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EXISTENCE OF SOLUTION FOR SOME SINGULAR KIRCHHOFF FRACTIONAL BOUNDARY VALUE PROBLEM

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Abstract

In this work, we investigate the question of the existence of multiple solutions for some nonlinear singular p-fractional problem with Riemann-Liouville fractional derivative and of Kirchhoff type. Precisely, we employ the method of the Nehari manifold combined with the analysis of the fibering map in order to show the existence of at least two non-trivial weak solutions.
Keywords: Nonlinear singular fractional differential equation, Riemann-Liouville Fractional Derivative, Fibreging maps, Nehari Manifold, Existence of solutions.

1. Introduction

In recent decades, fractional calculus has been investigated extensively. This is due to its importance and applications in many fields such as physics, aerodynamics, chemistry, electrodynamics of complex medium (see [9, 20, 23, 25]). Among all these subjects, there have been significant development boundary value problems involving different fractional operators. For details and examples, one can see the papers [2, 3, 8, 17, 18, 19, 21, 24] and references therein.

By combination between fibering maps and Nehari manifold, Saoudi et al. [26] proved the existence of at least two nontrivial solutions for the

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following problem

$$\begin{cases} -{}_tD_1^\alpha \left(\mathcal{J}_p \left({}_0D_t^\alpha u(t)\right)\right) = f(t,u(t)) + \lambda g(t)|u(t)^{q-2}u(t), \ t \in (0,1) \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where $0 < \frac{1}{2} < \alpha < 1$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. ${}_tD_{1},{}_0D_t$ are the left sided and right sided Riemann-Liouville derivatives of order α. $g \in C([0,1],\mathbb{R})$ and $2 < r < p < q$. While, $\mathcal{J}_p$ is the p-Laplacian operator which is defined by $\mathcal{J}_p(x) = |x|^{p-2}x$.

After that, Ghanmi and Zhang [19] extended the result in [26] to the following problem

$$\begin{cases} -{}_tD_1^\alpha \left(\mathcal{J}_p \left({}_0D_t^\alpha u(t)\right)\right) = \nabla W(t,u(t)) + \lambda g(t)|u(t)^{q-2}u(t), \ t \in (0,T) \\ u(0) = u(T) = 0, \end{cases} \tag{1.2}$$

where $0 < \frac{1}{p} < \alpha < 1$, and $\nabla W(t,u(t))$ is the gradian of $W(t,u(t))$ at u and $W \in C([0,T] \times \mathbb{R},\mathbb{R})$ is homogeneous of degree r, with $1 < r < p < q$.

Great attention has been devoted recently to the study of fractional Laplacian boundary value problems, we refer the reader to the works [5, 11, 8, 17, 18, 27].

Motivated by the above-mentioned papers, in this work, we want to contribute to the development of this new area on singular fractional differential equations involving both the Riemann Liouville and the p-Laplacian operators. Precisely, we will study the existence of nontrivial weak solutions for the following singular fractional boundary value problem

$$(\mathrm{P}_\lambda) \begin{cases} S(\varphi(t))\, {}_tD_T^\alpha \left(\mathcal{J}_p({}_0D_t^\alpha \varphi(t))\right) + M(t)\mathcal{J}_p({}_0D_t^\alpha \varphi(t)) \\ \qquad\qquad = \frac{f(t)}{u^\beta(t)} + \lambda g(t,\varphi(t)) \ \ t \in (0,T); \\ \varphi(0) = \varphi(T) = 0, \end{cases}$$

where λ is a positive parameter, $2 < q < p < r$, $\frac{1}{2} < \alpha \leq 1$, $0 < \beta < 1$ and $g \in C([0,1])$. The functional S is defined for a function φ by:

$$S(\varphi(t)) = \left(a + b\int_0^T |{}_0D_t^\alpha \varphi(t)|^p + M(t)|u(t)|^p \, dt\right)^{p-1}, \ a > 0, b > 0.$$

While $g \in C([0,T] \times \mathbb{R}, \mathbb{R})$ is positively homogeneous of degree $r-1$, that is $g(x,tu) = t^{r-1}x(x,u)$ holds for all $(x,u) \in [0,T] \times \mathbb{R}$. Moreover, if we put $G(x,s) := \int_0^s g(x,t)dt$, then we assume the following:

(H$_1$) $G : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is homogeneous of degree r that is

$$G(x,tu) = t^r G(x,u) \ (t > 0) \ \text{ for all } \ x \in [0,T], \ u \in \mathbb{R}.$$

Note that, from **(H$_1$)** g leads to the so-called Euler identity

$$ug(t,u) = rG(t,u).$$

Moreover, there exists $C_0 > 0$, such that

$$|G(t,u)| \leq C_0 |\varphi|^r. \tag{1.3}$$

Note that the problem (P$_\lambda$), has a solid theoretical significance and a sharp physical background. For example it is showen in [7, 10] that this problem describes the surface tension of the height of a thin liquid film on a solid surface in lubrication approximation. Moreover, the problem (P$_\lambda$) is related to the stationary version of the Kirchhoff problem which is introduced for the first time in 1983 by Kirchhoff [22]. Precisely, Kirchhoff was studying the following equation

$$\delta \frac{\partial^2 \varphi}{\partial s^2} - \left(\frac{\delta_0}{\rho} + \frac{E}{2T} \int_0^T |\frac{\partial \varphi}{\partial x}|^2 \, dx \right) \frac{\partial^2 \varphi}{\partial x^2} = 0, \tag{1.4}$$

where T is the length of the string, E is the young modulus of the material, ρ is the area of the cross-section, δ_0 is the initial tension, and δ is the mass density. Note that, if we consider the effects of the changes in the length of the strings during the vibrations, then we can ensure that the problem (1.4) extends the classical d'Alembert's wave equation.

Due to its importance in many fields, the study of the fractional boundary value problems of the Kirchhoff type has been attracted a lot of interest by researchers in the nonlinear analysis we refer the readers to [12, 13, 14].

Before presenting the main result of this work, let us introduce the notion of a weak solution for the problem (P$_\lambda$).

Definition 1.1. A function $u \in E_0^{\alpha,p}$ is said to be a weak solution of problem (P_λ), if for any $\psi \in E_0^{\alpha,p}$ we have:

$$S(\varphi(t)) \int_0^T J_p({}_0D_t^\alpha \varphi(t))\, {}_0D_t^\alpha \psi(t) + M(t) J_p({}_0D_t^\alpha \varphi(t))\, {}_0D_t^\alpha \psi(t)\, dt$$
$$- \int_0^T \frac{f(t)}{\varphi^\beta(t)} \psi(t)\, dt - \lambda \int_0^T g(t, \varphi(t))\, \psi(t)\, dt = 0$$

where $E_0^{\alpha,p}$ will be introduced later in Section 2.

In this paper, we want to use the Nehari manifold and fibering maps analysis combined with the variational method, in order to prove the following result.

Theorem 1.1. *Assume that* $\frac{1}{2} < \beta < 1$, $2 < q < p < r$, *and* $0 < \beta < 1$*. If* g *satisfies the hypotheses* **(H**$_1$**)-(H**$_2$**)***. Then there exists* $\lambda_* > 0$*, such that for all* $\lambda \in (0, \lambda_*)$*, problem* (P_λ) *admits at least two nontrivial solutions.*

The rest of the present paper is organized as follows. In Section 2, we present some preliminaries and important results on fractional calculus. In Section 3, the variational setting of the problem (P_λ) is presented. In Section 4, we prove the main result of this paper (Theorem 1.1).

2. Preliminaries

In this section, we present some important background theories and results on the concept of Riemann-Liouville fractional operators. Let us start by introduce the definitions of Riemann-Liouville fractional integral and Riemann-Liouville fractional derivative.

Definition 2.1. Let $\theta > 0$ and ψ be a function defined a.e. on $(0, T)$. The Left (resp. right) Riemann-Liouville fractional integral of order θ of ψ is given by

$${}_0I_t^\theta \psi(t) = \frac{1}{\beta(\theta)} \int_0^t (t-s)^{\theta-1} \psi(s) ds, \quad t \in (0, T],$$

respectively

$${}_tI_T^\theta \psi(t) = \frac{1}{\beta(\theta)} \int_t^T (t-s)^{\theta-1} \psi(s) ds, \quad t \in [0, T),$$

provided that the right side hands are point-wise defined on $[0,T]$, where β denotes Euler's Gamma function.
Note that, if $\psi \in L^1(0,T)$, then ${}_aI_t^\theta\psi$ and ${}_tI_T^\theta\psi$ are defined a.e. on $(0,T)$.

Definition 2.2. If $0 < \theta < 1$. Then, we define the Left (resp. right) Riemann-Liouville fractional derivative of order θ of a function ψ as follows:

$$ {}_0D_t^\theta\psi(t) = \frac{d}{dt}\left({}_0I_t^{1-\theta}\psi\right)(t),\ \forall\, t \in (0,T], $$

respectively

$$ {}_tD_T^\theta\psi(t) = \frac{d}{dt}\left({}_tI_T^{1-\theta}\psi\right)(t),\ \forall\, t \in [0,T], $$

provided that the right side hands are pointwise defined on $[0,T]$.

Remark 2.1. We note that, if ψ is an absolutely continuous function in $[0,T]$. Then, from [21], ${}_0D_t^\theta\psi$ (respectively ${}_tD_T^\theta\psi$) is defined a.e. on $(0,T)$ and satisfies

$$ {}_0D_t^\theta\psi(t) = {}_0I_t^{1-\theta}\psi'(t) + \frac{\psi(0)}{t^\theta\beta(1-\theta)}, \tag{2.1} $$

respectively

$$ {}_tD_T^\theta\psi(t) = -{}_t\,I_T^{1-\theta}\psi'(t) + \frac{\varphi(T)}{(T-t)^\theta\beta(1-\theta)}. \tag{2.2} $$

Moreover, if $\psi(0) = \psi(T) = 0$, then, we have

$$ {}_0D_t^\theta\psi(t) ={}_0 I_t^{1-\theta}\psi'(t) \text{ and } {}_tD_T^\theta\psi(t) = -{}_tI_T^{1-\theta}\psi'(t). $$

Note that the above information implies the equality between the Riemann-Liouville fractional derivative and the Caputo derivative.

Now, we will collect some properties of the left Riemann-Liouville fractional operators. One can easily derive the analgous version for the right one. Fore more details, one can see [5].

Proposition 2.1. *For each $\theta_1,\ \theta_2 > 0$, and for any $\psi \in L^1(0,T)$, one has*

$$ {}_0I_t^{\theta_1} \circ {}_0I_t^{\theta_2}\psi = {}_0I_t^{\theta_1+\theta_2}\psi. $$

Remark 2.2. Using Proposition 2.1, Equations (2.1) and (2.2), we can prove that for any $0 < \theta < 1$, and any $\psi \in L^1(0,T)$, we have

$$ {}_0D_t^\theta \circ {}_0I_t^\theta \psi = \psi, $$

moreover, if ψ is absolutely continuous with $\psi(0) = 0$. Then, we have

$$ {}_0I_t^\theta \circ {}_0D_t^\theta \psi = \psi. $$

Proposition 2.2. *Let $\theta > 0$ and $p \geq 1$, then the operator ${}_0I_t^\theta : L^p(0,T) \to L^p(0,T)$, is linear and continuous. Moreover for each $\psi \in L^p(0,T)$, we have*

$$ \|{}_0I_t^\theta \psi\|_p \leq \frac{T^\theta}{\beta(1+\theta)} \|\psi\|_p. $$

Next, we recall another classical result on the boundness of the left fractional integral in the sense of the supremum norm.

Proposition 2.3. *Let $0 < \frac{1}{p} < \theta < 1$ and $q = \frac{p}{p-1}$. Then, for each $\psi \in L^p(0,T)$, ${}_0I_t^\theta \psi$ is Hölder continuous on $(0,T]$ with exponent $\theta - \frac{1}{p} > 0$, Also, ${}_0I_t^\theta \psi$ can be continuously extended by 0 at $t = 0$. Finally, we can see that ${}_0I_t^\theta \psi \in C_0(0,T)$, moreover*

$$ \|{}_0I_t^\theta \psi\|_\infty \leq \frac{T^{\theta - \frac{1}{p}}}{\beta(\theta)\left((\theta-1)q+1\right)^{\frac{1}{q}}} \|\psi\|_p. \tag{2.3} $$

Since we will use the variational method in studying our main problem, we introduce in the following integration by parts formula:

Proposition 2.4. *Let $0 < \theta < 1$ and p, q are such that either*

$$ p \geq 1,\ q \geq 1 \text{ and } \frac{1}{p} + \frac{1}{q} < 1 + \theta, $$

or

$$ p \neq 1,\ q \neq 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1 + \theta. $$

Then, for each function $\psi \in L^p(0,T)$ and $\varphi \in L^q(0,T)$, we have

$$ \int_0^T \varphi(t)\, {}_0I_t^\theta \psi(t) dt = \int_0^T \psi(t)\, {}_0I_t^\theta \varphi(t) dt, $$

and

$$\int_0^T \varphi(t)\ {}_0^cD_t^\theta\psi(t)dt = \varphi(t)_tI_T^{1-\theta}\psi(t)|_{t=0}^{t=T} + \int_0^T \psi(t)\ {}_0D_t^\theta\varphi(t)dt. \tag{2.4}$$

Note that, from (2.4), *if* $\varphi(0) = \varphi(T) = 0$, *then we get*

$$\int_0^T \varphi(t)\ {}_0D_t^\theta\psi(t)dt = \int_0^T \psi(t)\ {}_0D_t^\theta\varphi(t)dt. \tag{2.5}$$

Now, we are in a position to discuss the variational setting associated with the problem (P_λ). We denote by $C_0^\infty([0,T],\mathbb{R})$ the set of all functions $v \in C^\infty([0,T],\mathbb{R})$ satisfying $v(0) = v(T) = 0$. For, $\theta > 0$ and $p > 1$, we denote by E_0^θ the closure of $C_0^\infty([0,T],\mathbb{R})$ with respect to the following norm

$$\|u\| = \left(\|u\|_p^p + \|{}_0^cD_t^\theta u\|_p^p\right)^{\frac{1}{p}}. \tag{2.6}$$

Remark 2.3. The following properties are useful for the rest of the paper:

(i) The fractional derivative space E_0^θ is the space of functions $u \in L^p([0,T])$ having an θ-order Caputo fractional derivative ${}_0^cD_t^\theta u \in L^p([0,T])$ and $\varphi(0) = \varphi(T) = 0$.

(ii) If $v \in E_0^\theta$ is such that $\varphi(0) = 0$, then the left and right Riemann-Liouville fractional derivatives of order θ are equivalent to the left and right Caputo fractional derivatives of order θ. That is

$${}_0^cD_t^\theta\varphi(t) = {}_0D_t^\theta\varphi(t),\ t \in [0,T].$$

(iii) The fractional space E_0^θ is a reflexive and a separable Banach space.

Lemma 2.1. *For any* $\varphi \in E_0^\theta$, *we have*

$$\|\varphi\|_p \leq \frac{T^\theta}{\Gamma(\theta+1)}\|{}_0D_t^\theta\varphi\|_p. \tag{2.7}$$

Moreover, if $\frac{1}{p} < \theta < 1$, *then we obtain*

$$\|\varphi\|_\infty \leq \frac{T^{\theta-\frac{1}{p}}}{\Gamma(\theta)\left((\theta-1)\widetilde{p}+1\right)^{\frac{1}{\widetilde{p}}}}\|{}_0D_t^\theta\varphi\|_p, \tag{2.8}$$

where $\widetilde{p} = \frac{p}{p-1}$.

Remark 2.4. From Equation (2.7), we can consider E_0^θ with respect to the following equivalent norm

$$\|\varphi\|_{\theta,p} = \|_0D_t^\theta \varphi\|_p.$$

Also from hypothesis **(H$_2$)**, can be equipped with the following equivalent norm

$$\|\varphi\|_M = \left(\|_0^cD_t^\theta \varphi\|_p^p + \|M^{\frac{1}{p}}\varphi\|_p^p\right)^{\frac{1}{p}}.$$

Moreover, we have

$$\min(1, M_0)\|\varphi\|_{\theta,p} \leq \|\varphi\|_M \leq \max(1, M_\infty)\|\varphi\|_{\theta,p}. \tag{2.9}$$

Lemma 2.2. *If $\frac{1}{p} < \theta < 1$, and the sequence $\{\varphi_n\} \rightharpoonup \varphi$ weakly in E_0^θ. Then $\{\varphi_n\} \to u$ strongly in $C([0,T])$, that is*

$$\|\varphi_n - \varphi\|_\infty \to 0 \quad \textit{as } n \to \infty.$$

3. A Variational Setting and the Analysis of the Fibering Map

In this section, we will prove the main result of this paper (Theorem 1.1). Our main tools are based on variational methods. Precisely, we will use the Nehari manifold method combined with the analysis of the fibering map. For this purpose we define the functional energy associated to problem (P_λ), as follows:

$$\begin{aligned}\mathcal{E}_\lambda(\varphi) = &\frac{1}{bp^2}\big(a + b\|\varphi\|_M^p\big)^p - \frac{\lambda}{r}\int_0^T G(t,\varphi)dt \\ &- \frac{1}{1-\beta}\int_0^T f(t)|\varphi|^{1-\beta}dt - \frac{a^p}{bp^2}, \qquad \varphi \in E_0^{\alpha,p}.\end{aligned} \tag{3.1}$$

Since the energy functional $\mathcal{E}_\lambda$ is not bounded below on the space $E_0^{\alpha,p}$, then we do not use the direct variational method to prove the existence of solutions. In order to guarantee the boundness of $\mathcal{E}_\lambda$, we will work in a suitable subset of $E_0^{\alpha,p}$. So we define the following constraint set

$$\mathcal{N}_\lambda := \Big\{ u \in E_0^{\alpha,p} \setminus \{0\} : \big(a + b\|\varphi\|_M^p\big)^{p-1}\|\varphi\|_M^p$$

$$= \lambda \int_0^T G(s, \varphi(s))ds + \int_0^T f(s)|\varphi(s)|^{1-\beta} ds \Big\}.$$

As introduced in [11], $\mathcal{N}_\lambda$ is closely related to the fibering map $J_\lambda : (0, \infty) \to \mathbb{R}$, which is defined by:

$$J_\varphi(t) = \mathcal{E}_\lambda(t\varphi).$$

A simple calculation shows that for each $\varphi \in E_0^{\alpha,p}$, we have

$$\begin{aligned} J'_\varphi(t) = & \, t^{p-1}\big(a + bt^p\|\varphi\|_M^p\big)^{p-1}\|\varphi\|_M^p - \lambda t^{r-1} \int_0^T G(s, \varphi(s))ds \\ & - t^{-\beta} \int_0^T f(s)|\varphi(s)|^{1-\beta} ds, \end{aligned}$$

and

$$\begin{aligned} J''_\varphi(t) = & \, (p-1)t^{p-2}\big(a + bt^p\|\varphi\|_M^p\big)^{p-1}\|\varphi\|_M^p \\ & + bp(p-1)t^{2p-2}\big(a + bt^p\|\varphi\|_M^p\big)^{p-2}\|\varphi\|_M^{2p} \\ & - \lambda(r-1)t^{r-2} \int_0^T G(s, \varphi(s))ds + \frac{\beta}{t^{\beta+1}} \int_0^T f(s)|\varphi(s)|^{1-\beta} ds. \end{aligned}$$

Remark 3.1. From the definition of $\mathcal{N}_\lambda$, we can see that $t\varphi \in \mathcal{N}_\lambda$, if and only if $J'_\varphi(t) = 0$, in particular, $\varphi \in \mathcal{N}_\lambda$, if and only if $J'_\varphi(1) = 0$. That is, $\varphi \in \mathcal{N}_\lambda$, if and only if

$$\big(a + b\|\varphi\|_M^p\big)^{p-1}\|\varphi\|_M^p - \lambda \int_0^T G(s, \varphi(s))ds - \int_0^T f(s)|\varphi(s)|^{1-\beta} ds = 0. \quad (3.2)$$

In order to prove the multiplicity of solutions, we split $\mathcal{N}_\lambda$ into the following three parts:

$$\mathcal{N}_\lambda^+ = \big\{\varphi \in \mathcal{N}_\lambda : J''_\varphi(1) > 0\big\} = \big\{\varphi \in E_0^{\alpha,p} : J'_\varphi(1) = 0 \text{ and } J''_\varphi(1) > 0\big\},$$
$$\mathcal{N}_\lambda^- = \big\{\varphi \in \mathcal{N}_\lambda : J''_\varphi(1) < 0\big\} = \big\{\varphi \in E_0^{\alpha,p} : J'_\varphi(1) = 0 \text{ and } J''_\varphi(1) < 0\big\},$$

and

$$\mathcal{N}_\lambda^0 = \big\{\varphi \in \mathcal{N}_\lambda : J''_\varphi(1) = 0\big\} = \big\{\varphi \in E_0^{\alpha,p} : J'_\varphi(1) = 0 \text{ and } J''_\varphi(1) = 0\big\}.$$

Remark 3.2.

(i) From Remark 3.1, if $t > 0$ is such that $J'_\varphi(t) = 0$ and $J''_\varphi(t) > 0$ (respectively $J'_\varphi(t) = 0$ and $J''_\varphi(t) < 0$), then $t\varphi \in \mathcal{N}_\lambda^+$ (respectively

$t\varphi \in \mathcal{N}_\lambda^-$).

(ii) Let $\varphi \in \mathcal{N}_\lambda$. Then, using Equations (1.3), (2.8) and (2.9), we obtain

$$\int_0^T G(t,\varphi(t))dt \le \zeta_1 \|\varphi\|_M^r, \tag{3.3}$$

and

$$\int_0^T f(t)|\varphi|^{1-\beta}dt \le \zeta_2 \|\varphi\|_M^{1-\beta}. \tag{3.4}$$

where

$$\zeta_1 = \frac{C_0 T^{1+r(\alpha-\frac{1}{p})}}{\left(\min(1,M_0)\Gamma(\alpha)\left((\alpha-1)\widetilde{p}+1\right)^{\frac{1}{\widetilde{p}}}\right)^r},$$

and

$$\zeta_2 = \frac{\|f\|_\infty T^{1+(1-\beta)(\alpha-\frac{1}{p})}}{\left(\min(1,M_0)\Gamma(\alpha)\left((\alpha-1)\widetilde{p}+1\right)^{\frac{1}{\widetilde{p}}}\right)^{1-\beta}}.$$

Now, we are in a position to prove the lower boundness of $\mathcal{E}_\lambda$ on $\mathcal{N}_\lambda$.

Lemma 3.1. *Assume that hypotheses* **(H$_1$)-(H$_2$)** *are satisfied, then, for all* $\lambda > 0$ *the functional* $\mathcal{E}_\lambda$ *is coercive and bounded below on* $\mathcal{N}_\lambda$.

Proof. Let $\varphi \in \mathcal{N}_\lambda$. Then from (3.2) and (3.4), we obtain

$$\begin{aligned}
\mathcal{E}_\lambda(\varphi) &= \frac{1}{bp^2}\left(a+b\|\varphi\|_M^p\right)^p - \frac{\lambda}{r}\int_0^T G(t,\varphi)dt - \frac{1}{1-\beta}\int_0^T f(t)|\varphi|^{1-\beta}dt - \frac{a^p}{bp^2} \\
&= \left(a+b\|\varphi\|_M^p\right)^{p-1}\Big(\frac{1}{bp^2}\left(a+b\|\varphi\|_M^p\right)^p - \frac{1}{r}\|\varphi\|_M^p\Big) \\
&\quad -\Big(\frac{1}{1-\beta}-\frac{1}{r}\Big)\int_0^T f(t)|\varphi|^{1-\beta}dt - \frac{a^p}{bp^2} \\
&\ge \left(a+b\|\varphi\|_M^p\right)^{p-1}\Big(\frac{a}{bp^2} + \Big(\frac{1}{p^2}-\frac{1}{r}\Big)\|\varphi\|_M^p\Big) \\
&\quad -\Big(\frac{1}{1-\beta}-\frac{1}{r}\Big)\frac{\|f\|_\infty T^{1+(1-\beta)(\alpha-\frac{1}{p})}}{\left(\min(1,M_0)\Gamma(\alpha)\left((\alpha-1)\widetilde{p}+1\right)^{\frac{1}{\widetilde{p}}}\right)^{1-\beta}}\|\varphi\|_M^{1-\beta} - \frac{a^p}{bp^2}.
\end{aligned}$$

Since we have $1-\beta < p^2 < r$, then from the above information, we can see that $\mathcal{E}_\lambda$ is coercive and bounded below on $\mathcal{N}_\lambda$. This ends the proof of Lemma 3.1. □

Lemma 3.2. *Under hypothesis of Theorem* 1.1, *there exists* $\lambda_0 > 0$ *such that, if* $\lambda \in (0,\lambda_0)$, *then* $\mathcal{N}_\lambda^0 = \emptyset$.

Proof. We proceed by contradiction. Assume that $\mathcal{N}_\lambda^0 \neq \emptyset$ and let $\varphi \in \mathcal{N}_\lambda^0$. From the definition of $\mathcal{N}_\lambda^0$, we have

$$
\begin{aligned}
&(p-1)\big(a+b\|\varphi\|_M^p\big)^{p-1}\|\varphi\|_M^p + bp(p-1)\big(a+b\|\varphi\|_M^p\big)^{p-2}\|\varphi\|_M^{2p} \\
&\qquad -\lambda(r-1)\int_0^T G(s,\varphi(s))ds + \beta\int_0^T f(s)|\varphi(s)|^{1-\beta}ds = 0.
\end{aligned}
$$

So from (3.2), we obtain

$$
\begin{aligned}
&(r+\beta-1)\int_0^T f(s)|\varphi(s)|^{1-\beta}ds - (r-p)\big(a+b\|\varphi\|_M^p\big)^{p-1}\|\varphi\|_M^{2p} \\
&\qquad + bp(p-1)\big(a+b\|\varphi\|_M^p\big)^{p-2}\|\varphi\|_M^p = 0, \qquad (3.5)
\end{aligned}
$$

and

$$
\begin{aligned}
&(p+\beta-1)\big(a+b\|\varphi\|_M^p\big)^{p-1}\|\varphi\|_M^p + bp(p-1)\big(a+b\|\varphi\|_M^p\big)^{p-2}\|\varphi\|_M^{2p} \\
&\qquad - \lambda(r+\beta-1)\int_0^T G(s,\varphi(s))ds = 0. \qquad (3.6)
\end{aligned}
$$

Now from Equation (3.3) and (3.4), we get

$$
\begin{aligned}
&(r-p^2)b^{p-1}\|\varphi\|_M^{p^2} \\
&\quad \leq (r-p)\big(a+b\|\varphi\|_M^p\big)^{p-1}\|\varphi\|_M^p - bp(p-1)\big(a+b\|\varphi\|_M^p\big)^{p-2}\|\varphi\|_M^{2p} \\
&\quad = (r+\beta-1)\int_0^T f(s)|\varphi(s)|^{1-\beta}ds \\
&\quad \leq \frac{(r+\beta-1)\|f\|_\infty T^{1+(1-\beta)(\alpha-\frac{1}{p})}}{\Big(\min(1,M_0)\Gamma(\alpha)\left((\alpha-1)\widetilde{p}+1\right)^{\frac{1}{\widetilde{p}}}\Big)^{1-\beta}}\|\varphi\|_M^{1-\beta}, \qquad (3.7)
\end{aligned}
$$

and

$$
\begin{aligned}
&(p+\beta-1)b^{p-1}\|\varphi\|_M^{p^2} \\
&\quad \leq (p+\beta-1)\big(a+b\|\varphi\|_M^p\big)^{p-1}\|\varphi\|_M^p + bp(p-1)\big(a+b\|\varphi\|_M^{2p}\big)^{p-2}\|\varphi\|_M^p \\
&\quad = \lambda(r+\beta-1)\int_0^T G(s,\varphi(s))ds \\
&\quad \leq \lambda\frac{C_0 T^{1+r(\alpha-\frac{1}{p})}(r+\beta-1)}{\Big(\min(1,M_0)\Gamma(\alpha)\left((\alpha-1)\widetilde{p}+1\right)^{\frac{1}{\widetilde{p}}}\Big)^r}\|\varphi\|_M^r. \qquad (3.8)
\end{aligned}
$$

From (3.7), we have

$$\|\varphi\|_M \leq \left(\frac{(r+\beta-1)\|f\|_\infty T^{1+(1-\beta)(\alpha-\frac{1}{p})}}{(r-p^2)b^{p-1}\big(\min(1,M_0)\Gamma(\alpha)\,((\alpha-1)\widetilde{p}+1)^{\frac{1}{\widetilde{p}}}\big)^{1-\beta}}\right)^{\frac{1}{p^2+\beta-1}}, \tag{3.9}$$

and from (3.8), we have

$$\|\varphi\|_M \geq \left(\frac{(p+\beta-1)b^{p-1}\big(\min(1,M_0)\Gamma(\alpha)\,((\alpha-1)\widetilde{p}+1)^{\frac{1}{\widetilde{p}}}\big)^{r}}{\lambda C_0 T^{1+r(\alpha-\frac{1}{p})}(r+\beta-1)}\right)^{\frac{1}{r-p^2}}. \tag{3.10}$$

So by combining (3.9) with (3.10), one has

$$\lambda \geq \lambda_0 := A^{\frac{r-p^2}{p^2+\beta-1}}B, \tag{3.11}$$

where

$$A = \frac{(r-p^2)b^{p-1}\big(\min(1,M_0)\Gamma(\alpha)\,((\alpha-1)\widetilde{p}+1)^{\frac{1}{\widetilde{p}}}\big)^{1-\beta}}{(r+\beta-1)\|f\|_\infty T^{1+(1-\beta)(\alpha-\frac{1}{p})}},$$

and

$$B = \frac{(p+\beta-1)b^{p-1}\big(\min(1,M_0)\Gamma(\alpha)\,((\alpha-1)\widetilde{p}+1)^{\frac{1}{\widetilde{p}}}\big)^{r}}{C_0 T^{1+r(\alpha-\frac{1}{p})}(r+\beta-1)}.$$

Finally, if we take λ small enough such that $0<\lambda<\lambda_0$, then from (3.11) we obtain a contradiction. So $\mathcal{N}_\lambda^0=\emptyset$ and the proof of Lemma 3.2 is now completed. □

Lemma 3.3. *Under hypotheses* **(H$_1$)-(H$_2$)**, *there exists* $\lambda_1>0$ *such that* $\mathcal{N}_\lambda^+$ *and* $\mathcal{N}_\lambda^-$ *are non-empty sets, provided that* $0<\lambda<\lambda_1$.

Proof. Let φ be a nontrivial function on $E_0^{\alpha,p}$. If $t>0$ then we have

$$\begin{aligned} J'_\varphi(t) &= t^{p-1}\big(a+bt^p\|\varphi\|_M^p\big)^{p-1}\|\varphi\|_M^p - \lambda t^{r-1}\int_0^T G(s,\varphi(s))ds \\ &\quad -t^{-\beta}\int_0^T f(s)|\varphi(s)|^{1-\beta}ds \\ &= t^{-\beta}\left[\mathcal{L}(t)-\int_0^T f(s)|\varphi(s)|^{1-\beta}ds\right], \end{aligned} \tag{3.12}$$

where

$$\mathcal{L}(t) := t^{p+\beta-1}\big(a + bt^p\|\varphi\|_M^p\big)^{p-1}\|\varphi\|_M^p - \lambda t^{r+\beta-1}\int_0^T G(s,\varphi(s))ds.$$

A simple calculation shows that

$$\mathcal{L}(t) \geq bt^{p^2+\beta-1}\|\varphi\|_M^{p^2} - \lambda t^{r+\beta-1}\int_0^T G(s,\varphi(s))ds := \mathcal{W}(t). \tag{3.13}$$

It is not difficult to show that $\mathcal{W}$ attains its maximum at

$$\overline{t} = \left(\frac{b(p^2+\beta-1)\|\varphi\|_M^{p^2}}{\lambda(r+\beta-1)\int_0^T G(s,\varphi(s))ds}\right)^{\frac{1}{r-p^2}}.$$

Morover, we have

$$\mathcal{W}(\overline{t}) = b\left(1 - \frac{p^2+\beta-1}{r+\beta-1}\right)\left(\frac{b(p^2+\beta-1)}{\lambda(r+\beta-1)}\right)^{\frac{p^2+\beta-1}{r-p^2}} \frac{\|\varphi\|_M^{p^2\frac{r+\beta-1}{r-p^2}}}{\left(\int_0^T G(s,\varphi(s))ds\right)^{\frac{p^2+\beta-1}{r-p^2}}}.$$

Now, using Equations (3.3) and (3.4), we obtain

$$\begin{aligned}
\mathcal{W}(\overline{t}) &= b\left(1 - \frac{p^2+\beta-1}{r+\beta-1}\right)\left(\frac{b(p^2+\beta-1)}{\lambda(r+\beta-1)}\right)^{\frac{p^2+\beta-1}{r-p^2}} \frac{\|\varphi\|_M^{p^2\frac{r+\beta-1}{r-p^2}}}{\left(\int_0^T G(s,\varphi(s))ds\right)^{\frac{p^2+\beta-1}{r-p^2}}}\\
&\geq b\left(1 - \frac{p^2+\beta-1}{r+\beta-1}\right)\left(\frac{b(p^2+\beta-1)}{\lambda(r+\beta-1)}\right)^{\frac{p^2+\beta-1}{r-p^2}} \frac{\|\varphi\|_M^{p^2\frac{r+\beta-1}{r-p^2}}}{\left(\zeta_1\|\varphi\|_M^r\right)^{\frac{p^2+\beta-1}{r-p^2}}}\\
&= b\left(1 - \frac{p^2+\beta-1}{r+\beta-1}\right)\left(\frac{b(p^2+\beta-1)}{\lambda(r+\beta-1)}\right)^{\frac{p^2+\beta-1}{r-p^2}} \frac{\|\varphi\|_M^{1-\beta}}{\zeta_1^{\frac{p^2+\beta-1}{r-p^2}}}\\
&\geq \zeta_2\|\varphi\|_M^{1-\beta},
\end{aligned} \tag{3.14}$$

provided that $0 < \lambda < \lambda_1$ where:

$$\lambda_1 = \frac{b(p^2+\beta-1)}{\zeta_2(r+\beta-1)}\left(\frac{b(r-p^2)}{\zeta_2(r+\beta-1)}\right)^{\frac{r-p^2}{p^2+\beta-1}}.$$

Hence, from the Equations (3.4), (3.13) and (3.14), we get

$$0 < \int_0^T f(s)|\varphi(s)|^{1-\beta} ds < \mathcal{W}(\bar{t}) \leq \mathcal{L}(\bar{t}). \tag{3.15}$$

So, there exist $\epsilon > 0$, $0 < s_1 < \bar{t}$, such that

$$\mathcal{L}(s_1) = \int_0^T f(s)|\varphi(s)|^{1-\beta} ds,$$

moreover, for all $s \in (s_1 - \epsilon, s_1)$, we have

$$\mathcal{L}(s) < \int_0^T f(s)|\varphi(s)|^{1-\beta} ds,$$

and for each $s \in (s_1, s_1 + \epsilon)$, we have

$$\mathcal{L}(s) > \int_0^T f(s)|\varphi(s)|^{1-\beta} ds.$$

From the above information and using Equation (3.12), we can easily prove that s_1 is a local minimum of the functional J_φ. That is $s_1\varphi \in \mathcal{N}_\lambda^+$. On the other hand, from (3.15) and from the fact that $\lim_{t\to\infty} \mathcal{L}(t) = -\infty$, we can deduce the existence of $s_2 > \bar{t}$ and $\delta > 0$, such that

$$\mathcal{L}(s_2) = \int_0^T f(s)|\varphi(s)|^{1-\beta} ds,$$

moreover, for all $s \in (s_2 - \delta, s_2)$, we have

$$\mathcal{L}(s) > \int_0^T f(s)|\varphi(s)|^{1-\beta} ds,$$

and for each $s \in (s_2, s_2 + \delta)$, we have

$$\mathcal{L}(s) < \int_0^T f(s)|\varphi(s)|^{1-\beta} ds.$$

From the above information and using Equation (3.12), we can easily prove that s_2 is a local maximum of the functional J_φ. That is $s_2\varphi \in \mathcal{N}_\lambda^-$. □

Now, we will prove that the energy functional has critical points in either $\mathcal{N}_\lambda^-$ and $\mathcal{N}_\lambda^-$. To prove that these critical points are solutions to the main problem, we need to prove the following lemma.

Lemma 3.4. *Let $\varphi \in \mathcal{N}_\lambda^-$ (respectively $\varphi \in \mathcal{N}_\lambda^+$), then for each $\psi \in E_0^{\alpha,p}$, there exists $\varepsilon > 0$ small enough and a continuous function Φ such that for all real number k with $|k| < \varepsilon$, we have*

$$\Phi(0) = 1 \quad \textit{and} \;\; \Phi(k)(\varphi + k\psi) \in \mathcal{N}_\lambda^- \;\; (\textit{respectively} \;\; \Phi(k)(\varphi + k\psi) \in \mathcal{N}_\lambda^+).$$

Proof. Let $\varphi \in \mathcal{N}_\lambda^-$ and $\psi \in E_0^{\alpha,p}$. In order to prove Lemma 3.4, let us introduce the following function $\Xi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by:

$$\begin{aligned}\Xi(t,k) &= t^{p+\beta-1}\big(a + bt^p\|\varphi + k\psi\|_M^p\big)^{p-1}\|\varphi + k\psi\|_M^p \\ &\quad -\lambda t^{r+\beta-1}\int_0^T G(s,\varphi(s) + k\psi(s))ds - \int_0^T f(s)|\varphi(s)+k\psi(s)|^{1-\beta}ds.\end{aligned}$$

A simple calculation shows that

$$\begin{aligned}\frac{\partial}{\partial t}\Xi(t,k) &= (p+\beta-1)t^{p+\beta-2}\big(a + bt^p\|\varphi + k\psi\|_M^p\big)^{p-1}\|\varphi + k\psi(s)\|_M^p \\ &\quad +pb(p-1)t^{2p+\beta-2}\big(a + bt^p\|\varphi + k\psi\|_M^p\big)^{p-2}\|\varphi + k\psi(s)\|_M^{2p} \\ &\quad \lambda(r+\beta-1)t^{r+\beta-2}\int_0^T G(s,\varphi(s) + k\psi(s))ds \\ &= J_\varphi'(1) + J_\varphi''(1).\end{aligned}$$

So the fact that $\varphi \in \mathcal{N}_\lambda^-$, implies that $\frac{\partial}{\partial t}\Xi(1,0) < 0$. Since Ξ is continuous and $\Xi(1,0) = 0$, then we can apply the implicit function theorem to the function Ξ at the point $(1,0)$, to obtain the existence of $\delta > 0$ and a positive continuous function Φ such that for all real number k with $|k| < \delta$, we have

$$\Phi(0) = 1,\; \Phi(k)(\varphi + k\psi) \in \mathcal{N}_\lambda.$$

Finally, by taking $\varepsilon > 0$ small enough if necessary, we also have

$$\Phi(k)(\varphi + k\psi) \in \mathcal{N}_\lambda^-.$$

If $\varphi \in \mathcal{N}_\lambda^+$, then we can proceed as in the first step to have the desired result. Since the proof is very similar to the case when $\varphi \in \mathcal{N}_\lambda^-$, then we omit it. □

Remark 3.3. Assume that hypotheses **(H$_1$)-(H$_2$)** are satisfied.

(i) Put $\lambda_* = \min(\lambda_0, \lambda_1)$, if $0 < \lambda < \lambda_*$, then all conclusions of Lemma 3.1, Lemma 3.2, and Lemma 3.3 hold.

(ii) From Lemma 3.2, $\mathcal{N}_\lambda$ can be decomposed as

$$\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-.$$

Moreover, since $J_0''(1) = 0$, then any $\varphi \in \mathcal{N}_\lambda^+$ or $\varphi \in \mathcal{N}_\lambda^-$, is nontrivial.

(iii) From Lemma 3.1, we can define the following expressions:

$$\overline{\sigma}_\lambda = \inf_{\varphi \in \mathcal{N}_\lambda^+} \mathcal{E}_\lambda(\varphi), \text{ and } \underline{\sigma}_\lambda = \inf_{\varphi \in \mathcal{N}_\lambda^-} \mathcal{E}_\lambda(\varphi).$$

4. Proof of The Main Result

In this section, we will continue to study the functional energy in the Nehari manifold sets and prove that this functional admits a critical point in each subset. We finish our proof by proving that these critical points are solutions to the main problem. So in the rest of this paper, we assume that conditions $(\mathbf{H}_1)$-$(\mathbf{H}_2)$ are fulfilled. We begin by proving the following propositions.

Proposition 4.1. *If conditions* $(\mathbf{H}_1)$-$(\mathbf{H}_2)$ *are satisfied, then for each* $\lambda \in (0, \lambda_*)$, *there exists* $\varphi_* \in \mathcal{N}_\lambda^+$ *such that*

$$\mathcal{E}_\lambda(\varphi_*) = \overline{\sigma}_\lambda = \inf_{\varphi \in \mathcal{N}_\lambda^+} \mathcal{E}_\lambda(\varphi).$$

Proof. Since the functional $\mathcal{E}_\lambda$ is bounded below on $\mathcal{N}_\lambda \supset \mathcal{N}_\lambda^+$, then there exists a minimizing sequence $\{\varphi_k\} \subset \mathcal{N}_\lambda^+$. That is a sequence that satisfies

$$\lim_{k\to\infty} \mathcal{E}_\lambda(\varphi_k) = \inf_{\varphi \in \mathcal{N}_\lambda^+} \mathcal{E}_\lambda(\varphi).$$

Also, from Lemma 3.1, we see that $\mathcal{E}_\lambda$ is coercive, which implies that $\{\varphi_k\}$ is bounded. On the other hand, from Remark 2.3, we see that the space $E_0^{\alpha,p}$

is reflexive. So up to a subsequence, there exists φ_* such that

$$\begin{cases} \varphi_n \rightharpoonup \varphi_*, & \text{weakly in } \quad E_0^{\alpha,p}, \\ \varphi_n \to \varphi_*, & \text{strongly in } \quad L^r([0,T],\mathbb{R}), \\ \varphi_n \to \varphi_*, & \text{a.e. in } [0,T]. \end{cases}$$

First af all, let us prove that

$$\lim_{n\to\infty} \int_0^T f(t)|\varphi_n|^{1-\beta} dt = \int_0^T f(t)|\varphi_*|^{1-\beta} dt. \tag{4.1}$$

From Lemma 2.2, as n large enough we have

$$\begin{aligned} \int_0^T f(t)|\varphi_n|^{1-\beta} dt &\leq \int_0^T f(t)|\varphi_*|^{1-\beta} dt + \int_0^T f(t)|\varphi_n - \varphi_*|^{1-\beta} dt \\ &\leq \int_0^T f(t)|\varphi_*|^{1-\beta} dt + T\|f\|_\infty \|\varphi_n - \varphi_*\|_\infty^{1-\beta} dt \\ &\leq \int_0^T f(t)|\varphi_*|^{1-\beta} dt + \circ(1). \end{aligned}$$

where $\circ(1)$ satisfies $\lim\limits_{n\to\infty} \circ(1) = 0$.
On the other hand, for n large enough, we get

$$\begin{aligned} \int_0^T f(t)|\varphi_*|^{1-\beta} dt &\leq \int_0^T f(t)|\varphi_n|^{1-\beta} dt + \int_0^T f(t)|\varphi_n - \varphi_*|^{1-\beta} dt \\ &\leq \int_0^T f(t)|\varphi_n|^{1-\beta} dt + T\|f\|_\infty \|\varphi_n - \varphi_*\|_\infty^{1-\beta} dt \\ &\leq \int_0^T f(t)|\varphi_n|^{1-\beta} dt + \int_0^T f(t)|\varphi_n - \varphi_*|^{1-\beta} dt \\ &\leq \int_0^T f(t)|\varphi_n|^{1-\beta} dt + \circ(1). \end{aligned}$$

Hence, the above information implies that

$$\int_0^T f(t)|\varphi_n|^{1-\beta} dt = \int_0^T f(t)|\varphi_*|^{1-\beta} dt + \circ(1).$$

That is, (4.1) holds.

Now, from [6], there exists $\rho \in L^r([0,T],\mathbb{R})$ such that for n large enough

$|\varphi_n(t)| \leq \rho(t)$. So, from the Dominated convergence theorem we obtain

$$\lim_{n\to\infty} \int_0^T G(t, \varphi_n(t))\, dt = \int_0^T G(t, \varphi_*(t))\, dt. \tag{4.2}$$

Now, by combining Equations (4.1), (4.2) with the weakly lower semi-continuity of the norm, we deduce that $\mathcal{E}_\lambda$ is weakly lower semi-continuous. Hence, we get

$$\mathcal{E}_\lambda(\varphi_*) \leq \lim_{n\to\infty} \mathcal{E}_\lambda(\varphi_n) = \overline{\sigma}_\lambda.$$

On the other hand, from the definition of $\overline{\sigma}_\lambda$, we have $\mathcal{E}_\lambda(\varphi_*) \geq \overline{\sigma}_\lambda$. Finally, the above information implies that

$$\mathcal{E}_\lambda(\varphi_*) = \overline{\sigma}_\lambda.$$

To finish the proof of Proposition 4.1, we need to prove that $\varphi_* \in \mathcal{N}_\lambda^+$. To this aim, let us prove that

$$\varphi_k \to \varphi_*, \ \text{ strongly in } \ E_0^{\alpha,p}.$$

If this is not true, then we get

$$\|\varphi_*\|_M < \liminf_{k\to\infty} \|\varphi_k\|_M. \tag{4.3}$$

On the other hand, from Lemma 3.3, there exists s_1 such that $s_1\varphi_* \in \mathcal{N}_\lambda^+$. So, by combining Equations (4.1), (4.2) with Equation (4.3), we obtain

$$\begin{aligned}
\lim_{k\to\infty} J'_{\varphi_k}(s_1) &= s_1^{p-1} \lim_{k\to\infty} \big(a + bs_1^p\|\varphi_k\|_M^p\big)^{p-1}\|\varphi_k\|_M^p \\
&\quad - \lambda s_1^{r-1} \lim_{k\to\infty}\int_0^T G(s, \varphi_k(s))ds - s_1^{-\beta} \lim_{k\to\infty}\int_0^T f(s)|\varphi_k(s)|^{1-\beta}ds \\
&> s_1^{p-1}\big(a + bs_1^p\|\varphi_*\|_M^p\big)^{p-1}\|\varphi_*\|_M^p - \lambda s_1^{r-1}\int_0^T G(s, \varphi_*(s))ds \\
&\quad - s_1^{-\beta}\int_0^T f(s)|\varphi_*(s)|^{1-\beta}ds \\
&= J'_{\varphi_*}(s_1) = 0.
\end{aligned}$$

Therefore, for k sufficiently large, we have $J'_{\varphi_k}(s_1) > 0$. On the other hand, since $\varphi_k \in \mathcal{N}_\lambda^+$, then we have, $J'_{\varphi_k}(1) = 0$ and $J''_{\varphi_k}(1) > 0$. Thus, by Lemma

3.3, we get $J'_{\varphi_k}(s) < 0$, for all $s \in (0,1)$. So we must have $s_1 > 1$. Since $s_1\varphi_* \in \mathcal{N}_\lambda^+$, then using again Lemma 3.3, we have $J_{\varphi_*}(t)$ is decreasing on $(0, s_1)$. Hence, we can conclude that

$$\mathcal{E}_\lambda(s_1\varphi_*) \leq \mathcal{E}_\lambda(\varphi_*) < \liminf_{k\to\infty} \mathcal{E}_\lambda(\varphi_k) = \overline{\sigma}_\lambda.$$

Which is a contradiction. Hence, $\varphi_k \to \varphi_*$, strongly in $E_0^{\alpha,p}$. Now, from the fact that $\varphi_k \in \mathcal{N}_\lambda^+$ implies that

$$J'_{\varphi_k}(1) = 0, \quad \text{and} \quad J''_{\varphi_k}(1) > 0.$$

So using (4.1), (4.2) and the strongly convergence we deduce that

$$J'_{\varphi_*}(1) = 0, \quad \text{and} \quad J''_{\varphi_*}(1) \geq 0.$$

That is

$$\varphi_* \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0.$$

Finally, Lemma 3.2, implies that $\varphi_* \in \mathcal{N}_\lambda^+$, and this finishes the proof of Proposition 4.1. □

Proposition 4.2. *If conditions* **(H$_1$)-(H$_2$)** *are satisfied, then for each* $\lambda \in (0, \lambda_*)$, *there exists* $\psi_* \in \mathcal{N}_\lambda^-$ *such that*

$$\mathcal{E}_\lambda(\psi_*) = \underline{\sigma}_\lambda = \inf_{\varphi\in\mathcal{N}_\lambda^-} \mathcal{E}_\lambda(\varphi).$$

Proof. Since the proof of Proposition 4.2 is very similar to the one in Proposition 4.1, then we omit it. □

***Proof of Theorem* 1.1.** From Proposition 4.1, there exists $\varphi_* \in \mathcal{N}_\lambda^+$ satisfying

$$\mathcal{E}_\lambda(\varphi_*) = \overline{\sigma}_\lambda = \inf_{\varphi\in\mathcal{N}_\lambda^-} \mathcal{E}_\lambda(\varphi).$$

Moreover, from Lemma 3.4, for each $\psi \in E_0^{\alpha,p}$, there exists $\varepsilon > 0$ small enough and a continuous function Φ such that for all real number t with $|t| < \varepsilon$, we have

$$\Phi(t)(\varphi_* + t\psi) \in \mathcal{N}_\lambda^+, \quad \text{and} \quad \Phi(t) \to 1, \quad \text{as} \quad t \to 0.$$

So for t small enough we have

$$\mathcal{E}_\lambda(\Phi(t)(\varphi_* + t\psi)) - \mathcal{E}_\lambda(\varphi_*) \geq 0.$$

Dividing the above inequality by $t > 0$, and letting t tend to *zero*, we obtain

$$S(\varphi_*) \int_0^T J_p({}_0D_t^\alpha \varphi_*(t))\, {}_0D_t^\alpha \psi(t) + M(t) J_p({}_0D_t^\alpha \varphi_*(t))\, {}_0D_t^\alpha \psi(t)\, dt \\ - \int_0^T \frac{f(t)}{\varphi_*^\beta(t)} \psi(t)\, dt - \lambda \int_0^T g(t, \varphi_*(t))\, \psi(t)\, dt \geq 0.$$

since ψ is arbitrary in $E_0^{\alpha,p}$, then the above inequality holds true if we replace ψ by $-\psi$, which yields to the equality instead of the inequality. So, from Definition 1.1, we see that $\varphi_* \in \mathcal{N}_\lambda^+$ is a weak solution for the problem (P_λ). Moreover, the fact that $J''_{\varphi_*}(1) > 0$ implies that φ_* is nontrivial.

On the other hand, if we proceed as above, we can use Proposition 4.2 and Lemma 3.4 to prove that $\psi_* \in \mathcal{N}_\lambda^-$ is a nontrivial weak solution for the problem (P_λ). Finally, the fact that $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$ are disjoint ends the proof of Theorem 1.1.

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