k} g(h)^{\chi}_{k+1} \dots g(h)^{\chi}_{m} \xi^{+}$$

where

$$\begin{aligned} g(h)^{\chi}_{\kappa} &= h(\kappa)^{\chi(\kappa)} f_{i_\kappa}^{(\chi(\kappa))} && \text{if } \kappa \in [k,m]', \\ g(h)^{\chi}_{\kappa} &= f_{i_\kappa}^{(c(\kappa,\chi|_{\geq\kappa})} && \text{if } \kappa \in [k,m]''. \end{aligned}$$

For $k \in [1,m]$ we show:

(b) $$g(h)_k g(h)_{k+1} \dots g(h)_m \xi^{+} = \sum_{\chi \in \mathcal{E}_{\geq k}} \mathcal{J}_{k,\chi}.$$

We argue by descending induction on k. Assume first that $k = m$. If $k \in [1,m]'$ then

$$g(h)_k \xi^{+} = \sum_{n \geq 0} h(k)^n f_{i_\kappa}^{(n)} \xi^{+} = \sum_{\chi \in \mathcal{E}_{\geq k}} \mathcal{J}_{k,\chi},$$

as required. If $k \in [1,m]''$, then $g(h)_k \xi^{+} = \dot{s}_{i_k} \xi^{+} = f_{i_k}^{(\langle i_k, \lambda \rangle)} \xi^{+}$, see 3.3(b).

Next we assume that $k < m$ and that (b) holds for k replaced by $k+1$. Let $\chi' = \chi_{\geq k+1}$. By the induction hypothesis, the left hand side of (b) is equal to

(c) $$g(h)_k \sum_{\chi \in \mathcal{E}_{\geq k+1}} \mathcal{J}_{k+1,\chi}.$$

If $k \in [1,m]'$, then clearly (c) is equal to the right hand side of (b). If $k \in [1,m]''$, then from the induction hypothesis we see that for any $\nu \in \mathcal{X}$ we have

$$(g(h)_{k+1} \dots g(h)_m \xi^{+})_{\nu} = \sum_{\chi \in \mathcal{E}_{\geq k+1}} (\mathcal{J}_{k+1,\chi})_{\nu} = \sum_{\chi \in \mathcal{E}_{\geq k+1;\nu}} \mathcal{J}_{k+1,\chi}$$

where $\mathcal{E}_{\geq k+1;\nu}$ is the set of all $\chi \in \mathcal{E}_{\geq k+1}$ such that

$$\nu = \lambda - \sum_{\kappa \in [k+1,m]'} \chi(\kappa) i'_{\kappa} - \sum_{\kappa \in [k+1,m]'', c(\kappa,\chi_{\geq\kappa}) \geq 0} c(\kappa, \chi_{\geq \kappa}) i'_{k}.$$

Using this and 3.3(c) we see that

$$g(h)_k g(h)_{k+1}\dots g(h)_m\xi^+ = \sum_{\nu\in\mathcal{X}} f_{i_k}^{(\langle i_k,\nu\rangle)}((g(h)_{k+1}g(h)_{k+2}\dots g(h)_m\xi^+)_\nu)$$
$$=\sum_{\nu\in\mathcal{X}} f_{i_k}^{(\langle i_k,\nu\rangle)} \sum_{\chi\in\mathcal{E}_{\geq k+1;\nu}} \mathcal{J}_{k+1,\chi} = \sum_{\chi\in\mathcal{E}_{\geq k}} f_{i_k}^{(c(k,\chi))}\mathcal{J}_{k+1,\chi|_{\geq k+1}} = \sum_{\chi\in\mathcal{E}_{\geq k}} \mathcal{J}_{k,\chi}.$$

This completes the inductive proof of (b).

In particular, we have

(d) $$g(h)_1 g(h)_2\dots g(h)_m\xi^+ = \sum_{\chi\in\mathcal{E}} \mathcal{J}_{1,\chi},$$

where $\mathcal{E}$ is the set of all maps $\chi:[1,m]'\to\mathbf{N}$. This shows that for any $b\in\beta$ there exists a polynomial P_b in the variables $x_k, k\in[1,m]'$ with coefficients in $\mathbf{N}$ such that the coefficient of b in $g(h)_1g(h)_2\dots g(h)_m\xi^+$ is obtained by substituting in P_b the variables x_k by $h(k)\in\mathbf{R}_{>0}$ for $k\in[1,m]', h\in A_{\geq 0}$. Each coefficient of this polynomial is a sum of products of expressions of the form $d_{b_1,b_2,i,n}\in\mathbf{N}$ (see 1.4); if one of these coefficients is $\neq 0$ then after the substitution $x_k\mapsto h(k)\in\mathbf{R}_{>0}$ we obtain an element in $\mathbf{R}_{>0}$ while if all these coefficients are 0 then the same substitution gives 0. Thus, there is a well defined subset $\beta_{v,\mathbf{i}}$ of β such that $P_b|_{x_k=h(k)}$ is in $\mathbf{R}_{>0}$ if $b\in\beta_{v,\mathbf{i}}$ and is 0 if $b\in\beta-\beta_{v,\mathbf{i}}$.

For a semifield K_1 we denote by $A(K_1)$ the set of maps $h:[1,m]'\to K_1$. For any $h\in K_1$ we can substitute in P_b the variables x_k by $h(k)\in K_1$ for $k\in[1,m]'$; the result is an element $P_{b,h,K_1}\in K_1^!$. Clearly, we have $P_{b,h,K_1}\in K_1$ if $b\in\beta_{v,\mathbf{i}}$ and $P_{b,h,K_1}=\circ$ if $b\in\beta-\beta_{v,\mathbf{i}}$.

From 3.2(b) we see that $b_w\in\beta_{v,\mathbf{i}}$.

We see that for a semifield K_1, $h\mapsto\sum_{b\in\beta}P_{b,h,K_1}b$ is a map $\theta_{K_1}: A(K_1)\to V(K_1)-\underline{\circ}$ and

(d) $$\theta_{K_1}(A(K_1))\subset\{\xi\in V(K_1);\mathrm{supp}(\xi)=\beta_{v,\mathbf{i}}\}.$$

(supp(ξ) as in 1.4.) Let $\omega_{K_1}: A(K_1)\to P(K_1)$ be the composition of θ_{K_1} with the obvious map $V(K_1)-\underline{\circ}\to P(K_1)$. From the definitions, if $K_1\to K_2$ is a homomorphism of semifields, then we have a commutative diagram

$$\begin{array}{ccc} A(K_1) & \xrightarrow{\omega_{K_1}} & P(K_1) \\ \downarrow & & \downarrow \\ A(K_2) & \xrightarrow{\omega_{K_2}} & P(K_2) \end{array}$$

where the vertical maps are induced by $K_1 \to K_2$.

3.5. In this subsection we assume that $m \geq 1$. We will consider two cases:

(I) $t_1 = s_{i_1}$,

(II) $t_1 = 1$.

In case (I) we set $(v', w') = (s_{i_1} v, s_{i_1} w)$, $\mathbf{i}' = (i_2, i_3, \dots, i_m) \in \mathcal{I}_{w'}$. We have $v' \leq w'$ and the analogue of the sequence $q_1, q_2, \dots, q_m$ in 3.2 for $(v', w', \mathbf{i}')$ is $q_2, q_3, \dots, q_m$.

In case (II) we set $(v', w') = (v, s_{i_1} w)$, $\mathbf{i}' = \mathbf{i}$. We have $v' \leq w'$ and the analogue of the sequence $q_1, q_2, \dots, q_m$ in 3.2 for $(v', w', \mathbf{i}')$ is $q_2, q_3, \dots, q_m$. For a semifield K_1 let $A'(K_1)$ be the set of maps $[2, m]' \to K_1$ (notation of 3.4) and let $\theta'_{K_1} : A'(K_1) \to V(K_1) - \underline{\circ}$, $\omega'_{K_1} : A'(K_1) \to P(K_1)$ be the analogues of $\theta_{K_1}, \omega_{K_1}$ in 3.4 when v, w is replaced by v', w'. From the definitions, in case (I), for $h \in A(K_1)$ we have

(a) $\theta_{K_1}(h) = \mathcal{T}_{i_1, K_1}(\theta'_{K_1}(h|_{[2,m]'})$

(notation of 2.6(a); in this case we have $\theta'_{K_1}(h|_{[2,m]'}) \in V(K_1)^{e_{i_1}}$ by 3.3(a) and the arguments following it); hence

(b) $\omega_{K_1}(h) = [\mathcal{T}_{i_1, K_1}](\omega'_{K_1}(h|_{[2,m]'})$

where $[\mathcal{T}_{i_1, K_1}]$ is the bijection $(V(K_1)^{e_{i_1}} - \underline{\circ})/K_1 \to (V(K_1)^{f_{i_1}} - \underline{\circ})/K_1$ induced by $\mathcal{T}_{i_1, K_1} : V(K_1)^{e_{i_1}} \to V(K_1)^{f_{i_1}}$ (the image of $\omega'_{K_1}(h|_{[2,m]'})$ is contained in $(V(K_1)^{e_{i_1}} - \underline{\circ})/K_1$).

From the definitions, in case (II), for $h \in A(K_1)$ we have

(c) $\theta_{K_1}(h) = (-i_1)^{h(i_1)}(\theta'_{K_1}(h|_{[2,m]'})$

(notation of 1.4).

3.6. In the remainder of this section we assume that $\lambda \in \mathcal{X}^{++}$. In the setup of 3.5, let $h, \tilde{h}$ be elements of $A(K_1)$. Let $\xi = \theta'_{K_1}(h|_{[2,m]'})$, $\tilde{\xi} = \theta'_{K_1}(\tilde{h}|_{[2,m]'})$ be such that $(-i_1)^{h(i_1)}(\xi)$, $(-i_1)^{\tilde{h}(i_1)}(\tilde{\xi})$ have the same image in $P(K)$. We show:

(a) *$h(i_1) = \tilde{h}(i_1)$ and $\xi, \tilde{\xi}$ have the same image in $P(K)$.*

By 3.2(a), (b) (for w' instead of w),

(b) *$b_{w'}$ appears in ξ with coefficient $c \in K_1$; if $b \in \beta$ appears in ξ with coefficient $\neq \circ$ then $\nu_b \neq \nu_{b_{w'}} + i'_1$.*

Similarly,

(c) *$b_{w'}$ appears in $\tilde{\xi}$ with coefficient $\tilde{c} \in K_1$; if $b \in \beta$ appears in $\tilde{\xi}$ with coefficient $\neq \circ$ then $\nu_b \neq \nu_{b_{w'}} + i'_1$.*

From our assumption on λ we have $b_{w'} \neq b_w = f_{i_0}^{(n)} b_{w'}$ and $f_{i_0}^{(1)} b_{w'} \neq \underline{\circ}$. By (b), (c) we have

$$(-i_1)^{h(i_1)}(\xi) = c\beta_{w'} + h(i_1) c f_{i_0}^{(1)} b_{w'} + K_1^!\text{-comb. of } b \in \beta \text{ of other weights},$$

$$(-i_1)^{\tilde{h}(i_1)}(\tilde{\xi}) = \tilde{c}\beta_{w'} + \tilde{c}\tilde{h}(i_1) f_{i_0}^{(1)} b_{w'} + K_1^!\text{-comb. of } b \in \beta \text{ of other weights}.$$

We deduce that for some $k \in K_1$ we have $\tilde{c} = kc$, $\tilde{c}\tilde{h}(i_1) = kch(i_1)$. It follows that $h(i_1) = \tilde{h}(i_1)$. Using this and our assumption, we see that for some $k \in K_1$ we have $(-i_1)^{h(i_1)}(\xi) = (-i_1)^{h(i_1)}(c\tilde{\xi})$. Using 1.4(a) we deduce $\xi = c\tilde{\xi}$. This proves (a).

3.7. In the setup of 3.4 we show:

(a) *$\omega_{K_1} : A(K_1) \to P(K_1)$ is injective.*

We argue by induction on m. If $m = 0$ there is nothing to prove. We now assume that $m \geq 1$. Let $\omega'_{K_1} : A'(K_1) \to P(K_1)$ be as in 3.5. By the induction hypothesis, ω'_{K_1} is injective. In case I (in 3.5), we use 3.5(b) and the bijectivity of $[\mathcal{T}_{i_1,K_1}]$ to deduce that ω_{K_1} is injective. In case II (in 3.5), we use 3.5(c) and 3.6(a) to deduce that ω_{K_1} is injective. This proves (a).

3.8. According to [10],

(a) *$h \mapsto \sigma(h)B^+\sigma(h)^{-1}$ defines an isomorphism τ from A to an open subvariety of $\mathcal{B}_{v,w}$ containing $(\mathcal{B}_{v,w})_{\geq 0}$ and τ restricts to a bijection $A_{\geq 0} \xrightarrow{\sim} (\mathcal{B}_{v,w})_{\geq 0}$.*

(The existence of a homeomorphism $\mathbf{R}_{>0}^{|w|-|v|} \xrightarrow{\sim} (\mathcal{B}_{v,w})_{\geq 0}$ was conjectured in [5].)

We define $\tilde{A}_{\geq 0}$ in terms A and its subset $A_{\geq 0}$ as in 1.9. Note that $\tilde{A}_{\geq 0}$ can be identified with the set of maps $h : [1, m]' \to K$ that is, with $A(K)$ (notation of 3.4). Now $\tau : A \to \mathcal{B}_{v,w}$ (see (a)) carries $A_{\geq 0}$ onto the subset $(\mathcal{B}_{v,w})_{\geq 0}$ of $\mathcal{B}_{v,w}$ hence it induces a map

(b) $A(K) = \tilde{A}_{\geq 0} \to \widetilde{\mathcal{B}_{v,w}}_{\geq 0}$ which is a bijection.

(We use (a) and 1.9(a)).

3.9. From the definition we deduce that we have canonically

(a) $\tilde{\mathcal{B}}_{\geq 0} = \sqcup_{v,w \text{ in } W, v \leq w} \widetilde{\mathcal{B}_{v,w}}_{\geq 0}$.

The left hand side is identified in 1.10 with $P^\bullet(K)$, a subspace of $P(K)$. Hence the subset $\widetilde{\mathcal{B}_{v,w}}_{\geq 0}$ of $\tilde{\mathcal{B}}_{\geq 0}$ can be viewed as a subset $P_{v,w}(K)$ of $P(K)$ and 3.8(b) defines a bijection of $A(K)$ onto $P_{v,w}(K)$. The composition of this bijection with the imbedding $P_{v,w}(K) \subset P(K)$ coincides with the map $\omega_K : A \to P(K)$ in 3.4. (This follows from definitions.)

Similarly, the composition of the imbeddings

$$(\mathcal{B}_{v,w})_{\geq 0} \subset \mathcal{B}_{\geq 0} = P^\bullet_{\geq 0} \subset P_{\geq 0} = P(\mathbf{R}_{>0})$$

(see 1.7(a)) can be identified via 3.8(a) with the imbedding $\omega_{\mathbf{R}_{>0}} : A_{\geq 0} \to P(\mathbf{R}_{>0})$ whose image is denoted by $P_{v,w}(\mathbf{R}_{>0})$.

Recall that $P^\bullet(\mathbf{Z})$ is the image of $P^\bullet(K)$ under the map $P(K) \to P(\mathbf{Z})$ induced by $r : K \to \mathbf{Z}$ (see 1.11). For $v \leq w$ in W let $P_{v,w}(\mathbf{Z})$ be the image of $P_{v,w}(K)$ under the map $P(K) \to P(\mathbf{Z})$. We have clearly $P^\bullet(\mathbf{Z}) = \cup_{v \leq w} P_{v,w}(\mathbf{Z})$. From the commutative diagram in 3.4 attached to $r : K \to \mathbf{Z}$

we deduce a commutative diagram

$$\begin{array}{ccc} A(K) & \longrightarrow & P_{v,w}(K) \\ \downarrow & & \downarrow \\ A(\mathbf{Z}) & \longrightarrow & P_{v,w}(\mathbf{Z}) \end{array}$$

in which the vertical maps are surjective and the upper horizontal map is a bijection. It follows that the lower horizontal map is surjective; but it is also injective (see 3.7(a)) hence bijective.

3.10. We return to the setup of 3.4. If K_1 is one of the semifields $\mathbf{R}_{>0}, K, \mathbf{Z}$, then the elements of $P_{v,w}(K_1)$ are represented by elements of $\xi \in V(K_1) - \underline{o}$ with $\mathrm{supp}(\xi) = \beta_{v,\mathbf{i}}$. In the case where $K_1 = \mathbf{R}_{>0}$, $P_{v,w}(K_1)$ depends only on v, w and not on $\mathbf{i}$. It follows that $\beta_{v,\mathbf{i}}$ depends only on v, w not on $\mathbf{i}$ hence we can write $\beta_{v,w}$ instead of $\beta_{v,\mathbf{i}}$.

Note that in [9, 2.4] it was conjectured (for $\mathbf{R}_{>0}$) that the set $[[v, w]]$ defined in [9, 2.3(a)] in type A_2 should make sense in general. This conjecture is now established for $\mathbf{R}_{>0}$ by taking $[[v, w]] = \beta_{v,w}$ (and the analogue of the conjecture for K_1 as above is also established).

Using 2.4(a) and the definitions we see that

(a) $$\beta_{v,w} \subset \beta^w \cap \phi(\beta^{vw_I}).$$

We expect that this is an equality (a variant of a conjecture in [9, 2.4], see also [9, 2.3(a)]). From 3.4 we see that

(b) $$b_w \in \beta_{v,w}.$$

From 2.3(d) we deduce:

(c) $$\phi(\beta_{ww_I,vw_I}) = \beta_{v,w}.$$

Using (b), (c) we deduce:

(d) $$\phi(b_{vw_I}) \in \beta_{v,w}.$$

3.11. For K_1 as in 3.10 and for $v \le w$ in W, $v' \le w'$ in W, we show:

(a) *If* $P_{v,w}(K_1) \cap P_{v',w'}(K_1) \ne \emptyset$, *then* $v = v'$, $w = w'$.

If K_1 is $\mathbf{R}_{>0}$ or K this is already known. We will give a proof of (a) which applies also when $K_1 = \mathbf{Z}$. From the results in 3.10 we see that it is enough to show:

(b) *If* $\beta_{v,w} = \beta_{v',w'}$, *then* $v = v'$, $w = w'$.

From 3.10(b) we have $b_{w'} \in \beta_{v',w'}$ hence $b_{w'} \in \beta_{v,w}$ so that (using 3.10(a)) we have $b_{w'} \in \beta^w$. Using 2.1(a) we deduce that $b_{w'} \in V'^{\mathbf{i}}$ (with $\mathbf{i}$ as in 2.1). It follows that either $b_{w'} = b_w$ or $\nu_{b_{w'}} - \nu_{b_w}$ is of the form $j'_1 + j'_2 + \cdots + j'_k$ with $j_t \in I$ and $k \ge 1$. Interchanging the roles of w, w' we see that either $b_w = b_{w'}$ or $\nu_{b_w} - \nu_{b_{w'}}$ is of the form $\tilde{j}'_1 + \tilde{j}'_2 + \cdots + \tilde{j}'_{k'}$ with $\tilde{j}_t \in I$ and $k' \ge 1$. If $b_w \ne b_{w'}$ then we must have $j'_1 + j'_2 + \cdots + j'_k + \tilde{j}'_1 + \tilde{j}'_2 + \cdots + \tilde{j}'_{k'} = 0$, which is absurd. Thus we have $b_w = b_{w'}$. Since $\lambda \in \mathcal{X}^{++}$ this implies $w = w'$.

Now applying ϕ to the first equality in (a) and using 3.10(c) we see that $\beta_{ww_I, vw_I} = \beta_{w'w_I, v'w_I}$. Using the first part of the argument with v, w, v', w' replaced by $ww_I, vw_I, w'w_I, v'w_I$, we see that $vw_I = v'w_I$ hence $v = v'$. This completes the proof of (b) hence that of (a).

Now the proof of Theorem 0.2 is complete.

3.12. Now $\phi : \mathcal{B} \to \mathcal{B}$ (see 2.3) induces an involution $\tilde{\mathcal{B}} \to \tilde{\mathcal{B}}$ and an involution $\tilde{\mathcal{B}}_{\ge 0} \to \tilde{\mathcal{B}}_{\ge 0}$ denoted again by ϕ. From 2.3(a), (d) we deduce that this involution restricts to a bijection $\widetilde{\mathcal{B}_{ww_I, vw_I}}_{\ge 0} \to \widetilde{\mathcal{B}_{v,w}}_{\ge 0}$ for any $v \le w$ in W. The involution $\phi : \tilde{\mathcal{B}}_{\ge 0} \to \tilde{\mathcal{B}}_{\ge 0}$ can be viewed as an involution of $P^\bullet(K)$ which coincides with the restriction of the involution $\phi : P(K) \to P(K)$ in 2.7. The last involution is compatible with the involution $\phi : P(\mathbf{Z}) \to P(\mathbf{Z})$ in 2.7 under the map $P(K) \to P(\mathbf{Z})$ induced by $r : K \to \mathbf{Z}$. It follows the image $P^\bullet(\mathbf{Z})$ of $P^\bullet(K)$ under $P(K) \to P(\mathbf{Z})$ is stable under $\phi : P(\mathbf{Z}) \to P(\mathbf{Z})$. Thus there is an induced involution ϕ on $\mathcal{B}(\mathbf{Z}) = P^\bullet(\mathbf{Z})$ which carries $P_{ww_I, vw_I}(\mathbf{Z})$ onto $P_{v,w}(\mathbf{Z})$ for any $v \le w$ in W.

4. Independence on λ

4.1. For λ, λ' in $\mathcal{X}^+$ let ${}^{\lambda,\lambda'}P$ be the set of lines in ${}^{\lambda}V \otimes {}^{\lambda'}V$. We define a linear map $E : {}^{\lambda}V \times {}^{\lambda'}V \to {}^{\lambda}V \otimes {}^{\lambda'}V$ by $(\xi, \xi') \mapsto \xi \otimes \xi'$. This induces a map $\bar{E} : {}^{\lambda}P \times {}^{\lambda'}P \to {}^{\lambda,\lambda'}P$.

Let K_1 be a semifield. Let $\mathcal{S} = {}^{\lambda}\beta \times {}^{\lambda'}\beta$. Let ${}^{\lambda,\lambda'}V(K_1)$ be the set of formal sums $u = \sum_{s\in\mathcal{S}} u_s s$ where $u_s \in K_1^!$. This is a monoid under addition (component by component) and we define scalar multiplication

$$K_1^! \times {}^{\lambda,\lambda'}V(K_1) \to {}^{\lambda,\lambda'}V(K_1)$$

by $(k, \sum_{s\in\mathcal{S}} u_s s) \mapsto \sum_{s\in\mathcal{S}} (ku_s)s$. Let $\mathrm{End}({}^{\lambda,\lambda'}V(K_1))$ be the set of maps $\zeta : {}^{\lambda,\lambda'}V(K_1) \to {}^{\lambda,\lambda'}V(K_1)$ such that $\zeta(\xi + \xi') = \zeta(\xi) + \zeta(\xi')$ for ξ, ξ' in ${}^{\lambda,\lambda'}V(K_1)$ and $\zeta(k\xi) = k\zeta(\xi)$ for $\xi \in {}^{\lambda,\lambda'}V(K_1), k \in K_1^!$. This is a monoid under composition of maps.

We define a map

$$E(K_1) : {}^{\lambda}V(K_1) \times {}^{\lambda'}V(K_1) \to {}^{\lambda,\lambda'}V(K_1)$$

by

$$\Big(\sum_{b_1\in{}^{\lambda}\beta} \xi_{b_1} b_1\Big), \Big(\sum_{b'_1\in{}^{\lambda'}\beta} \xi'_{b'_1} b'_1\Big) \mapsto \sum_{(b_1,b'_1)\in\mathcal{S}} \xi_{b_1}\xi'_{b'_1}(b_1, b'_1).$$

We define a map

$$\mathrm{End}({}^{\lambda}V(K_1)) \times \mathrm{End}({}^{\lambda'}V(K_1)) \to \mathrm{End}({}^{\lambda,\lambda'}V(K_1))$$

by $(\tau, \tau') \mapsto [(b_1, b'_1) \mapsto E(K_1)(\tau(b_1), \tau'(b'_1))]$. Composing this map with the map

$$\mathfrak{G}(K_1) \to \mathrm{End}({}^{\lambda}V(K_1)) \times \mathrm{End}({}^{\lambda'}V(K_1))$$

whose components are the maps

$$\mathfrak{G}(K_1) \to \mathrm{End}({}^{\lambda}V(K_1)), \quad \mathfrak{G}(K_1) \to \mathrm{End}({}^{\lambda'}V(K_1))$$

in 1.5 we obtain a map $\mathfrak{G}(K_1) \to \mathrm{End}({}^{\lambda,\lambda'}V(K_1))$ which is a monoid homomorphism. Thus $\mathfrak{G}(K_1)$ acts on ${}^{\lambda,\lambda'}V(K_1)$; it also acts on ${}^{\lambda}V(K_1) \times {}^{\lambda'}V(K_1)$ (by 1.5) and the two actions are compatible with $E(K_1)$.

Let $\underline{\circ}$ be the element $u \in {}^{\lambda,\lambda'}V(K_1)$ such that $u_s = \circ$ for all $s \in \mathcal{S}$. Let ${}^{\lambda,\lambda'}P(K_1)$ be the set of orbits of the free K_1 action (scalar multiplication) on ${}^{\lambda,\lambda'}V(K_1) - \underline{\circ}$. Now $E(K_1)$ restricts to a map

$$({}^{\lambda}V(K_1)) - \underline{\circ}) \times ({}^{\lambda'}V(K_1) - \underline{\circ}) \to {}^{\lambda,\lambda'}V(K_1) - \underline{\circ}$$

and induces an (injective) map

$$\bar{E}(K_1) : {}^{\lambda}P(K_1) \times {}^{\lambda'}P(K_1) \to {}^{\lambda,\lambda'}P(K_1).$$

Now $\mathfrak{G}(K_1)$ acts naturally on ${}^{\lambda}P(K_1) \times {}^{\lambda'}P(K_1)$ and on ${}^{\lambda,\lambda'}P(K_1)$; these $\mathfrak{G}(K_1)$-actions are compatible with $\bar{E}(K_1)$.

4.2. For λ, λ' in $\mathcal{X}^+$ there is a unique linear map

$$\Gamma : {}^{\lambda+\lambda'}V \to {}^{\lambda}V \otimes {}^{\lambda'}V$$

which is compatible with the G-actions and takes ${}^{\lambda+\lambda'}\xi^+$ to ${}^{\lambda}\xi^+ \otimes {}^{\lambda'}\xi^+$. This induces a map $\bar{\Gamma} : {}^{\lambda+\lambda'}P \to {}^{\lambda,\lambda'}P$.

For $b \in {}^{\lambda+\lambda'}\beta$ we have

$$\Gamma(b) = \sum_{(b_1,b'_1)\in\mathcal{S}} e_{b,b_1,b'_1} b_1 \otimes b'_1$$

where $e_{b,b_1,b'_1} \in \mathbf{N}$. (This can be deduced from the positivity property [3, 14.4.13(b)] of the homomorphism r in [3, 1.2.12].) There is a unique map

$$\Gamma(K_1) : {}^{\lambda+\lambda'}V(K_1) \to {}^{\lambda,\lambda'}V(K_1)$$

compatible with addition and scalar multiplication and such that for $b \in {}^{\lambda+\lambda'}\beta$ we have

$$\Gamma(K_1)(b) = \sum_{(b_1,b'_1)\in\mathcal{S}} e_{b,b_1,b'_1}(b_1, b'_1)$$

where e_{b,b_1,b'_1} are viewed as elements of $K_1^!$. Since Γ is injective, for any $b \in {}^{\lambda+\lambda'}\beta$ we have $e_{b,b_1,b'_1} \in \mathbf{N} - \{0\}$ for some b_1, b'_1, hence $e_{b,b_1,b'_1} \in K_1$, when viewed as an element of $K_1^!$. It follows that $\Gamma(K_1)$ maps ${}^{\lambda+\lambda'}V(K_1) - \underline{\circ}$ into

${}^{\lambda,\lambda'}V(K_1) - \underline{\circ}$. Hence $\Gamma(K_1)$ defines an (injective) map

$$\bar{\Gamma}(K_1) : {}^{\lambda+\lambda'}P(K_1) \to {}^{\lambda,\lambda'}P(K_1)$$

which is compatible with the action of $\mathfrak{G}(K_1)$ on the two sides.

4.3. We now assume that K_1 is either K as in 0.1(i) or $\mathbf{Z}$ as in 0.1(ii) and that $\lambda \in \mathcal{X}^{++}, \lambda' \in \mathcal{X}^{+}$ so that $\lambda + \lambda' \in \mathcal{X}^{++}$. We have the following result.

(a) *Let* $\mathcal{L} \in {}^{\lambda+\lambda'}P^\bullet(K_1)$. *Then* $\bar{\Gamma}(K_1)(\mathcal{L}) = \bar{E}(K_1)(\mathcal{L}_1, \mathcal{L}'_1)$ *for some* $(\mathcal{L}_1, \mathcal{L}'_1) \in {}^{\lambda}P^\bullet(K_1) \times {}^{\lambda'}P(K_1)$ *(which is unique, by the injectivity of* $\bar{E}(K_1)$*). Thus,* $\mathcal{L} \mapsto \mathcal{L}_1$ *is a well defined map* $H(K_1) : {}^{\lambda+\lambda'}P^\bullet(K_1) \to {}^{\lambda}P^\bullet(K_1)$.

We shall prove (a) for $K_1 = \mathbf{Z}$ assuming that it is true for $K_1 = K$. We can find $\tilde{\mathcal{L}} \in {}^{\lambda+\lambda'}P^\bullet(K)$ such that $\mathcal{L} \in {}^{\lambda+\lambda'}P^\bullet(\mathbf{Z})$ is the image of $\tilde{\mathcal{L}}$ under the map ${}^{\lambda+\lambda'}P^\bullet(K) \to {}^{\lambda+\lambda'}P^\bullet(\mathbf{Z})$ induced by $r : K \to \mathbf{Z}$. By our assumption we have $\bar{\Gamma}(K)(\tilde{\mathcal{L}}) = \bar{E}(K)(\tilde{\mathcal{L}}_1, \tilde{\mathcal{L}}'_1)$ with $(\tilde{\mathcal{L}}_1, \tilde{\mathcal{L}}'_1) \in {}^{\lambda}P^\bullet(K) \times {}^{\lambda'}P(K)$. Let $\mathcal{L}_1$ (resp. $\mathcal{L}'_1$) be the image of $\tilde{\mathcal{L}}_1$ (resp. $\tilde{\mathcal{L}}'_1$) under the map ${}^{\lambda}P^\bullet(K) \to {}^{\lambda}P^\bullet(\mathbf{Z})$ (resp. ${}^{\lambda'}P(K) \to {}^{\lambda'}P(\mathbf{Z})$) induced by $r : K \to \mathbf{Z}$. From the definitions we see that $\bar{\Gamma}(\mathbf{Z})(\mathcal{L}) = \bar{E}(\mathbf{Z})(\mathcal{L}_1, \mathcal{L}'_1)$. This proves the existence of $(\mathcal{L}_1, \mathcal{L}'_1)$. The proof of (a) in the case where $K_1 = K$ will be given in 4.6.

Assuming that (a) holds, we have a commutative diagram

$$\begin{array}{ccc} {}^{\lambda+\lambda'}P^\bullet(K) & \xrightarrow{H(K)} & {}^{\lambda}P^\bullet(K) \\ \downarrow & & \downarrow \\ {}^{\lambda+\lambda'}P^\bullet(\mathbf{Z}) & \xrightarrow{H(\mathbf{Z})} & {}^{\lambda}P^\bullet(\mathbf{Z}) \end{array}$$

in which the vertical maps are induced by $r : K \to \mathbf{Z}$.

4.4. We preserve the setup of 4.3. For each $w \in W$ we assume that a sequence $\mathbf{i}_w = (i_1, i_2, \dots, i_m) \in \mathcal{I}_w$ has been chosen (here $m = |w|$). Let $\mathcal{Z}(K_1) = \sqcup_{v \le w \text{ in } W} A_{v,w}(K_1)$ where $A_{v,w}(K_1)$ is the set of all maps $[1, m]' \to K_1$ (with $[1, m]'$ defined as in 3.2 in terms of v, w and $\mathbf{i} = \mathbf{i}_w$). From the results in 3.9 we have a bijection

$${}^{\lambda}D(K_1) : \mathcal{Z}(K_1) \xrightarrow{\sim} {}^{\lambda}P^\bullet(K_1)$$

whose restriction to $A_{v,w}(K_1)$ is as in the last commutative diagram in 3.9 (with $\mathbf{i} = \mathbf{i}_w$). Replacing here λ by $\lambda + \lambda'$ we obtain an analogous bijection

$$^{\lambda+\lambda'}D(K_1) : \mathcal{Z}(K_1) \xrightarrow{\sim} {}^{\lambda+\lambda'}P^\bullet(K_1).$$

From the commutative diagram in 3.4 we deduce a commutative diagram

$$\begin{array}{ccc} \mathcal{Z}(K) & \xrightarrow{^\lambda D(K)} & {}^\lambda P^\bullet(K) \\ \downarrow & & \downarrow \\ \mathcal{Z}(\mathbf{Z}) & \xrightarrow{^\lambda D(\mathbf{Z})} & {}^\lambda P^\bullet(\mathbf{Z}) \end{array}$$

and a commutative diagram

$$\begin{array}{ccc} \mathcal{Z}(K) & \xrightarrow{^{\lambda+\lambda'} D(K)} & {}^{\lambda+\lambda'} P^\bullet(K) \\ \downarrow & & \downarrow \\ \mathcal{Z}(\mathbf{Z}) & \xrightarrow{^{\lambda+\lambda'} D(\mathbf{Z})} & {}^{\lambda+\lambda'} P^\bullet(\mathbf{Z}) \end{array}$$

in which the vertical maps are induced by $r : K \to \mathbf{Z}$.

4.5. We preserve the setup of 4.3. We assume that 4.3(a) holds. From the commutative diagrams in 4.3, 4.4 we deduce a commutative diagram

$$\begin{array}{ccc} \mathcal{Z}(K) & \xrightarrow{(^\lambda D(K))^{-1} H(K) {}^{\lambda+\lambda'} D(K)} & \mathcal{Z}(K) \\ \downarrow & & \downarrow \\ \mathcal{Z}(\mathbf{Z}) & \xrightarrow{(^\lambda D(\mathbf{Z}))^{-1} H(\mathbf{Z}) {}^{\lambda+\lambda'} D(\mathbf{Z})} & \mathcal{Z}(\mathbf{Z}) \end{array}$$

in which the vertical maps are induced by $r : K \to \mathbf{Z}$. Recall that K_1 is K or $\mathbf{Z}$. We have the following result.

(a) $(^\lambda D(K_1))^{-1} H(K_1) {}^{\lambda+\lambda'} D(K_1)$ *is the identity map* $\mathcal{Z}(K_1) \to \mathcal{Z}(K_1)$.

If (a) holds for $K_1 = K$ then it also holds for $K_1 = \mathbf{Z}$, in view of the commutative diagram above in which the vertical maps are surjective. The proof of (a) in the case $K_1 = K$ will be given in 4.7.

From (a) we deduce:

(b) $H(K_1)$ *is a bijection.*

4.6. In this subsection we assume that $K_1 = K$. Let $\mathbf{k} = \mathbf{C}(x)$ where x is an indeterminate. We have $K^! \subset \mathbf{k}$. For any $\lambda \in \mathcal{X}^+$ we set ${}^\lambda V_{\mathbf{k}} = \mathbf{k} \otimes {}^\lambda V$. This is naturally a module over the group $G(\mathbf{k})$ of $\mathbf{k}$ points of G. Let $\mathcal{B}(\mathbf{k})$ be the set of subgroups of $G(\mathbf{k})$ that are $G(\mathbf{k})$-conjugate to $B^+(\mathbf{k})$, the group of $\mathbf{k}$-points of B^+. We identify ${}^\lambda V(K)$ with the set of vectors in ${}^\lambda V_{\mathbf{k}}$ whose coordinates in the $\mathbf{k}$-basis ${}^\lambda\beta$ are in $K^!$. In the case where $\lambda \in \mathcal{X}^{++}$, we identify ${}^\lambda V^\bullet(K) - 0$ with the set of all $\xi \in {}^\lambda V(K) - 0$ such that the stabilizer in $G(\mathbf{k})$ of the line $[\xi]$ belongs to $\mathcal{B}(\mathbf{k})$. (For a nonzero vector ξ in a $\mathbf{k}$-vector space we denote by $[\xi]$ the $\mathbf{k}$-line in that vector space that contains ξ.)

Now let $\lambda \in \mathcal{X}^{++}, \lambda' \in \mathcal{X}^+$. We show that 4.3(a) holds for λ, λ'. We identify ${}^{\lambda,\lambda'}V(K)$ with the set of vectors in ${}^\lambda V_{\mathbf{k}} \otimes_{\mathbf{k}} {}^{\lambda'}V_{\mathbf{k}}$ whose coordinates in the $\mathbf{k}$-basis ${}^\lambda\beta \otimes {}^{\lambda'}\beta$ are in $K^!$.

Then $E(K)$ becomes the restriction of the homomorphism of $G(\mathbf{k})$-modules $E' : {}^\lambda V_{\mathbf{k}} \times {}^{\lambda'}V_{\mathbf{k}} \to {}^\lambda V_{\mathbf{k}} \otimes_{\mathbf{k}} {}^{\lambda'}V_{\mathbf{k}}$ given by $(\xi, \xi') \mapsto \xi \otimes_{\mathbf{k}} \xi'$ and $\Gamma(K)$ becomes the restriction of the homomorphism of $G(\mathbf{k})$-modules Γ' : ${}^{\lambda+\lambda'}V_{\mathbf{k}} \to {}^\lambda V_{\mathbf{k}} \otimes_{\mathbf{k}} {}^{\lambda'}V_{\mathbf{k}}$ obtained from Γ by extension of scalars.

Let $L_\lambda = [{}^\lambda\xi^+] \subset {}^\lambda V_{\mathbf{k}}$, $L_{\lambda'} = [{}^{\lambda'}\xi^+] \subset {}^{\lambda'}V_{\mathbf{k}}$, $L_{\lambda+\lambda'} = [{}^{\lambda+\lambda'}\xi^+] \subset {}^{\lambda+\lambda'}V_{\mathbf{k}}$. Now let $\xi \in {}^{\lambda+\lambda'}V^\bullet(K) - 0$. Then $[\xi] = gL_{\lambda+\lambda'}$ for some $g \in G(\mathbf{k})$ hence

$$\begin{aligned}\Gamma'([\xi]) =& g(L_\lambda \otimes L_{\lambda'}) = (gL_\lambda) \otimes (g(L_{\lambda'}) = E'(gL_\lambda, g(L_{\lambda'}) \\ =& E'([g({}^\lambda\xi^+)], [g({}^{\lambda'}\xi^+)]).\end{aligned}$$

To prove 4.3(a) in our case it is enough to prove that for some c, c' in $\mathbf{k}^*$ we have $cg({}^\lambda\xi^+) \in {}^\lambda V(K)$, $c'g({}^{\lambda'}\xi^+) \in {}^{\lambda'}V(K)$. We have $\xi = c_0 g({}^{\lambda+\lambda'}\xi^+)$ for some $c_0 \in \mathbf{k}^*$ and $\Gamma'(\xi) = \Gamma(\xi) \in {}^{\lambda,\lambda'}V(K)$. Thus, $c_0\Gamma'(g({}^{\lambda+\lambda'}\xi) \in {}^{\lambda,\lambda'}V(K)$ that is, $c_0(g{}^\lambda\xi^+) \otimes (g{}^{\lambda'}\xi^+) \in {}^{\lambda,\lambda'}V(K)$. It is enough to show:

(a) *If* $z \in {}^\lambda V_{\mathbf{k}}$, $z' \in {}^{\lambda'}V_{\mathbf{k}}$, $c_0 \in \mathbf{k}^*$ *satisfy* $c_0 z \otimes z' \in {}^{\lambda,\lambda'}V(K) - 0$, *then* $cz \in {}^\lambda V(K) - 0$, $c'z' \in {}^{\lambda'}V(K) - 0$ *for some* c, c' *in* $\mathbf{k}^*$.

We write $z = \sum_{b \in {}^\lambda\beta} z_b b$, $z' = \sum_{b' \in {}^{\lambda'}\beta} z'_{b'} b'$ with $z_b, z'_{b'}$ in $\mathbf{k}$. We have $c_0 z_b z'_{b'} \in K^!$ for all b, b'. Replacing z by $c_0 z$ we can assume that $c_0 = 1$ so that $z_b z'_{b'} \in K^!$ for all b, b' and $z_b z'_{b'} \neq 0$ for some b, b'. Thus we can find $b'_0 \in {}^{\lambda'}\beta$ such that $z'_{b'_0} \in K$. We have $z_b z'_{b'_0} \in K^!$ for all b. Replacing z by $z'_{b'_0} z$ we can assume that $z_b \in K^!$ for all b. We can find $b_0 \in {}^\lambda\beta$ such that

$z_{b_0} \in K$. We have $z_{b_0} z'_{b'} \in K^!$ for all b'. It follows that $z'_{b'} \in K^!$ for all b'. This proves (a) and completes the proof of 4.3(a).

4.7. We preserve the setup of 4.3 and assume that $K_1 = K$. We show that 4.5(a) holds in this case. Let $v \le w, \mathbf{i}$ be as in 3.2 and let $A(K_1)$ be as in 3.4. Let $h \in A(K_1)$. We have ${}^{\lambda+\lambda'}D(K_1)(h) = [\sigma_{K_1}(h)^{\lambda+\lambda'}\xi^+]$ where $\sigma_{K_1} : A(K_1) \to G(\mathbf{k})$ is defined by the same formula as σ in 3.2. (Note that for $i \in I$, $y_i(t) \in G(\mathbf{k})$ is defined for any $t \in \mathbf{k}$.) Hence

$$\begin{aligned}\bar{\Gamma}(K_1)^{\lambda+\lambda'}D(K_1)(h) &= [(\sigma_{K_1}(h)^{\lambda}\xi^+) \otimes (\sigma_{K_1}(h)^{\lambda'}\xi^+)] \\ &= \bar{E}(K_1)([\sigma_{K_1}(h)^{\lambda}\xi^+], [\sigma_{K_1}(h)^{\lambda'}\xi^+])\end{aligned}$$

so that

$$H(K_1)^{\lambda+\lambda'}D(K_1)(h) = [\sigma_{K_1}(h)^{\lambda}\xi^+] = {}^{\lambda}D(K_1)(h).$$

This shows that the map in 4.5(a) takes h to h for any $h \in A(K_1)$. This proves 4.5(a).

4.8. We now assume that K_1 is either K as in 0.1(i) or $\mathbf{Z}$ as in 0.1(ii) and that $\lambda \in \mathcal{X}^{++}, \lambda' \in \mathcal{X}^{++}$. From 4.3(a),4.5(a) we have a well defined bijection $H(K_1) : {}^{\lambda+\lambda'}P^{\bullet}(K_1) \xrightarrow{\sim} {}^{\lambda}P^{\bullet}(K_1)$. Interchanging λ, λ' we obtain a bijection $H'(K_1) : {}^{\lambda+\lambda'}P^{\bullet}(K_1) \xrightarrow{\sim} {}^{\lambda'}P^{\bullet}(K_1)$. Hence we have a bijection

$$\gamma_{\lambda,\lambda'} = H'(K_1)H(K_1)^{-1} : {}^{\lambda}P^{\bullet}(K_1) \xrightarrow{\sim} {}^{\lambda'}P^{\bullet}(K_1).$$

From the definitions we see that $H(K_1)$ is compatible with the $\mathfrak{G}(K_1)$-actions. Similarly, $H'(K_1)$ is compatible with the $\mathfrak{G}(K_1)$-actions. It follows that $\gamma_{\lambda,\lambda'}$ is compatible with the $\mathfrak{G}(K_1)$-actions. From the definitions we see that if λ'' is third element of $\mathcal{X}^{++}$, we have

$$\gamma_{\lambda,\lambda''} = \gamma_{\lambda',\lambda''}\gamma_{\lambda,\lambda'}.$$

This shows that our definition of $\mathcal{B}(K_1)$ is independent of the choice of λ.

5. The Non-simply Laced Case

5.1. Let $\delta : G \to G$ be an automorphism of G such that $\delta(B^+) = B^+, \delta(B^-) = B^-$ and $\delta(x_i(t)) = x_{i'}(t)$, $\delta(y_i(t)) = y_{i'}(t)$ for all $i \in I, t \in \mathbf{C}$ where $i \mapsto i'$ is a permutation of I denoted again by δ. We define an automorphism of W by $s_i \mapsto s_{\delta(i)}$ for all $i \in I$; we denote this automorphism again by δ. We assume further that $s_i s_{\delta(i)} = s_{\delta(i)} s_i$ for any $i \in I$. The fixed point set G^δ of $\delta : G \to G$ is a connected simply connected semisimple group over $\mathbf{C}$. The fixed point set W^δ of $\delta : W \to W$ is the Weyl group of G^δ and as such it has a length function $w \mapsto |w|_\delta$.

Now δ takes any Borel subgroup of G to a Borel subgroup of G hence it defines an automorphism of $\mathcal{B}$ denoted by δ, with fixed point set denoted by $\mathcal{B}^\delta$. This automorphism restricts to a bijection $\mathcal{B}_{\geq 0} \to \mathcal{B}_{\geq 0}$. We can identify $\mathcal{B}^\delta$ with the flag manifold of G^δ by $B \mapsto B \cap G^\delta$. Under this identification, the totally positive part of the flag manifold of G^δ (defined in [5]) becomes $\mathcal{B}^\delta_{\geq 0} = \mathcal{B}_{\geq 0} \cap \mathcal{B}^\delta$. For $\lambda \in \mathcal{X}$ we define $\delta(\lambda) \in \mathcal{X}$ by $\langle \delta(i), \delta(\lambda) \rangle = \langle i, \lambda \rangle$ for all $i \in I$. In the setup of 1.4 assume that $\lambda \in \mathcal{X}^{++}$ satisfies $\delta(\lambda) = \lambda$. There is a unique linear isomorphism $\delta : V \to V$ such that $\delta(g\xi) = \delta(g)\delta(\xi)$ for any $g \in G, \xi \in V$ and such that $\delta(\xi^+) = \xi^+$. This restricts to a bijection $\beta \to \beta$ denoted again by δ. For any semifield K_1 we define a bijection $V(K_1) \to V(K_1)$ by $\sum_{b \in \beta} \xi_b b \mapsto \sum_{b \in \beta} \xi_{\delta^{-1}(b)} b$ where $\xi_b \in K_1^!$. This induces a bijection $P(K_1) \to P(K_1)$ denoted by δ. We now assume that K_1 is as in 0.1(i), (ii). Then the subset $P^\bullet(K_1)$ of $P(K_1)$ is defined and is stable under δ; let $P^\bullet(K_1)^\delta$ be the fixed point set of $\delta : P^\bullet(K_1) \to P^\bullet(K_1)$. Recall that $\mathfrak{G}(K_1)$ acts naturally on $P(K_1)$. This restricts to an action on $P^\bullet(K_1)^\delta$ of the monoid $\mathfrak{G}(K_1)^\delta$ (the fixed point set of the isomorphism $\mathfrak{G}(K_1) \to \mathfrak{G}(K_1)$ induced by δ) which is the same as the monoid associated in [8] to G^δ and K_1. We set $\mathcal{B}^\delta(K_1) = P^\bullet(K_1)^\delta$.

The following generalization of Theorem 0.2 can be deduced from Theorem 0.2.

(a) *The set $\mathcal{B}^\delta(\mathbf{Z})$ has a canonical partition into pieces $P_{v,w;\delta}(\mathbf{Z})$ indexed by the pairs $v \leq w$ in W^δ. Each such piece $P_{v,w;\delta}(\mathbf{Z})$ is in bijection with $\mathbf{Z}^{|w|_\delta - |v|_\delta}$; in fact, there is an explicit bijection $\mathbf{Z}^{|w|_\delta - |v|_\delta} \xrightarrow{\sim} P_{v,w;\delta}(\mathbf{Z})$ for any reduced expression of w in W^δ.*

References

1. G. Lusztig, Canonical bases arising from quantized enveloping algebras, *J. Amer. Math. Soc.*, **3** (1990), 447-498

2. G. Lusztig, Canonical bases arising from quantized enveloping algebras, II, *Progr. of Theor. Phys. Suppl.*, **102** (1990), 175-201.

3. G. Lusztig, *Introduction to Quantum Groups*, Progr. in Math. 110, Birkhäuser, Boston, 1993.

4. G. Lusztig, Problems on canonical bases, Algebraic groups and their generalizations: quantum and infinite dimensionalmethods, *Proc. Symp. Pure Math.* **56**(2), *Amer. Math. Soc.*, (1994), 169-176.

5. G. Lusztig, Total positivity in reductive groups, *Lie Theory and Geometry*, Progr. in Math. 123, Birkhäuser, Boston, (1994), 531-568.

6. G. Lusztig, Total positivity in partial flag manifolds, *Represent. Th.*, **2** (1998), 70-78.

7. G. Lusztig, Positive structures in Lie theory, arxiv:1812.09313.

8. G. Lusztig, Total positivity in reductive groups, II, *Bull. Inst. Math. Acad. Sin.* (*N.S.*), **14** (2019), 403-460.

9. G. Lusztig, Total positivity in Springer fibres, arxiv:1909.00896.

10. R. Marsh and K. Rietsch, Parametrizations of flag varieties, *Represent. Th.*, **8** (2004), 212-242.