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INVOLUTIONS IN WEYL GROUPS AND NIL-HECKE ALGEBRAS

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Abstract

In a previous article we have defined an action of the Iwahori-Hecke algebra of a Coxeter group W on a free module with basis indexed by the involutions in W. In this paper we show that the specialization of this action at the parameter 0 has a simple description.

0.1. Let W be a Coxeter group and let S be the set of simple reflections of W; we assume that S is finite. Let $w \mapsto |w|$ be the length function on W. Let H be the Iwahori-Hecke algebra attached to W. Recall that H is the free $\mathbf{Z}[u]$-module with basis $\{\mathbf{T}_w; w \in W\}$ (u is an indeterminate) with (associative) multiplication characterized by $\mathbf{T}_s\mathbf{T}_w = \mathbf{T}_{sw}$ if $s \in S, w \in W, |sw| = |w|+1$, $\mathbf{T}_s\mathbf{T}_w = u^2\mathbf{T}_{sw} + (u^2-1)\mathbf{T}_w$ if $s \in S, w \in W, |sw| = |w|-1$.

Let $w \mapsto w^*$ be an automorphism with square 1 of W preserving S and let $\mathbf{I}_* = \{w \in W; w^* = w^{-1}\}$ be the set of "twisted involutions" in W. Let M be the free $\mathbf{Z}[u]$-module with basis $\{\mathbf{a}_x; x \in \mathbf{I}_*\}$. For any $s \in S$ we define a $\mathbf{Z}[u]$-linear map $\mathbf{T}_s : M \to M$ by

$$\mathbf{T}_s\mathbf{a}_x = u\mathbf{a}_x + (u+1)\mathbf{a}_{sx} \text{ if } x \in \mathbf{I}_*, sx = xs^*, |sx| = |x|+1,$$

$$\mathbf{T}_s\mathbf{a}_x = (u^2-u-1)\mathbf{a}_x + (u^2-u)\mathbf{a}_{sx} \text{ if } x \in \mathbf{I}_*, sx = xs^*, |sx| = |x|-1,$$

$$\mathbf{T}_s\mathbf{a}_x = \mathbf{a}_{sxs^*} \text{ if } x \in \mathbf{I}_*, sx \neq xs^*, |sx| = |x|+1,$$

$$\mathbf{T}_s\mathbf{a}_x = (u^2-1)\mathbf{a}_x + u^2\mathbf{a}_{sxs^*} \text{ if } x \in \mathbf{I}_*, sx \neq xs^*, |sx| = |x|-1.$$

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It is known that the maps $\mathbf{T}_s$ define an H-module structure on M. (See [2] for the case where W is a Weyl group or an affine Weyl group and [3] for the general case; the case where W is a Weyl group and u is specialized to 1 was considered earlier in [1].) When u is specialized to 0, H becomes the free $\mathbf{Z}$-module H_0 with basis $\{T_w; w \in W\}$ with (associative) multiplication characterized by

$T_sT_w = T_{sw}$ if $s \in S, w \in W, |sw| = |w| + 1$,

$T_sT_w = -T_w$ if $s \in S, w \in W, |sw| = |w| - 1$

(a nil-Hecke algebra). From these formulas we see that there is a well defined monoid structure $w, w' \mapsto w \bullet w'$ on W such that for any w, w' in W we have

$T_wT_{w'} = (-1)^{|w|+|w'|+|w \bullet w'|}T_{w \bullet w'}$

(equality in H_0). In this monoid we have

(a) $(w \bullet w')^{-1} = w'^{-1} \bullet w^{-1}$,

(b) $(w \bullet w')^* = w^* \bullet w'^*$,

for any w, w' in W.

0.2. When u is specialized to 0, the H-module M becomes the H_0-module M_0 with $\mathbf{Z}$-basis $\{a_w; w \in \mathbf{I}_*\}$ in which for $s \in S$ we have

$$\begin{aligned} &T_sa_x = a_{sx} \text{ if } x \in \mathbf{I}_*, sx = xs^*, |sx| = |x| + 1,\\ &T_sa_x = a_{sxs^*} \text{ if } x \in \mathbf{I}_*, sx \neq xs^*, |sx| = |x| + 1, \\ &T_sa_x = -a_x \text{ if } x \in \mathbf{I}_*, |sx| = |x| - 1. \end{aligned} \tag{a}$$

By [3, 4.5] there is a unique function $\phi : \mathbf{I}_* \to \mathbf{N}$ such that $\phi(1) = 0$ and such that for any $s \in S, x \in \mathbf{I}_*$ with $|sx| = |x| - 1$ we have $\phi(x) = \phi(sx) + 1$ if $sx = xs^*$, $\phi(sxs^*) = \phi(x)$ if $sx \neq xs^*$; moreover we have $|x| = \phi(x) \bmod 2$ for all $x \in \mathbf{I}_*$. For $x \in \mathbf{I}_*$ we set $||x|| = (1/2)(|x| + \phi(x)) \in \mathbf{N}$. (See also [5, p.92].)

One of our main observations is that the action of any $T_w, w \in W$ on a basis element $a_x, x \in \mathbf{I}_*$ of M_0 has a simple description in terms of the monoid $(W, \bullet)$, namely

(b) $T_wa_x = (-1)^{|w|+||x||+||w \bullet x \bullet w^{*-1}||}a_{w \bullet x \bullet w^{*-1}}$.

(Note by 0.1(a),(b), we have $w \bullet x \bullet w^{*-1} \in \mathbf{I}_*$ whenever $x \in \mathbf{I}_*$.) See §1 for a proof.

0.3. In [4] it is shown that there is a unique map $\pi : W \to \mathbf{I}_*$ such that for any $w \in W$ we have $T_w a_1 = \pm a_{\pi(w)}$ in M_0. This can be deduced also from 0.2(b), which gives a closed formula for π, namely

$$\pi(w) = w \bullet w^{*-1}. \tag{a}$$

By [4, 1.8(c)],

(b) the map $\pi : W \to \mathbf{I}_*$ is surjective.

More generally, let $J \subset S$ be such that the subgroup W_J generated by J is finite. Let w_J be the longest element of W_J. Let J^* be the image of J under $*$. Let ${}^J W = \{w \in W; |w| = |w_J| + |w_J w|\}$, $W^{J^*} = \{w \in W; |w| = |w_{J^*}| + |w w_{J^*}|\}$, ${}^J W^{J^*} = {}^J W \cap W^{J^*}$. The following extension of (b) is verified in §2.

(c) π restricts to a surjective map ${}^J\pi : {}^J W \to \mathbf{I}_* \cap {}^J W^{J^*}$.

0.4. In 2.3 it is shown that when W is an irreducible affine Weyl group then for a suitable $J, *$, the map in 0.3(c) can be interpreted as a (surjective) map from the set of translations in W to the set of dominant translations in W. This map is bijective if W is of affine type A_1 (see 2.4) but is not injective if W is of affine type A_2. This map takes any dominant translation to its square. It would be interesting to find a simple formula for this map extending the formulas in 2.4.

1. Proof of 0.2(b)

We prove 0.2(b) by induction on $|w|$. If $w = 1$ we have $w \bullet x \bullet w^{*-1} = x$ hence the desired result holds. Assume now that $w = s \in S$. If $sx = xs^*, |sx| = |x|+1$, we have $s \bullet x \bullet s^* = (sx) \bullet s^* = x \bullet s^* \bullet s^* = x \bullet s^* = xs^* = sx$ and

$$\begin{aligned}&1 + ||x|| + ||sx|| = 1 + (1/2)(|x| + \phi(x)) + 1/2(|sx| + \phi(sx))\\ &= 1 + (1/2)(|x| + \phi(x)) + 1/2(|x| + 1 + \phi(x) + 1) = |x| + \phi(x) + 2 = 0 \bmod 2.\end{aligned}$$

If $sx \neq xs^*, |sx| = |x| + 1$, we have $|sxs^*| = |sx| + 1$ hence $s \bullet x \bullet s^* = (sx) \bullet s^* = sxs^*$ and

$$1 + ||x|| + ||sxs^*|| = 1 + (1/2)(|x| + \phi(x)) + 1/2(|sxs^*| + \phi(sxs^*))$$

$= 1 + (1/2)(|x| + \phi(x)) + 1/2(|x| + 2 + \phi(x)) = |x| + \phi(x) + 2 = 0 \bmod 2.$

If $|sx| = |x| - 1$, we have $|xs^*| = |x| - 1$ hence $s \bullet x \bullet s^* = x \bullet s^* = x$ and $1 + ||x|| + ||x|| = 1 \bmod 2$ hence the desired result holds.

Assume now that $w \neq 1$. We can find $s \in S$ such that $|sw| = |w| - 1$. By the induction hypothesis we have

$$T_{sw}a_x = (-1)^{|sw|+||x||+||(sw)\bullet x\bullet (sw)^{*-1}||}a_{(sw)\bullet x\bullet (sw)^{*-1}}.$$

Using the earlier part of the proof we have

$$T_w a_x = T_s T_{sw} a_x = (-1)^{|sw|+||x||+||(sw)\bullet x\bullet (sw)^{*-1}||}T_s a_{(sw)\bullet x\bullet (sw)^{*-1}}$$
$$= (-1)^{|sw|+||x||+||(sw)\bullet x\bullet (sw)^{*-1}||}(-1)^{1+||(sw)\bullet x\bullet (sw)^{*-1}||+||w\bullet x\bullet w^{*-1}||}$$
$$a_{s\bullet (sw)\bullet x\bullet (sw)^{*-1}\bullet s^*}$$
$$= (-1)^{|w|+||x||+||w\bullet x\bullet w^{*-1}||}a_{w\bullet x\bullet w^{*-1}}.$$

This completes the proof of 0.2(b).

2. The map $^J\pi$

2.1. For w_1, w_2 in W we say that w_1 is an initial segment of w_2 if there exist $s_1, s_2, \dots s_n$ in S and $k \in [0, n]$ such that $w_1 = s_1 s_2 \dots s_k$, $w_2 = s_1 s_2 \dots s_n$, $|w_1| = k, |w_2| = n$; we say that w_1 is a final segment of w_2 if w_1^{-1} is an initial segment of w_2^{-1}. We show:

(a) For w, w' in W, w is an initial segment of $w \bullet w'$ and w' is a final segment of $w \bullet w'$.

We argue by induction on $|w'|$. If $w' = 1$ the result is obvious. Assume now that $w' \neq 1$. We can find $s \in S$ such that $|w'| = |sw'| + 1$. We have $w \bullet w' = w \bullet s \bullet (sw')$. If $|ws| = |w| + 1$ then $w \bullet w' = (ws) \bullet (sw')$ and by the induction hypothesis ws is an initial segment of $(ws) \bullet (sw') = w \bullet w'$. Since w is an initial segment of ws it follows that w is an initial segment of $w \bullet w'$. If $|ws| = |w| - 1$ then $w \bullet w' = w \bullet (sw')$ and by the induction hypothesis w is an initial segment of $w \bullet (sw') = w \bullet w'$. This proves the first assertion of (a). The second assertion of (a) follows from the first using 0.1(a).

2.2. We now fix $J \subset S$ as in 0.3. Let $w \in {}^JW$. Then w_J is an initial segment of w and (by 2.1(a)) w is an initial segment of $w \bullet w^{*-1}$ hence w_J is an initial segment of $w \bullet w^{*-1}$ so that $w \bullet w^{*-1} \in {}^JW$. Since $w \bullet w^{*-1} \in \mathbf{I}_*$, we see that w_{J^*} is a final segment of $w \bullet w^{*-1}$ so that $w \bullet w^{*-1} \in W^{J^*}$. Thus, we have $w \bullet w^{*-1} \in \mathbf{I}_* \cap {}^JW^{J^*}$. We see that the map ${}^J\pi : {}^JW \to \mathbf{I}_* \cap {}^JW^{J^*}$ in 0.3(c) is well defined.

We now prove that this map is surjective. Let $x \in \mathbf{I}_* \cap {}^JW^{J^*}$. Let z be the unique element of minimal length in $W_J x W_{J^*}$. Now z^{*-1} is again an element of minimal length in $W_J x W_{J^*}$, so it must be equal to z. Thus, we have $z \in \mathbf{I}_*$. By 0.2(b) we have $T_{w_J} a_z = \pm a_{w_J \bullet z \bullet w_{J^*}}$. Note that

$$w_J \bullet z \bullet w_{J^*} \in W_J x W_{J^*} = W_J z W_{J^*}.$$

By 2.1(a), w_J is an initial segment of $w_J \bullet z \bullet w_{J^*}$ so that

$$w_J \bullet z \bullet w_{J^*} \in {}^JW.$$

Since $w_J \bullet z \bullet w_{J^*} \in \mathbf{I}_*$, we have also $w_J \bullet z \bullet w_{J^*} \in W^{J^*}$ so that $w_J \bullet z \bullet w_{J^*} \in {}^JW^{J^*}$. Thus, $w_J \bullet z \bullet w_{J^*}$ is the element of maximal length in $W_J x W_{J^*}$ so that it must be equal to x and we have $T_{w_J} a_z = \pm a_x$. By 0.2(b) we have $T_e a_1 = \pm a_z$ for some $e \in W$. We then have $\pm a_x = \pm T_{w_J} T_e a_1 = \pm T_w a_1$ where $w = w_J \bullet e$ has w_J as initial segment (see 2.1(a)) so that $w \in {}^JW$. This proves the surjectivity of ${}^J\pi$.

2.3. In the remainder of this section we assume that W is an irreducible affine Weyl group. Let $\mathcal{T}$ be the (normal) subgroup of W consisting of translations. We fix a proper subset $J \subset S$ such that $W = W_J\mathcal{T}$. Then W_J is finite. We assume that $w \mapsto w^*$ is the unique automorphism $w \mapsto w^*$ of W such that $w^* = w_J w w_J$ for $w \in W_J$ and $t^* = w_J t^{-1} w_J$ for $t \in \mathcal{T}$. This automorphism preserves S and has square 1. We have $J^* = J$ and ${}^JW^J \subset \mathbf{I}_*$ (see [3, 8.2]). Let $\mathcal{T}_{dom} = \{t \in \mathcal{T}; |w_J t| = |w_J| + |t|\} = \{t \in \mathcal{T}; w_J t \in {}^JW\}$.

(a) If $t \in \mathcal{T}_{dom}$ we have $|w_J t w_J| = |t|$; hence $|w_J t| = |w_J t w_J| + |w_J|$ and $w_J t = (w_J t w_J) \bullet w_J$.

Indeed, $|w_J t w_J| = |(t^*)^{-1}| = |t^*| = |t|$.

It is known that

(b) if t, t' are in $\mathcal{T}_{dom}$ then $tt' \in \mathcal{T}_{\rm dom}$ and $|tt'| = |t| + |t'|$ hence $t \bullet t' = tt'$ and $w_J \bullet (tt') = w_J tt'$.

From (a) we see that $\{w_J t; t \in \mathcal{T}_{dom}\} \subset {}^JW^J$; in fact this inclusion is an equality. For $t \in \mathcal{T}$ we define $[t] \in {}^JW$ by $W_J t = W_J[t]$; now $t \mapsto [t]$ is a bijection $\mathcal{T} \leftrightarrow {}^JW$. Under this bijection and the bijection $\mathcal{T}_{\rm dom} \leftrightarrow {}^JW^J$, $t \leftrightarrow w_J t$, the map ${}^J\pi : {}^JW \to {}^JW^J$ becomes a map

(c) $\pi' : \mathcal{T} \to \mathcal{T}_{dom}$.

The following result describes explicitly the restriction of π' to $\mathcal{T}_{dom}$.

(d) For $t \in \mathcal{T}_{dom}$ we have ${}^J\pi(w_J t) = w_J t^2$. Hence $\pi'(t) = t^2$.

Using (a), (b) and the definitions we have

$$\begin{aligned} {}^J\pi(w_J t) &= (w_J t) \bullet (w_J t)^{*-1} = (w_J t w_J) \bullet w_J \bullet w_J \bullet (w_J t w_J)^{*-1} \\ &= (w_J t w_J) \bullet w_J \bullet (w_J t^{*-1} w_J) = (w_J t) \bullet t = w_J \bullet t \bullet t = w_J t^2. \end{aligned}$$

This proves (d).

2.4. In the setup of 2.3 we assume that W is of affine type A_1. We can assume that $S = \{s_1, s_2\}$ and $J = \{s_1\}$; now $*$ is the identity map. We shall write $i_1 i_2 i_3 \ldots$ instead of $s_{i_1} s_{i_2} s_{i_3} \ldots$. Then $A = 21 \in \mathcal{T}_{dom}$ and in fact the elements of $\mathcal{T}_{dom}$ are precisely the powers $A^m, m \in \mathbf{N}$. The elements of ${}^JW$ are $1t, 1t2$ with $t \in \mathcal{T}_{dom}$. The elements of ${}^JW^{J^*}$ are $1t$ with $t \in \mathcal{T}_{dom}$. If $t = A^m, m \in \mathbf{N}$, we have

$${}^J\pi(1t) = 1A^{2m},$$

$${}^J\pi(1t2) = 1A^{2m+1}.$$

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