

EXISTENCE OF RELAXED OPTIMAL CONTROL FOR G -NEUTRAL STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS WITH UNCONTROLLED DIFFUSION

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Abstract

In this paper, we study under refined Lipschitz hypothesis, the question of existence and uniqueness of solution of controlled neutral stochastic functional differential equations driven by G -Brownian motion (G -NSFDEs in short). An existence of a relaxed optimal control where the neutral and diffusion terms do not depend on the control variable was the main result of the article. The latter is done by using tightness techniques and the weak convergence techniques for each probability measure in the set of all possible probabilities of our dynamic. A motivation of our work is presented and a numerical analysis for the uncontrolled G -NSFDE is given.

1. Introduction

Due to the important ambiguous concepts in the study of optimal control problems in finance under the principles of uncertainty, it appears in different typical fields that contain incomplete or inaccurate parameters, especially

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financial crises and risks resulting from dark fluctuations and their impact on the movement of asset prices and liquidity in the markets.

The concepts of uncertainty in fluctuations were studied by [13, 14], who established a type of non-linear expectation theory or expectancy theory within the framework of G -Brownian motion, and [6, 7] did that through the capacity theory, and then relied on the G -Brownian movement under G -expectation to create G -stochastic calculus and this is what led both to prove the existence and uniqueness of the stochastic differential equations driven by the G -Brownian motion by [10, 13]. In addition, [8, 9] studied the existence and uniqueness of neutral stochastic functional differential equations within the framework of the G -Brownian motion (G -NSFDEs in short), is given by

$$\begin{cases} d[X(t) - Q(t, X_t)] = b(t, X_t) dt + \gamma(t, X_t) d\langle B \rangle_t + \sigma(t, X_t) dB_t, & t \in [0, T] \\ X_0 = \eta, \end{cases} \quad (1)$$

where, $\eta \in BC([- \tau, 0]; \mathbb{R})$, and $\tau \geq 0$, $X_t = \{X(t + \theta) : -\tau \leq \theta \leq 0\}$, $(B_t, t \geq 0)$ is a one-dimensional G -Brownian motion defined on some space of sublinear expectation $(\Omega, \mathcal{H}, \widehat{E}, \mathbb{F}^{\mathcal{P}})$, with a universal filtration $\mathbb{F}^{\mathcal{P}} = \{\widehat{\mathcal{F}}_t^{\mathcal{P}}\}_{t \geq 0}$, and $\{\langle B \rangle_t, t \geq 0\}$ is the quadratic variation process of G -Brownian motion, Q, b, γ , and σ are functions on $[0, T] \times BC([- \tau, 0]; \mathbb{R})$. With what the G -expectation permits

$$\widehat{E}[\cdot] = \sup_{\mathbb{P} \in \mathcal{P}} E^{\mathbb{P}}[\cdot],$$

where $E^{\mathbb{P}}$ are ordinary expectations, and \mathcal{P} is a tight family of possibly mutually singular probability measures. For more details see [6, 7]. Recently, [4, 11, 12, 15] considered an optimal control problem with the uncertainty of G -Brownian motion and its quadratic variation $\langle B \rangle$. In this paper we consider the following G -NSFDE

$$\begin{cases} d[X^u(t) - Q(t, X_t^u)] = b(t, X_t^u, u(t)) dt + \gamma(t, X_t^u, u(t)) d\langle B \rangle_t + \sigma(t, X_t^u) dB_t \\ X_0^u = \eta, t \in [0, T] \end{cases} \quad (2)$$

where $u(\cdot) \in \mathbb{A}$ stands for the control variable for each $t \in [0, T]$, and \mathbb{A} is a compact polish space of \mathbb{R} . Let $\mathcal{P}(\mathbb{A})$ denote the space of probability

measures on $\mathcal{B}(\mathbb{A})$, the σ -algebra of Borel subsets of the set \mathbb{A} of values taken by the strict control. The set $\mathcal{U} = \mathcal{U}([0, T])$ is a set of strict controls. The cases of controlled of various stochastic systems driven by a classical Brownian motion has been treated by different authors, see e.g [1, 3, 18]. In this paper, we study under the concepts presented in [13, 14] the existence of a relaxed optimal control that minimize the cost functional:

$$\widehat{E} \left[\int_0^T \mathcal{L}(t, X_t^u, u(t)) dt + \Psi(X_T^u) \right], \quad (3)$$

The proof is based on the tightness arguments of the distribution of the control problem.

Motivation: To motivate our work let consider a Brownian particle moving in an unbounded medium. Let $X(t)$ be the position and $Y(t)$ be the velocity of the particle at time t . So the dynamic is represented by

$$X'(t) = Y(t) \quad mdY(t) = b(t)dt + \sigma d\xi_t, \quad (4)$$

where m is the mass of the particle and $\sigma d\xi_t$ is the noise part of the medium on the particle. According to Boussinesq representation in [5], $b(t) = -hY(t) - Y'(t)\sigma\sqrt{\frac{hm_1^3}{\pi}} \int_0^\infty Y'^2(s) ds$, which represents the systematic action of the medium on the particle, where $-hY(t)$ is the Stokes friction force at time t and m the apparent additional mass which is half the mass of the material of the medium ousted by the body. The $\int_0^\infty Y'^2(s) ds$ is the viscous hydrodynamic aftereffect. These models represent a NSFDE in the classical case.

In reality, it is difficult to estimate exactly the noise parameter σ , and what we can have as information is only a range interval $[\sigma_{min}, \sigma_{max}]$ where σ belongs, and so, the question is to study the worse-case scenario, which is difficult to analyse it by direct methods. The worst scenario system can be transformed to a *G*-NSFDE, and if we want to control the dynamic of the particle subject to some constrain, this will lead to a stochastic optimal control driven by a *G*-NSFDE.

The rest of the paper is formed as follows. In Section 2, we introduce some preliminaries which will be used to establish our result. In Section 3, it is related to three topics, first, we are concentrated to introduce the Problem of *G*-NSFDEs relaxed control, secondly, we prove the existence and

uniqueness of solution of G -NSFDEs with uncontrolled diffusion, we established the existence of a minimizer of the cost functional in third. Finally, we study the approximation of the relaxed control and we prove the existence of relaxed control. The last section is devoted to some numerical analysis.

2. Preliminaries

The main purpose of this section is to introduce some basic notions and results in G -stochastic calculus that are used in the subsequent sections. More details can be found in [6, 7, 13, 14, 16, 17].

We set $\Omega := \{\omega \in C([0, T], \mathbb{R}) : \omega(0) = 0\}$, the space of real valued continuous functions on $[0, T]$ such that $\omega(0) = 0$, equipped with the following distance

$$d(w^1, w^2) := \sum_{N=1}^{\infty} 2^{-N} \left(\left(\max_{0 \leq t \leq N} |w_t^1 - w_t^2| \right) \wedge 1 \right),$$

$\Omega_t := \{w_{\cdot \wedge t} : w \in \Omega\}$, $B_t(w) = w_t, t \geq 0$ the canonical process on Ω and let $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ be the natural filtration generated by $(B_t)_{t \geq 0}$. Moreover, we set, for each $t \in [0, \infty)$

$$\begin{aligned} \mathcal{F}_{t+} &:= \bigcap_{s > t} \mathcal{F}_s, \\ \mathbb{F}^+ &:= (\mathcal{F}_{t+})_{t \geq 0}, \\ \mathcal{F}_t^{\mathbb{P}} &:= \mathcal{F}_{t+} \vee \mathcal{N}^{\mathbb{P}}(\mathcal{F}_{t+}), \\ \widehat{\mathcal{F}}_t^{\mathbb{P}} &:= \mathcal{F}_{t+} \vee \mathcal{N}^{\mathbb{P}}(\mathcal{F}_{\infty}), \end{aligned}$$

where $\mathcal{N}^{\mathbb{P}}(\mathcal{G})$ is a \mathbb{P} -negligible set on a σ -algebra \mathcal{G} given by

$$\mathcal{N}^{\mathbb{P}}(\mathcal{G}) := \{D \subset \Omega : \text{there exists } \widetilde{D} \in \mathcal{G} \text{ such that } D \subset \widetilde{D} \text{ and } \mathbb{P}[\widetilde{D}] = 0\},$$

where \mathbb{P} is a probability measure on the Borel σ -algebra $\mathcal{B}(\Omega)$ of Ω . Consider the following spaces: for $0 \leq t \leq T$

$$\begin{aligned} Lip(\Omega_t) &:= \{\varphi(B_{t_1}, \dots, B_{t_n}) : \varphi \in C_{b,Lip}(\mathbb{R}^n) \text{ and } t_1, t_2, \dots, t_n \in [0, t]\}, \\ Lip(\Omega) &:= \bigcup_{n \in \mathbb{N}} Lip(\Omega_n), \end{aligned}$$

where $C_{b,Lip}(\mathbb{R}^n)$ is the space of bounded and Lipschitz on \mathbb{R}^n . Let $T > 0$ be a fixed time.

Peng in [13] has constructed the G -expectation $\widehat{E} : \mathcal{H} := Lip(\Omega_T) \rightarrow \mathbb{R}$ which is a consistent sublinear expectation on the lattice \mathcal{H} of real functions *i.e.* it satisfies:

1. Sub-additivity: $\widehat{E}[X + Y] \leq \widehat{E}[X] + \widehat{E}[Y]$, for all $X, Y \in \mathcal{H}$,
2. Monotonicity: $X \geq Y \Rightarrow \widehat{E}[X] \geq \widehat{E}[Y]$, for all $X, Y \in \mathcal{H}$,
3. Constant preserving: $\widehat{E}[c] = c$, for all $c \in \mathbb{R}$,
4. Positive homogeneity: $\widehat{E}[\lambda X] = \lambda \widehat{E}[X]$, for all $\lambda \geq 0, X \in \mathcal{H}$,

The triple $(\Omega, \mathcal{H}, \widehat{E})$ is said to be sub-linear expectation space, if **1** and **2** are only satisfied. Moreover, $\widehat{E}[\cdot]$ is called a nonlinear expectation and the triple $(\Omega, \mathcal{H}, \widehat{E})$ is called a nonlinear expectation space.

We assume that, if $Y = (Y_1, \dots, Y_n), Y_i \in \mathcal{H}$, then $\varphi(Y_1, \dots, Y_n) \in \mathcal{H}$ for all $\varphi \in C_{b,Lip}(\mathbb{R}^n)$.

Definition 1. A random vector $Y = (Y_1, \dots, Y_n)$ is said to be independent from another random vector $X = (X_1, \dots, X_m)$ under \widehat{E} if for any $\varphi \in C_{b,Lip}(\mathbb{R}^{n+m})$

$$\widehat{E}[\varphi(X, Y)] = \widehat{E}\left[\widehat{E}[\varphi(x, Y)]_{x=X}\right].$$

Definition 2. A process X on $(\Omega, \mathcal{H}, \widehat{E})$ is said to be G -normally distributed under the G -expectation $\widehat{E}[\cdot]$ if for any $\varphi \in C_{b,Lip}(\mathbb{R})$ the function

$$u(t, x) := \widehat{E}\left[\varphi\left(x + \sqrt{t}X\right)\right], (t, x) \in [0, T] \times \mathbb{R},$$

is the unique viscosity solution of the parabolic equation

$$\begin{cases} \frac{\partial u}{\partial t} = G(u_{xx}) \\ u(0, x) = \varphi(x) \end{cases}$$

where the nonlinear function G is defined by $G(a) := \frac{1}{2}\widehat{E}[aX^2] = \frac{1}{2}(\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-)$, $a \in \mathbb{R}$, with $\overline{\sigma}^2 := \widehat{E}[X^2]$, $\underline{\sigma}^2 := -\widehat{E}[-X^2]$, $a^+ = \max\{0, a\}$ and $a^- = -\min\{0, a\}$. This G -normal distribution is denoted by $\mathcal{N}(0, [\underline{\sigma}^2, \overline{\sigma}^2])$.

Definition 3. (G-Brownian Motion) The canonical process $(B_t)_{t \geq 0}$ on $(\Omega, \mathcal{H}, \widehat{E})$ is called a G -Brownian motion if the following properties are satisfied:

- $B_0 = 0$.
- For each $t, s \geq 0$ the increment $B_{t+s} - B_t$ is $\mathcal{N}(0, [s\underline{\sigma}^2, s\overline{\sigma}^2])$ -distributed.
- $B_{t_1}, B_{t_2}, \dots, B_{t_n}$ is independent of B_t , for $n \geq 1$ and $t_1, t_2, \dots, t_n \in [0, t]$.

For $p \geq 1$, we denote by $L_G^p(\Omega_T)$ the completion of $Lip(\Omega_T)$ under the natural norm

$$\|X\|_{L_G^p(\Omega_T)}^p := \widehat{E}[|X|^p],$$

and define the space $M_G^{0,p}(0, T)$ of \mathbb{F} -progressively measurable, \mathbb{R} -valued simple processes of the form

$$\eta(t) = \eta(t, \omega) = \sum_{i=0}^{n-1} \xi_{t_i}(\omega) \mathbb{1}_{[t_i, t_{i+1})}(t),$$

where $\{t_0, \dots, t_n\}$ is a subdivision of $[0, T]$. Denoted by $M_G^p(0, T)$ the closure of $M_G^{0,p}(0, T)$ with respect to the norm

$$\|\eta\|_{M_G^p(0, T)}^p := \widehat{E}\left[\int_0^T |\eta(t)|^p ds\right].$$

Note that $M_G^q(0, T) \subset M_G^p(0, T)$ if $1 \leq p < q$. For each $t \geq 0$, let $L^0(\Omega_t)$ be the set of F_t -measurable functions. We set

$$Lip(\Omega_t) := Lip(\Omega) \cap L^0(\Omega_t), \quad L_G^p(\Omega_t) := L_G^p(\Omega) \cap L^0(\Omega_t).$$

For each $\eta \in M_G^{0,2}(0, T)$, the related Itô integral of $(B_t)_{t \geq 0}$ is defined by

$$I(\eta) = \int_0^T \eta(s) dB_s := \sum_{j=0}^{N-1} \eta_j (B_{t_{j+1}} - B_{t_j}),$$

where the mapping $I : M_G^{0,2}(0, T) \rightarrow L_G^2(\Omega_T)$ is continuously extended to $M_G^2(0, T)$. The quadratic variation process $\langle B \rangle_t$ of $(B_t)_{t \geq 0}$, defined by

$$\langle B \rangle_t := B_t^2 - 2 \int_0^t B_s dB_s \tag{5}$$

For each $\eta \in M_G^{0,1}(0, T)$, let the mapping $\mathcal{J}_{0,T}(\eta) : M_G^{0,1}(0, T) \mapsto \mathbb{L}_G^1(\Omega_T)$ given by:

$$\mathcal{J}_{0,T}(\eta) = \int_0^T \eta(t) d\langle B \rangle_t := \sum_{j=0}^{N-1} \xi_j (\langle B \rangle_{t_{j+1}} - \langle B \rangle_{t_j}).$$

Then $\mathcal{J}_{0,T}(\eta)$ can be extended continuously to

$$\mathcal{J}_{0,T}(\eta) : M_G^1(0, T) \rightarrow \mathbb{L}_G^1(\Omega_T).$$

Lemma 4 ([14]). *We have for each $p \geq 1$*

$$\begin{aligned} \widehat{E} \left[\int_0^T \eta(t) d\langle B \rangle_t \right] &\leq \overline{\sigma}^2 \widehat{E} \left[\int_0^T |\eta(t)| dt \right], \text{ for each } \eta \in M_G^1(0, T). \\ \widehat{E} \left[\left(\int_0^T \eta(t) dB_t \right)^2 \right] &= \widehat{E} \left[\int_0^T \eta^2(t) d\langle B \rangle_t \right], \text{ for each } \eta \in M_G^2(0, T) \text{ (isometry)}. \\ \widehat{E} \left[\int_0^T |\eta(t)|^p dt \right] &\leq \int_0^T \widehat{E} [|\eta(t)|^p] dt, \text{ for each } \eta \in M_G^p(0, T). \end{aligned}$$

Proposition 5 ([7]). *For each $\xi \in \mathbb{L}_G^1(\Omega)$, there exists a weakly compact family of probability measures \mathcal{P} on $(\Omega, \mathcal{B}(\Omega))$ such that*

$$\widehat{E}[\xi] = \sup_{\mathbb{P} \in \mathcal{P}} E^{\mathbb{P}}[\xi].$$

Then, we define the associated regular choquet capacity related to \mathbb{P} :

$$c(C) := \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P}(C), \quad C \in \mathcal{B}(\Omega).$$

Definition 6. *A set $C \in \mathcal{B}(\Omega)$ is polar if $c(C) = 0$ or equivalently if $\mathbb{P}(C) = 0$ for all $\mathbb{P} \in \mathcal{P}$. A property holds quasi surely (*q.s.* in short) if it holds outside a polar set.*

Let define $\mathcal{N}_{\mathcal{P}}$ the \mathcal{P} -polar sets, as following

$$\mathcal{N}_{\mathcal{P}} := \bigcap_{\mathbb{P} \in \mathcal{P}} \mathcal{N}^{\mathbb{P}}(\mathcal{F}_{\infty}).$$

We must use the following universal filtration $\mathbb{F}^{\mathcal{P}}$ for the possibly mu-

tually singular probability measures $\mathbb{P}, \mathbb{P} \in \mathcal{P}$ in [17].

$$\begin{aligned}\mathbb{F}^{\mathcal{P}} &:= \{\widehat{\mathcal{F}}_t^{\mathcal{P}}\}_{t \geq 0}, \\ \widehat{\mathcal{F}}_t^{\mathcal{P}} &:= \bigcap_{\mathbb{P} \in \mathcal{P}} (\mathcal{F}_t^{\mathbb{P}} \vee \mathcal{N}_{\mathcal{P}}) \quad \text{for } t \geq 0.\end{aligned}$$

In view of the dual formulation of the G -expectation, we end this section by the following Burkholder-Davis-Gundy-type estimates, formulated in one dimension.

Proposition 7 ([10]).

- For each $p \geq 2$ and $\eta \in M_G^p(0, T)$, then there exists some constant C_p depending only on p and T such that

$$\widehat{E} \left[\sup_{s \leq u \leq t} \left| \int_s^u \eta_r dB_r \right|^p \right] \leq C_p |t - s|^{\frac{p}{2} - 1} \int_s^t \widehat{E}[|\eta_r|^p] dr.$$

- For each $p \geq 1$ and $\eta \in M_G^p(0, T)$, then there exists a positive constant $\bar{\sigma}$ such that $\frac{d\langle B \rangle_t}{dt} \leq \bar{\sigma}$ q.s., we have

$$\widehat{E} \left[\sup_{s \leq u \leq t} \left| \int_s^u \eta_r d\langle B \rangle_r \right|^p \right] \leq \bar{\sigma}^p |t - s|^{p-1} \int_s^t \widehat{E}[|\eta_r|^p] dr.$$

3. Formulation of the Problem

We study the existence of optimal control problem for G -NSFDE, given the following integral equation

$$\begin{aligned}X(t) &= \eta(0) + Q(t, X_t) - Q(0, \eta) + \int_0^t b(s, X_s, u(s)) ds \\ &\quad + \int_0^t \gamma(s, X_s, u(s)) d\langle B \rangle_s + \int_0^t \sigma(s, X_s) dB_s, t \in [0, T] \quad (6)\end{aligned}$$

with random initial data

$$\eta = \{\eta(\theta)\}_{-\tau \leq \theta \leq 0} \in BC([-\tau, 0]; \mathbb{R}),$$

with $BC([-\tau, 0]; \mathbb{R})$ is the space of \mathbb{R} -valued functions defined on $[-\tau, 0]$ and $\tau > 0$, where $X_t = \{X(t + \theta) : -\tau \leq \theta \leq 0\}$, and $u(t) \in \mathbb{A}$ is called a

strict control variable for each $t \in [0, T]$. Let the space

$$\tilde{\mathcal{H}}_T := \left\{ X = (X(t))_{t \in [0, T]}, \mathbb{F}^{\mathcal{P}} - \text{adapted such that: } \int_0^T \widehat{E} \left[|X(s)|^2 \right] ds < \infty \right\},$$

equipped with the norms $N_C(X) := \left(\int_0^T e^{-2Cs} \widehat{E} \left(|X(s)|^2 \right) ds \right)^{\frac{1}{2}}$, where $C \geq 0$. Since

$$e^{-2CT} N_0(X) \leq N_C(X) \leq N_0(X),$$

then these norms are equivalent. Moreover, the functions

$$\begin{aligned} Q, \sigma &: [0, T] \times BC([- \tau, 0]; \mathbb{R}) \times \Omega \rightarrow \mathbb{R}, \\ b, \gamma &: [0, T] \times BC([- \tau, 0]; \mathbb{R}) \times \mathbb{A} \times \Omega \rightarrow \mathbb{R}, \end{aligned}$$

are measurable, the random variable $Q(0, 0) \in L_G^2(\Omega_T)$ as well as $Q(\cdot, x)$, $\sigma(\cdot, x)$, $b(\cdot, x, u(\cdot))$, $\gamma(\cdot, x, u(\cdot)) \in \tilde{\mathcal{H}}_T$ for each $x \in BC([- \tau, 0]; \mathbb{R})$ and for each strict control u .

3.1. Problem of G-NSFDE relaxed control

In the absence of convexity assumptions, the strict control problem may not have an optimal solution because \mathbb{A} is too small to contain a minimizer. Then the space of strict controls must be injected into a wider space that has good properties of compactness and convexity. The set \mathbb{A} is a compact Polish space, and $\mathcal{P}(\mathbb{A})$ be the space of probability measures on \mathbb{A} , endowed with its Borel σ -algebra $\mathcal{B}(\mathbb{A})$. (For more details see [15]).

Next, we introduce the class of relaxed stochastic controls on $(\Omega, \mathcal{H}, \widehat{E})$.

Definition 8. A relaxed stochastic control on $(\Omega, \mathcal{H}, \widehat{E})$ is an $\mathbb{F}^{\mathcal{P}}$ -progressively measurable random measure of the form $q(\omega, dt, d\xi) = \mu_t(\omega, d\xi)dt$ such that

$$\begin{aligned} X(t) = & \eta(0) + Q(t, X_t) - Q(0, \eta) + \int_0^t \int_{\mathbb{A}} b(s, X_s, \xi) \mu_s(d\xi) ds \\ & + \int_0^t \int_{\mathbb{A}} \gamma(s, X_s, \xi) \mu_s(d\xi) d\langle B \rangle_s + \int_0^t \sigma(s, X_s) dB_s, \quad t \in [0, T]. \quad (7) \end{aligned}$$

Note that each strict control can be considered as a relaxed control via the mapping

$$\Phi(u)(dt, d\xi) = \delta_{u(t)}(d\xi) dt, \quad (8)$$

where $\delta_{u(t)}$ is a Dirac measure charging $u(t)$ for each t .

Remark 9. We mean by “the process $q(\omega, dt, d\xi)$ is $\mathbb{F}^{\mathcal{P}}$ - progressively measurable” that for every $C \in \mathcal{B}(\mathbb{A})$ and for every $t \in [0, T]$, the mapping $(s, \omega) \mapsto \mu_s(\omega, C)$ is $B([0, t]) \otimes \widehat{\mathcal{F}}_t^{\mathcal{P}}$ -measurable. In particular, the process $(\mu_t(C))_{t \in [0, T]}$ is adapted to $\mathbb{F}^{\mathcal{P}}$.

We denote by \mathcal{R} the class of relaxed stochastic controls.

3.2. Existence and uniqueness of solution for G-NSFDE

In order to consider control problem (6), we first study the question of existence and uniqueness of solution to the following equation

$$\begin{aligned} X(t) = & \eta(0) + Q(t, X_t) - Q(0, \eta) + \int_0^t \int_{\mathbb{A}} b(s, X_s, \xi) \mu_s(d\xi) ds \\ & + \int_0^t \int_{\mathbb{A}} \gamma(s, X_s, \xi) \mu_s(d\xi) d\langle B \rangle_s + \int_0^t \sigma(s, X_s) dB_s, t \in [0, T]. \end{aligned} \quad (9)$$

where $\mu_t(d\xi) = \delta_{u(t)}(d\xi)$.

To guarantee existence and uniqueness of the solution of the equation (6), we need the following assumptions:

(A₁) There exists $K_1 > 0$ such that

$$|H(t, x, u) - H(t, y, u)| \leq K_1 |x(0) - y(0)|,$$

uniformly with respect to (t, ω) for each $x, y \in BC([- \tau, 0]; \mathbb{R})$, where $H = b, \gamma, \sigma$.

(A₂) There exists $0 < k_0 < \frac{1}{4}$ such that

$$|Q(t, x) - Q(t, y)| \leq k_0 |x(0) - y(0)|, \quad (10)$$

uniformly with respect to (t, ω) for each $x, y \in BC([- \tau, 0]; \mathbb{R})$.

Note that, since $|Q(0, \eta)| \leq k_0 |\eta(0)| + |Q(0, 0)|$, then $Q(0, \eta) \in L_G^2(\Omega_T)$ for all $\eta \in BC([- \tau, 0]; \mathbb{R})$.

Remark 10. Indeed, the functions Q, b, γ , and σ defined by

$$\begin{aligned} b(t, x, u) &:= c(t, x(0), u), \\ \gamma(t, x, u) &:= \alpha(t, x(0), u), \\ \sigma(t, x) &:= \beta(t, x(0)), \end{aligned}$$

and

$$Q(t, x) := \lambda(t, x(0)),$$

such that, the functions c, α and β are K_1 -Lipschitz, and λ is k_0 -Lipschitz uniformly with respect to (t, ω) for each $x \in BC([- \tau, 0]; \mathbb{R})$, satisfies the assumptions (A_1) and (A_2) .

Theorem 11. Let the assumptions (A_1) and (A_2) be satisfied. Then, for each $u(t) \in \mathbb{A}$, the integral equation (6) has an unique solution $X^u \in \tilde{\mathcal{H}}_T$.

Proof. Let the mapping $\Theta : \tilde{\mathcal{H}}_T \rightarrow \tilde{\mathcal{H}}_T$ defined by: for each $t \in [0, T]$,

$$\begin{aligned} \Theta(X)(t) &= \eta(0) + Q(t, X_t) - Q(0, \eta) + \int_0^t b(s, X_s, u(s)) ds \\ &\quad + \int_0^t \gamma(s, X_s, u(s)) d\langle B \rangle_s + \int_0^t \sigma(s, X_s) dB_s. \end{aligned} \tag{11}$$

We have for all $X, \bar{X} \in \tilde{\mathcal{H}}_T$

$$\begin{aligned} &|\Theta(X)(t) - \Theta(\bar{X})(t)| \\ &\leq |Q(t, X_t) - Q(t, \bar{X}_t)| + \left| \int_0^t [b(s, X_s, u(s)) - b(s, \bar{X}_s, u(s))] ds \right| \\ &\quad + \left| \int_0^t [\gamma(s, X_s, u(s)) - \gamma(s, \bar{X}_s, u(s))] d\langle B \rangle_s \right| \\ &\quad + \left| \int_0^t [\sigma(s, X_s) - \sigma(s, \bar{X}_s)] dB_s \right|. \end{aligned} \tag{12}$$

Taking G -expectation on both sides, and using the following inequality

$$\left(\sum_{i=1}^k d_i \right)^2 \leq 2^{k-1} \sum_{i=1}^k d_i^2, \text{ for each } d_1 \dots d_k > 0 \quad (13)$$

we have

$$\begin{aligned} & \widehat{E} \left[\left| \Theta(X)(t) - \Theta(\overline{X})(t) \right|^2 \right] \\ & \leq 8\widehat{E} \left[\left| Q(t, X_t) - Q(t, \overline{X}_t) \right|^2 \right] \\ & \quad + 8\widehat{E} \left[\left| \int_0^t [b(s, X_s, u(s)) - b(s, \overline{X}_s, u(s))] ds \right|^2 \right] \\ & \quad + 8\widehat{E} \left[\left| \int_0^t [\gamma(s, X_s, u(s)) - \gamma(s, \overline{X}_s, u(s))] d\langle B \rangle_s \right|^2 \right] \\ & \quad + 8\widehat{E} \left[\left| \int_0^t [\sigma(s, X_s) - \sigma(s, \overline{X}_s)] dB_s \right|^2 \right] \\ & := 8 \sum_{i=1}^4 U_i. \end{aligned} \quad (14)$$

Now, we have by assumption (A_2)

$$U_1 \leq k_0^2 \widehat{E} \left[|X(t) - \overline{X}(t)|^2 \right] \quad (15)$$

By applying Hölder inequality and (A_1) , we have

$$\begin{aligned} U_2 & \leq T \int_0^T \widehat{E} \left[\left| [b(s, X_s, u(s)) - b(s, \overline{X}_s, u(s))] \right|^2 \right] ds \\ & \leq TK_1^2 \int_0^T \widehat{E} \left[|X(s) - \overline{X}(s)|^2 \right] ds. \end{aligned} \quad (16)$$

Similarly, by using the G -BDG inequalities, we obtain

$$\begin{aligned} U_3 + U_4 & \leq T\overline{\sigma}^2 \int_0^T \widehat{E} \left[\left| [\gamma(s, X_s, u(s)) - \gamma(s, \overline{X}_s, u(s))] \right|^2 \right] ds \\ & \quad + C_2 \int_0^T \widehat{E} \left[\left| [\sigma(s, X_s) - \sigma(s, \overline{X}_s)] \right|^2 \right] ds, \\ & = K_1^2 [T\overline{\sigma}^2 + C_2] \int_0^T \widehat{E} \left[|X(s) - \overline{X}(s)|^2 \right] ds. \end{aligned} \quad (17)$$

Combining (15), (16), and (17), we get

$$\begin{aligned} \widehat{E} \left[|\Theta(X)(t) - \Theta(\overline{X})(t)|^2 \right] &\leq 8k_0^2 \widehat{E} \left[|X(t) - \overline{X}(t)|^2 \right] \\ &\quad + C \int_0^T \widehat{E} \left[|X(s) - \overline{X}(s)|^2 \right] ds. \end{aligned} \quad (18)$$

where $C = 8K_1^2 (T + T\sigma^2 + C_2)$.

Multiplying by e^{-2Ct} both sides of inequality (18) and integrating on $[0, T]$, we obtain

$$\begin{aligned} N_C^2 [\Theta(X) - \Theta(\overline{X})] &\leq 8k_0^2 N_C^2 [X - \overline{X}] + C \int_0^T e^{-2Ct} \left(\int_0^t \widehat{E} \left[|X(s) - \overline{X}(s)|^2 \right] ds \right) dt \\ &\leq 8k_0^2 N_C^2 [X - \overline{X}] + C \int_0^T \left(\widehat{E} \left[|X(s) - \overline{X}(s)|^2 \right] \int_s^T e^{-2Ct} dt \right) ds \\ &\leq 8k_0^2 N_C^2 [X - \overline{X}] + \int_0^T \widehat{E} \left[|X(s) - \overline{X}(s)|^2 \right] \left(\frac{e^{-2Cs} - e^{-2CT}}{2} \right) ds \\ &\leq 8k_0^2 N_C^2 [X - \overline{X}] + \frac{1}{2} N_C^2 [X - \overline{X}]. \end{aligned} \quad (19)$$

Thus, we obtain the following estimation

$$N_C [\Theta(X) - \Theta(\overline{X})] \leq \sqrt{8k_0^2 + \frac{1}{2}} N_C [X - \overline{X}].$$

We have, by using Hölder inequality,

$$\begin{aligned} N_0^2 \left(\int_0^{\cdot} b(s, 0, u(s)) ds \right) &= \int_0^T \widehat{E} \left[\left| \int_0^t b(s, 0, u(s)) ds \right|^2 \right] dt \\ &\leq T \int_0^T \int_0^t \widehat{E} \left[|b(s, 0, u(s))|^2 \right] ds dt \\ &\leq T^2 N_0^2 (b(\cdot, 0, u(\cdot))). \end{aligned}$$

Similarly, it is easy to check, by G -BDG inequalities, that

$$N_0^2 \left(\int_0^\cdot \gamma(s, 0, u(s)) d\langle B \rangle_s \right) \leq \bar{\sigma}^2 T^2 N_0^2(\gamma(\cdot, 0, u(\cdot)))$$

and

$$N_0^2 \left(\int_0^\cdot \sigma(s, 0) dB_s \right) \leq C_2 T N_0^2(\sigma(\cdot, 0)).$$

Now observe that,

$$\begin{aligned} \Theta(0)(t) &= \eta(0) + Q(t, 0) - Q(0, \eta) + \int_0^t b(s, 0, u(s)) ds \\ &\quad + \int_0^t \gamma(s, 0, u(s)) d\langle B \rangle_s + \int_0^t \sigma(s, 0) dB_s. \end{aligned}$$

It follows that

$$\begin{aligned} N_0(\Theta(0)) &\leq \sqrt{T} \left(\|Q(0, \eta)\|_{L_G^2(\Omega_T)} + |\eta(0)| \right) + N_0(Q(\cdot, 0)) \\ &\quad + TN_0(b(\cdot, 0, u(\cdot))) + \bar{\sigma} T N_0(\gamma(\cdot, 0, u(\cdot))) + \sqrt{C_2 T} N_0(\sigma(\cdot, 0)), \end{aligned}$$

then the process $\Theta(0) \in \tilde{\mathcal{H}}_T$, so that if $X \in \tilde{\mathcal{H}}_T$ then

$$N_C(\Theta(X)) \leq N_C(\Theta(X) - \Theta(0)) + N_C(\Theta(0)) \leq N_0(X) + N_0(\Theta(0)) < \infty.$$

This means that $\Theta(X) \in \tilde{\mathcal{H}}_T$, which implies that Θ is well defined.

Finally, taking into account the fact that $\sqrt{8k_0^2 + \frac{1}{2}} < 1$ and assumption (A_2) , we deduce that $\Theta(X)$ is a contraction on $\tilde{\mathcal{H}}_T$, then the fixed point $X^u \in \tilde{\mathcal{H}}_T$ is the unique solution of (7). The proof is completed. \square

3.3. Relaxed control problem

In this section, we consider a relaxed control problem (7). Let X^μ denote the solution of equation (7) associated with the relaxed control. We establish

the existence of a minimizer of the cost corresponding to μ .

$$J(\mu) = \widehat{E} \left[\int_0^T \int_{\mathbb{A}} \mathcal{L}(t, X_t^\mu, \xi) \mu_t(d\xi) dt + \Psi(X_T^\mu) \right],$$

the functions,

$$\begin{aligned} \mathcal{L} &: [0, T] \times BC([-\tau, 0]; \mathbb{R}) \times \mathbb{A} \rightarrow \mathbb{R}, \\ \Psi &: BC([-\tau, 0]; \mathbb{R}) \rightarrow \mathbb{R}, \end{aligned}$$

satisfy the following assumption:

(A₃) \mathcal{L}, Ψ are bounded and for each $t \in [0, T]$ and $x \in BC([-\tau, 0]; \mathbb{R})$ the functions $\mathcal{L}(t, x, \cdot), \Psi(x)$ are continuous. Additionally, we suppose that:

$$|\mathcal{L}(t, x, u) - \mathcal{L}(t, y, u)| + |\Psi(x) - \Psi(y)| \leq K_1 |x(0) - y(0)|.$$

We recall that in the strict control problem

$$J(u) = \widehat{E} \left[\int_0^T \mathcal{L}(t, X_t^u, u(t)) dt + \Psi(X_T^u) \right] \tag{20}$$

over the set \mathcal{U} ,

$$\begin{aligned} X^u(t) &= \eta(0) + Q(t, X_t^u) - Q(0, \eta) + \int_0^t b(s, X_s^u, u(s)) ds \\ &\quad + \int_0^t \gamma(s, X_s^u, u(s)) d\langle B \rangle_s + \int_0^t \sigma(s, X_s^u) dB_s \end{aligned} \tag{21}$$

then, we have

$$\begin{aligned} X^\mu(t) &= \eta(0) + Q(t, X_t^\mu) - Q(0, \eta) + \int_0^t \int_{\mathbb{A}} b(s, X_s^\mu, \xi) \mu_s(d\xi) ds \\ &\quad + \int_0^t \int_{\mathbb{A}} \gamma(s, X_s^\mu, \xi) \mu_s(d\xi) d\langle B \rangle_s + \int_0^t \sigma(s, X_s^\mu) dB_s. \end{aligned} \tag{22}$$

We suppose as well that the coefficients of the *G*-NSFDE verify the following condition

(A₄) The coefficients b, γ, σ are bounded and for every fixed $t \in [0, T]$ and $x \in BC([-\tau, 0]; \mathbb{R})$ the functions $b(t, x, \cdot), \gamma(t, x, \cdot)$ and $\sigma(t, x)$ are continuous *q.s.*

3.4. Approximation and existence of relaxed optimal control

By introducing the relaxed control problem, the next lemma, which extends the celebrated Chattering Lemma, states that each relaxed control in \mathcal{R} can be approximated by strict controls.

Definition 12. (stable convergence) Let $\mu^n, \mu \in \mathcal{R}, n \in \mathbb{N}^*$. We say that, we have a stable convergence, if for any continuous function $f : [0, T] \times \mathbb{A} \rightarrow \mathbb{R}$, we have

$$\lim_{n \rightarrow \infty} \int_{[0, T] \times \mathbb{A}} f(t, \xi) \mu^n(dt, d\xi) = \int_{[0, T] \times \mathbb{A}} f(t, \xi) \mu(dt, d\xi) \quad (23)$$

Lemma 13 ([15], (G-Chattering Lemma)). Let (\mathbb{A}, d) be a separable compact metric space. Let $(\mu_t)_{t \geq 0}$ be an $\mathbb{F}^{\mathcal{P}}$ -progressively measurable process taking values in $\mathcal{P}(\mathbb{A})$. Then there exists a sequence $(u^n(t))_{n \geq 0}$ of $\mathbb{F}^{\mathcal{P}}$ -progressively measurable processes taking values in \mathbb{A} , such that the sequence of random measures $\delta_{u^n(t)}(d\xi)dt$ converges in the sense of stable convergence (thus weakly) to $\mu_t(d\xi)dt$ q.s.

Taking use of the fact that under $\mathbb{P} \in \mathcal{P}$, B is a continuous martingale with a quadratic variation process $\langle B \rangle$ such that $c_t := \frac{d\langle B \rangle_t}{dt}$ is bounded. Let X^μ and X^n the corresponding solutions satisfy the following integral equations type of G-NSFDEs:

$$\begin{aligned} X^\mu(t) &= \eta(0) + Q(t, X_t^\mu) - Q(0, \eta) \\ &\quad + \int_0^t \int_{\mathbb{A}} (b(s, X_s^\mu, \xi) + c_s \gamma(s, X_s^\mu, \xi)) \mu_s(d\xi) ds + \int_0^t \sigma(s, X_s^\mu) dB_s \end{aligned} \quad (24)$$

and

$$\begin{aligned} X^n(t) &= \eta(0) + Q(t, X_t^n) - Q(0, \eta) \\ &\quad + \int_0^t \int_{\mathbb{A}} (b(s, X_s^n, \xi) + c_s \gamma(s, X_s^n, \xi)) \delta_{u^n(s)}(d\xi) ds + \int_0^t \sigma(s, X_s^n) dB_s \end{aligned} \quad (25)$$

with random initial data

$$X_0^\mu = X_0^n = \eta \in BC([- \tau, 0]; \mathbb{R}). \quad (26)$$

Lemma 14 (stability results). Let μ be a relaxed control, and let (u^n) be a sequence defined as in (**G-Chattering Lemma**). Then we have

(i) For every $\mathbb{P} \in \mathcal{P}$, it holds that

$$\lim_{n \rightarrow \infty} E^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} |X^n(t) - X^\mu(t)|^2 \right] = 0 \tag{27}$$

and

$$\lim_{n \rightarrow \infty} \widehat{E} \left[\sup_{0 \leq t \leq T} |X^n(t) - X^\mu(t)|^2 \right] = 0. \tag{28}$$

(ii) Let $J(u^n)$ and $J(\mu)$ be the corresponding cost functionals to u^n and μ respectively. Then, there exists a subsequence (u^{n_k}) of (u^n) such that for every $\mathbb{P} \in \mathcal{P}$

$$\lim_{k \rightarrow \infty} J^{\mathbb{P}}(u^{n_k}) = J^{\mathbb{P}}(\mu) \tag{29}$$

and

$$\lim_{k \rightarrow \infty} J(u^{n_k}) = J(\mu). \tag{30}$$

Moreover,

$$\inf_{u \in \mathcal{U}} J^{\mathbb{P}}(u) = \inf_{\mu \in \mathcal{R}} J^{\mathbb{P}}(\mu) \tag{31}$$

and there exists a relaxed control $\widehat{\mu}_{\mathbb{P}} \in \mathcal{R}$ such that

$$J^{\mathbb{P}}(\widehat{\mu}_{\mathbb{P}}) = \inf_{\mu \in \mathcal{R}} J^{\mathbb{P}}(\mu). \tag{32}$$

Proof.

(i) The proof of this result is inspired by [15]. Subtracting (24) from (25) term by term, we have

$$\begin{aligned} X^n(t) - X^\mu(t) &= [Q(t, X_t^n) - Q(t, X_t^\mu)] \\ &\quad + \int_0^t \int_{\mathbb{A}} (b(s, X_s^n, \xi) + c_s \gamma(s, X_s^n, \xi)) \delta_{u^n(s)}(d\xi) ds \\ &\quad - \int_0^t \int_{\mathbb{A}} (b(s, X_s^\mu, \xi) + c_s \gamma(s, X_s^\mu, \xi)) \mu_s(d\xi) ds \\ &\quad + \int_0^t [\sigma(s, X_s^n) - \sigma(s, X_s^\mu)] dB_s \\ &= [Q(t, X_t^n) - Q(t, X_t^\mu)] + \mathcal{I}_n(t). \end{aligned} \tag{33}$$

Taking G -expectation on both sides and using the assumptions (A_1) and (A_2) , it follows that

$$\begin{aligned} \widehat{E} \left[\sup_{0 \leq t \leq T} |X^n(t) - X^\mu(t)|^2 \right] &\leq 2k_0^2 \widehat{E} \left[\sup_{0 \leq t \leq T} |X^n(t) - X^\mu(t)|^2 \right] \\ &\quad + 2\widehat{E} \left[\sup_{0 \leq t \leq T} |\mathcal{I}_n(s)|^2 \right] \end{aligned}$$

then

$$\widehat{E} \left[\sup_{0 \leq t \leq T} |X^n(t) - X^\mu(t)|^2 \right] \leq \frac{2}{(1 - 2k_0^2)} \widehat{E} \left[\sup_{0 \leq t \leq T} |\mathcal{I}_n(s)|^2 \right]. \quad (34)$$

We have

$$\begin{aligned} &\widehat{E} \left[\sup_{0 \leq t \leq T} |\mathcal{I}_n(s)|^2 \right] \\ &\leq \widehat{E} \left(\sup_{0 \leq t \leq T} \left| \int_0^t \int_{\mathbb{A}} (b(s, X_s^n, \xi) + c_s \gamma(s, X_s^n, \xi)) \delta_{u^n(s)}(d\xi) ds \right. \right. \\ &\quad \left. \left. - \int_0^t \int_{\mathbb{A}} (b(s, X_s^\mu, \xi) + c_s \gamma(s, X_s^\mu, \xi)) \mu_s(d\xi) ds \right|^2 \right) \\ &\quad + \widehat{E} \left(\sup_{0 \leq t \leq T} \left| \int_0^t [\sigma(s, X_s^n) - \sigma(s, X_s^\mu)] dB_s \right|^2 \right) \\ &\leq \widehat{E} \left(\sup_{0 \leq t \leq T} \left| \int_0^t \int_{\mathbb{A}} (b(s, X_s^n, \xi) + c_s \gamma(s, X_s^n, \xi)) \delta_{u^n(s)}(d\xi) ds \right. \right. \\ &\quad \left. \left. - \int_0^t \int_{\mathbb{A}} (b(s, X_s, \xi) + c_s \gamma(s, X_s, \xi)) \delta_{u^n(s)}(d\xi) ds \right|^2 \right) \\ &\quad + \widehat{E} \left(\sup_{0 \leq t \leq T} \left| \int_0^t \int_{\mathbb{A}} (b(s, X_s, \xi) + c_s \gamma(s, X_s, \xi)) \delta_{u^n(s)}(d\xi) ds \right. \right. \\ &\quad \left. \left. - \int_0^t \int_{\mathbb{A}} (b(s, X_s^\mu, \xi) + c_s \gamma(s, X_s^\mu, \xi)) \mu_s(d\xi) ds \right|^2 \right) \\ &\quad + \widehat{E} \left(\sup_{0 \leq t \leq T} \left| \int_0^t [\sigma(s, X_s^n) - \sigma(s, X_s^\mu)] dB_s \right|^2 \right). \end{aligned} \quad (35)$$

Let $\varepsilon > 0$. Then, there exists $\mathbb{P}^\varepsilon \in \mathcal{P}$ such that

$$\begin{aligned}
& \widehat{E} \left[\sup_{0 \leq t \leq T} |\mathcal{I}_n(s)|^2 \right] \\
& \leq E^{\mathbb{P}^\varepsilon} \left(\sup_{0 \leq t \leq T} \left| \int_0^t \int_{\mathbb{A}} (b(s, X_s^n, \xi) + c_s \gamma(s, X_s^n, \xi)) \delta_{u^n(s)}(d\xi) ds \right. \right. \\
& \quad \left. \left. - \int_0^t \int_{\mathbb{A}} (b(s, X_s, \xi) + c_s \gamma(s, X_s, \xi)) \delta_{u^n(s)}(d\xi) ds \right|^2 \right) \\
& \quad + E^{\mathbb{P}^\varepsilon} \left(\sup_{0 \leq t \leq T} \left| \int_0^t \int_{\mathbb{A}} b(s, X_s, \xi) \delta_{u^n(s)}(d\xi) ds \right. \right. \\
& \quad \left. \left. - \int_0^t \int_{\mathbb{A}} b(s, X_s^\mu, \xi) \mu_s(d\xi) ds \right|^2 \right) \\
& \quad + E^{\mathbb{P}^\varepsilon} \left(\sup_{0 \leq t \leq T} \left| \int_0^t \int_{\mathbb{A}} c_s \gamma(s, X_s, \xi) \delta_{u^n(s)}(d\xi) ds \right. \right. \\
& \quad \left. \left. - \int_0^t \int_{\mathbb{A}} c_s \gamma(s, X_s^\mu, \xi) \mu_s(d\xi) ds \right|^2 \right) \\
& \quad + E^{\mathbb{P}^\varepsilon} \left(\sup_{0 \leq t \leq T} \left| \int_0^t [\sigma(s, X_s^n) - \sigma(s, X_s^\mu)] dB_s \right|^2 \right) + \varepsilon. \tag{36}
\end{aligned}$$

Then, we have

$$\begin{aligned}
& \widehat{E} \left[\sup_{0 \leq t \leq T} |\mathcal{I}_n(s)|^2 \right] \\
& \leq 16E^{\mathbb{P}^\varepsilon} \left(\sup_{0 \leq t \leq T} \left| \int_0^t \int_{\mathbb{A}} (b(s, X_s^n, \xi) + c_s \gamma(s, X_s^n, \xi)) \delta_{u^n(s)}(d\xi) ds \right. \right. \\
& \quad \left. \left. - \int_0^t \int_{\mathbb{A}} (b(s, X_s, \xi) + c_s \gamma(s, X_s, \xi)) \delta_{u^n(s)}(d\xi) ds \right|^2 \right) \\
& \quad + 16E^{\mathbb{P}^\varepsilon} \left(\sup_{0 \leq t \leq T} \left| \left[\int_0^t \int_{\mathbb{A}} b(s, X_s, \xi) \delta_{u^n(s)}(d\xi) ds \right. \right. \right. \\
& \quad \left. \left. \left. - \int_0^t \int_{\mathbb{A}} b(s, X_s^\mu, \xi) \mu_s(d\xi) ds \right] \right|^2 \right)
\end{aligned}$$

$$\begin{aligned}
& +16E^{\mathbb{P}^\varepsilon} \left(\sup_{0 \leq t \leq T} \left| \left[\int_0^t \int_{\mathbb{A}} c_s \gamma(s, X_s, \xi) \delta_{u^n(s)}(d\xi) ds \right. \right. \right. \\
& \quad \left. \left. \left. - \int_0^t \int_{\mathbb{A}} c_s \gamma(s, X_s^\mu, \xi) \mu_s(d\xi) ds \right] \right|^2 \right) \\
& +16E^{\mathbb{P}^\varepsilon} \left(\sup_{0 \leq t \leq T} \left| \int_0^t [\sigma(s, X_s^n) - \sigma(s, X_s^\mu)] dB_s \right|^2 \right) + 16\varepsilon^2 \\
& = 16 \{ (\mathcal{I}_{(n,1)} + \mathcal{I}_{(n,2)} + \mathcal{I}_{(n,3)} + \mathcal{I}_{(n,4)}) + \varepsilon^2 \}. \tag{37}
\end{aligned}$$

Since b, γ are bounded and continuous in the control variable ξ , then, by using the dominated convergence theorem, and the stable convergence of $\delta_{u^n(t)}(d\xi) dt$ to $\mu_t(d\xi) dt$, we have

$$\lim_{n \rightarrow \infty} \mathcal{I}_{(n,2)} = \lim_{n \rightarrow \infty} \mathcal{I}_{(n,3)} = 0. \tag{38}$$

Similarly, we use the assumption (A_1) , then

$$\lim_{n \rightarrow \infty} (\mathcal{I}_{(n,1)} + \mathcal{I}_{(n,4)}) \leq K_1^2 \lim_{n \rightarrow \infty} \left[E^{\mathbb{P}^\varepsilon} \left(\int_0^T |X^n(s) - X^\mu(s)|^2 ds \right) + \varepsilon^2 \right]. \tag{39}$$

It follows, by using dominated convergence theorem, that

$$\begin{aligned}
& \lim_{n \rightarrow \infty} E^{\mathbb{P}^\varepsilon} \left[\int_0^T |X^n(s) - X^\mu(s)|^2 ds \right] \\
& \leq \int_0^T \lim_{n \rightarrow \infty} E^{\mathbb{P}^\varepsilon} \left[|X^n(s) - X^\mu(s)|^2 \right] ds \\
& \leq \int_0^T \lim_{n \rightarrow \infty} \widehat{E} \left[\sup_{0 \leq \nu \leq s} |X^n(\nu) - X^\mu(\nu)|^2 \right] ds. \tag{40}
\end{aligned}$$

Taking $Z(\delta) = \lim_{n \rightarrow \infty} \widehat{E} \left[\sup_{0 \leq t \leq \delta} |X^n(t) - X^\mu(t)|^2 \right]$, for each $\delta > 0$, then we deduce from the formulas (39) and (40), that

$$Z(T) \leq \frac{32K_1^2}{1 - 2k_0^2} \left(\int_0^T Z(s) ds + \varepsilon^2 \right) \tag{41}$$

using Gronwall's lemma, we conclude that

$$\lim_{n \rightarrow \infty} \widehat{E} \left[\sup_{0 \leq t \leq T} |X^n(t) - X^\mu(t)|^2 \right] = 0. \tag{42}$$

(ii) Property (i) implies that there exists a subsequence $(X^{n_k}(t))_{n_k}$ that converges to $X^\mu(t)$ *q.s.*, and uniformly in t . We have, for all $\mathbb{P} \in \mathcal{P}$

$$\begin{aligned} & \left| J^\mathbb{P}(u^{n_k}) - J^\mathbb{P}(\mu) \right| \\ & \leq E^\mathbb{P} \left[\int_0^T \int_{\mathbb{A}} |\mathcal{L}(t, X_t^{n_k}, \xi) - \mathcal{L}(t, X_t, \xi)| \delta_{u^{n_k}(t)}(d\xi) dt \right] \\ & \quad + E^\mathbb{P} \left[\left| \int_0^T \int_{\mathbb{A}} \mathcal{L}(t, X_t, \xi) \delta_{u^{n_k}(t)}(d\xi) dt - \int_0^T \int_{\mathbb{A}} \mathcal{L}(t, X_t^\mu, \xi) \mu_t(d\xi) dt \right| \right] \\ & \quad + E^\mathbb{P} [|\Psi(X_T^{n_k}) - \Psi(X_T^\mu)|]. \end{aligned} \tag{43}$$

The first and third terms in the right-hand side converge to 0 as a result of the continuity and boundness assumptions on \mathcal{L} and Ψ with respect to X that. And, the second term on the right-hand side tends to 0, due to the continuity and the boundness of \mathcal{L} in the variable ξ , and by the weak convergence of $\delta_{u^{n_k}(t)}(d\xi)dt$ to $\mu_t(d\xi)dt$, we use the dominated convergence theorem to conclude.

Using **Lemma 14 (stability results)**, we obtain for all $\mathbb{P} \in \mathcal{P}$,

$$\lim_{k \rightarrow \infty} J^\mathbb{P}(u^{n_k}) = J^\mathbb{P}(\mu) \tag{44}$$

then,

$$\lim_{k \rightarrow \infty} J(u^{n_k}) = J(\mu), \tag{45}$$

we have $J^\mathbb{P}(u) = J^\mathbb{P}(\delta_u)$, This yields $\inf_{u \in \mathcal{U}} J^\mathbb{P}(u) \geq \inf_{\mu \in \mathcal{R}} J^\mathbb{P}(\mu)$. Given an arbitrary $\mu \in \mathcal{R}$. From **Lemma 13 (G-Chattering Lemma)**, to obtain a sequence of strict controls $(u^{n_k}) \subset \mathcal{U}$ such that $\delta_{u^{n_k}(t)}(d\xi)dt$ converges weakly to $\mu_t(d\xi)dt$, we obtain

$$J^\mathbb{P}(\mu) = \lim_{n \rightarrow \infty} J^\mathbb{P}(u^n) \geq \inf_{u \in \mathcal{U}} J^\mathbb{P}(u) \tag{46}$$

since μ is arbitrary, we have:

$$\inf_{\mu \in \mathcal{R}} J^{\mathbb{P}}(\mu) \geq \inf_{u \in \mathcal{U}} J^{\mathbb{P}}(u). \quad (47)$$

□

The main result is to give the following theorem. Note that this result extends to G -NSFDEs with an uncontrolled diffusion coefficient. We show that an optimal solution for the relaxed control problem exists, the proof is based on the existence of optimal relaxed control for each $\mathbb{P} \in \mathcal{P}$ and a tightness argument.

Theorem 15. *For every $u \in \mathcal{U}$ and $\mu \in \mathcal{R}$, we have*

$$\inf_{u \in \mathcal{U}} J(u) = \inf_{\mu \in \mathcal{R}} J(\mu). \quad (48)$$

Moreover, there exists a relaxed control $\hat{\mu} \in \mathcal{R}$ such that

$$J(\hat{\mu}) = \min_{\mu \in \mathcal{R}} J(\mu) \quad (49)$$

recall that

$$J(\mu) = \sup_{\mathbb{P} \in \mathcal{P}} J^{\mathbb{P}}(\mu) \quad (50)$$

where for each $\mathbb{P} \in \mathcal{P}$, the relaxed cost functional is given as follow

$$J^{\mathbb{P}}(\mu) = E^{\mathbb{P}} \left[\int_0^T \int_{\mathbb{A}} \mathcal{L}(t, X_t^{\mu}, \xi) \mu_t(d\xi) dt + \Psi(X_T^{\mu}) \right]. \quad (51)$$

Let $(\mu^n, X^{\mu^n})_{n \geq 0}$ be a minimizing sequence of $\inf_{\mu \in \mathcal{R}} J^{\mathbb{P}}(\mu)$ such that

$$\lim_{n \rightarrow \infty} J^{\mathbb{P}}(\mu^n) = \inf_{\mu \in \mathcal{R}} J^{\mathbb{P}}(\mu) \quad (52)$$

where X^{μ^n} is the unique solution of (7), corresponding to the random variables μ^n which belongs to the compact set M .

The proof of the existence of an optimal relaxed control entails demonstrating that the sequence of distributions of the processes $(\mu^n, X^{\mu^n})_{n \geq 0}$ is tight for a given topology on the state space and then proving that we can extract a subsequence that converges in law to a process (μ, X^{μ}) , that

satisfies (22). To achieve the proof, we show that under some regularity conditions of $(J^{\mathbb{P}}(\mu^n))_n$ converges to $J^{\mathbb{P}}(\hat{\mu})$ which is equal to $\inf_{\mu \in \mathcal{R}} J^{\mathbb{P}}(\mu)$ and then $(\hat{\mu}, X^{\hat{\mu}})$ is optimal.

Lemma 16 ([2]). *The sequence of distributions of the relaxed controls $(\mu^n)_{n \geq 0}$ is relatively compact in M .*

Proof of Theorem 15. The relaxed controls μ^n are random variables in the compact set M . Then by Prohorov’s theorem the associated family of distribution $(\mu^n)_{n \geq 0}$ is tight on the space M , then it is relatively compact in M . Thus, there exists a subsequence $(\mu^{n_k}, X^{\mu^{n_k}})_{k \geq 0}$ of $(\mu^n, X^{\mu^n})_{n \geq 0}$ that weakly converges to $(\hat{\mu}, X^{\hat{\mu}})$ which solves (22). Using Skorohod’s embedding theorem, the continuity and boundness assumptions of the functions \mathcal{L} and Ψ , and Lebesgue Dominated Convergence Theorem, we finally obtain :

$$\inf_{\mu \in \mathcal{R}} J^{\mathbb{P}}(\mu) = \lim_{k \rightarrow \infty} J^{\mathbb{P}}(\mu^{n_k}) = J^{\mathbb{P}}(\hat{\mu}).$$

Then, from **Lemma 14 (stability results)**, for every $\mathbb{P} \in \mathcal{P}$ there exists a relaxed control $\hat{\mu} \in \mathcal{R}$ such that

$$\hat{\mu}_{\mathbb{P}} = \arg \min_{\mu \in \mathcal{R}} J^{\mathbb{P}}(\mu).$$

Then, we conclude that

$$J(\hat{\mu}) = \min_{\mu \in \mathcal{R}} J(\mu). \quad \square$$

Remark 17. The relaxed model is a real extension of the strict model, as the infimum of the two cost functions are equal, and the relaxed model has an optimal solution, as shown by the prior results.

4. Numerical Analysis: Euler-Maruyama Method for G-Neutral SFDEs

In this section we present a numerical analysis of a G -NSFDE. The idea is to use the Euler-Maruyama scheme to solve the G -NSFDE (1). Let $\tau > 0, T > \tau, N \in \mathbb{N}, h = \frac{T+\tau}{N}$ and $t_0 = -\tau, t_1 = -\tau + h, \dots, t_{N_0} = 0, \dots, t_N =$

T be a discretization of the interval $[-\tau, T]$. Consider the following Euler-Maruyama scheme:

$$\left\{ \begin{array}{l} \text{Given an initial data } \eta : [-\tau, 0] \rightarrow \mathbb{R}^n, \text{ and put } X(t) = \eta(t) \\ \hspace{15em} \text{for } t = t_0, t_1, \dots, t_{N_0} \\ \text{Now for } i = N_0, N_0 + 1 \dots N \\ X(t_{i+1}) = X(t_i) + Q(t_{i+1}, X_{t_{i+1}}) - Q(t_i, X_{t_i}) + b(t_i, X_{t_i})h \\ \hspace{10em} + \gamma(t_i, X_{t_i}) \langle B \rangle_{t_i} + \sigma(t_i, X_{t_i}) (B_{t_{i+1}} - B_{t_i}) \end{array} \right. \quad (53)$$

where, $X_{t_i} := \{X_{t_i}(\lambda) : -\tau \leq \lambda \leq 0\}$, $X_{t_i}(\lambda) := X(t_{i+k}) + \frac{\lambda - t_k}{h} [X(t_{i+k+1}) - X(t_{i+k})]$, k is such that $t_k \leq \lambda < t_{k+1}$. In order that our algorithm works we have to give a value for X_{t_0-h} , we can set it equal $X_{t_0-h} = X_{t_0} = \eta(-\tau)$.

For the simulation of the increments of the G -Brownian motion and its quadratic variation we follow the same method given by [19] by simulating its corresponding G -PDE using finite difference.

In Figure 1 (resp. Figure 2) we represent the simulation of the density (resp. distribution) of the G -Normal BM for $\sigma_{\min} = 0.8$ and different values of σ_{\max} .

Simulation of the G-Normal density and distribution for $\sigma_{\min} = 0.8$ and different σ_{\max}

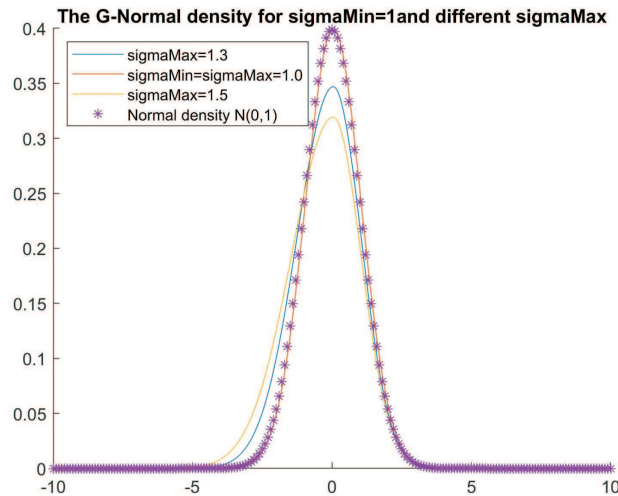


Figure 1: G-Normal density.

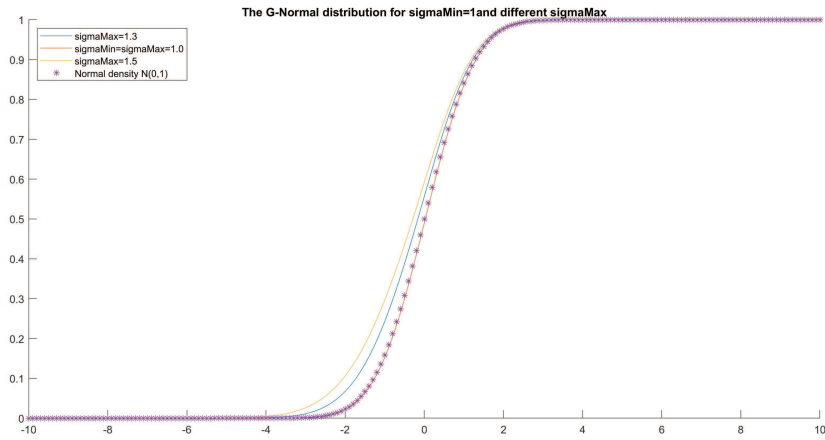


Figure 2: *G*-Normal distribution

Also, in Figure 3 (resp. Figure 4) we represent the simulation of the density (resp. distribution) of the *G*-Normal BM for $\sigma_{\max} = 1.3$ and different values of σ_{\min} .

Simulation of the *G*-Normal density and distribution for $\sigma_{\max} = 1.3$ and different σ_{\min}

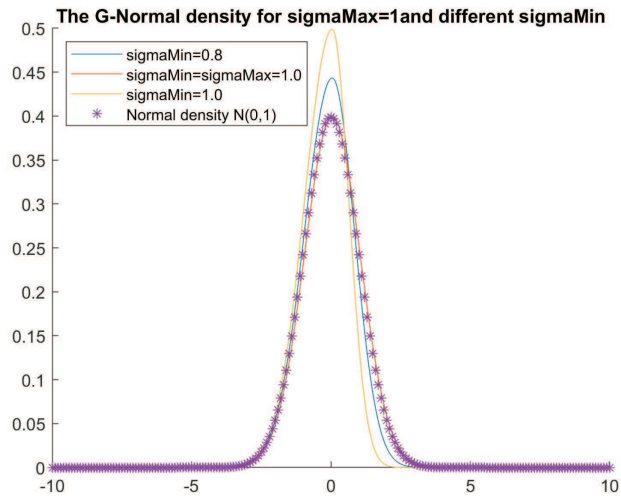


Figure 3: *G*-Normal density.

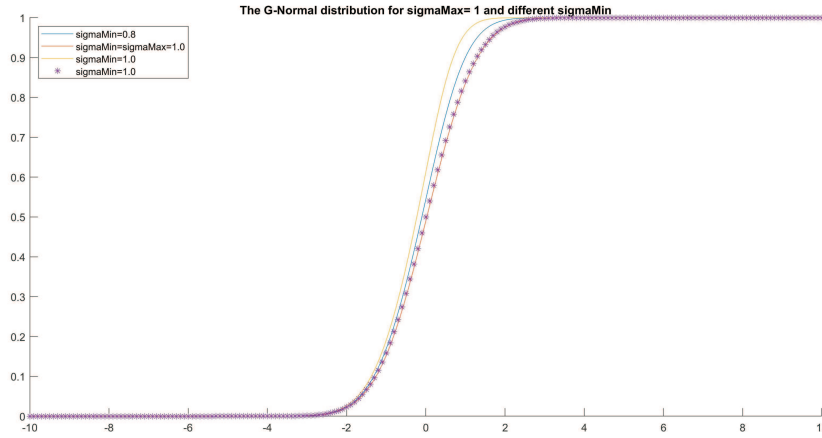


Figure 4: G-Normal distribution

Now, let take in this part of this section, $T = 1, \tau = 0.1$ and the coefficients of the G -NSFDE (1) given by:

$$\begin{aligned}
 Q(t, X_t) &:= 0.3 \int_{t-\tau}^t X(s) ds \\
 b(t, X_t) &:= 10 \int_{t-\tau}^t X(s) ds \\
 \gamma(t, X_t) &:= 0.4 \int_{t-\tau}^t X(s) ds \\
 \sigma(t, X_t) &:= 5 \int_{t-\tau}^t X(s) ds.
 \end{aligned}$$

For these given data and coefficients we get the following results:

In Figure 5 (resp. Figure 6) we represent the trajectories of the solution of the G -NSFDE where the G -Brownian motion is with $\sigma_{\max} = 1$ (resp. $\sigma_{\max} = 3$), $\sigma_{\min} = 0.65$ and the initial condition $(X_0(t))_{-\tau \leq t \leq 0}$ solution of $dX_0(t) = dW(t)$ with $X_0(-\tau) = 0$ where W is the standard Brownian motion.

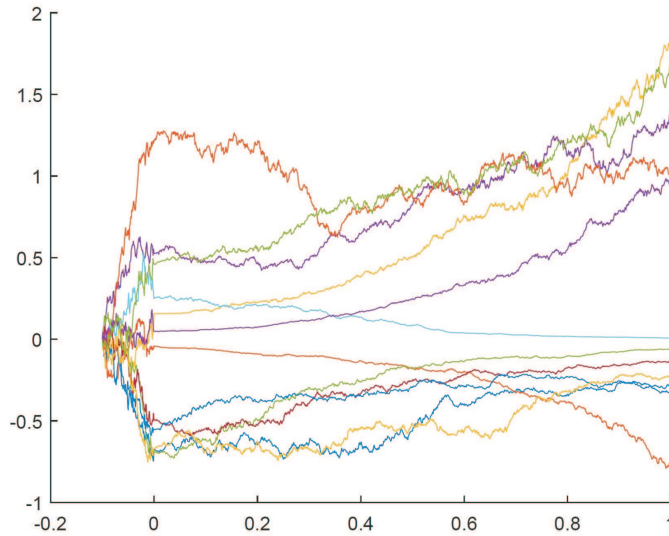


Figure 5: Solution *G*-NSFDE with random initial condition and $\sigma_{\max} = 1, \sigma_{\min} = 0.65$.

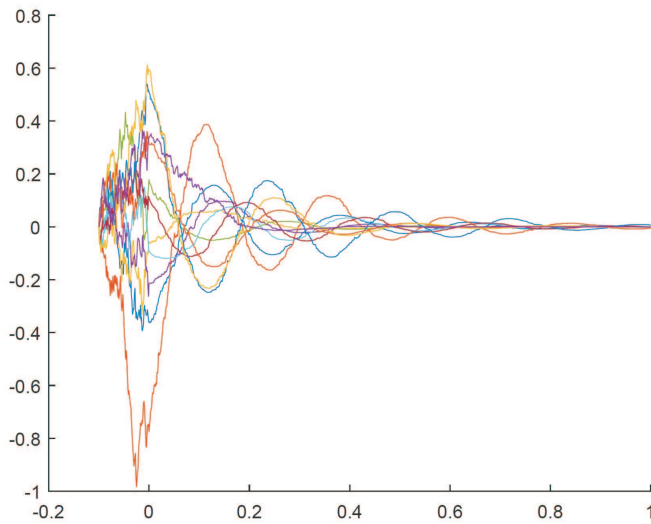


Figure 6: Solution *G*-NSFDE with random initial condition BM and $\sigma_{\max} = 3, \sigma_{\min} = 0.65$.

In Figure 7 we represent the trajectories of the solution of the *G*-NSFDE where the *G*-Brownian motion is with $\sigma_{\max} = 1, \sigma_{\min} = 0.65$ and deterministic initial condition $(X_0(t))_{-\tau \leq t \leq 0}$ given by $X_0(t) = \exp(t)$.

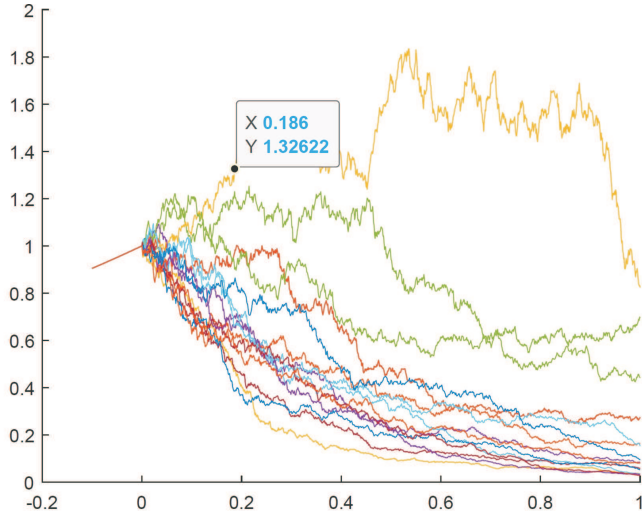


Figure 7: Solution G-NSFDE with deterministic initial condition $\exp(t)$ and $\sigma_{\max} = 1, \sigma_{\min} = 0.65$.

In Figure 8 we represent the trajectories of the solution of the G -NSFDE where the G -Brownian motion is with $\sigma_{\max} = 1, \sigma_{\min} = 0.65$ and deterministic initial condition $(X_0(t))_{-\tau \leq t \leq 0}$ given by: $\forall t \in [-\tau, 0], X_0(t, \omega)$ a fixed value between $[-0.2, 0.2]$.

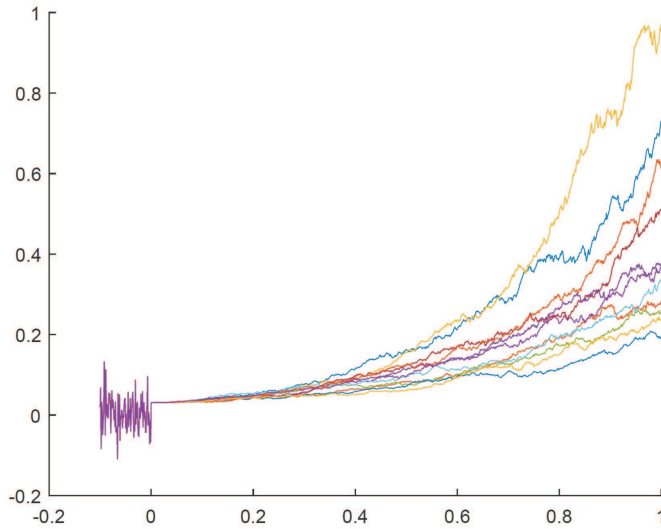


Figure 8: Solution G-NSFDE with deterministic initial condition $X_0(t)$ take values between $[-0.2, 0.2]$ for $t \in [-\tau, 0]$ and $\sigma_{\max} = 1, \sigma_{\min} = 0.65$.

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