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STABILITY OF POSITIVE SOLUTIONS FOR A CLASS OF NONLINEAR HADAMARD TYPE FRACTIONAL DIFFERENTIAL EQUATIONS WITH p-LAPLACIAN

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Abstract

In this paper, using fixed point methods, we will establish some existence theorems of solutions for integral boundary value problems of nonlinear Hadamard fractional differential equations with p-Laplacian. Also, the Hyers-Ulam stability of this class of problem is studied. An example is included to show the applicability of our results.

1. Introduction

The theory of fractional calculations has attracted the attention of researchers from various disciplines, such as engineering, mathematics, physics, chemistry, biology and bioengineering and other applied sciences, processing, control theory, signals, fluid dynamics, modern physics, set theory, hydrodynamics, viscoelastic theory, computer networking, information processing system networking, notable and picture processing; see the remarkable monographs [5, 6, 15, 17, 18, 20, 22, 23] over the past two decades. Various mathematical procedures as the Banach contract principle, Schauder's fixed point theorem, Schaefera's fixed point theorem, the Leray-Schauder nonlinear alternative, Mönch's fixed point and the measure of noncompactness, have been examined by scientists through different aspects oriented towards the search for differential fractional equations, in the books [1, 2] and the papers [7, 8, 9]. Another remarkable line of research, which attracts more

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attention, is the analysis of the stability of fractional order differential equation. The first work was initiated by Ulam himself and was later confirmed by Hyers in [16], hence the Ulam-Hyers unstability. Afterwards, Rassias introduced Ulam-Hyers-Rassias stability; see recently reported stability results in the sense of Ulam [3, 4, 19, 21, 25]. It should be noted that the above, the said areas of interest (existence and stability) have often been the subject of deliberations the parameters of the Riemann-Liouville, Caputo and Hilfer derivatives.

Some authors have worked on the existence of solutions for fractional differential equations with p-Laplacian operator. For example, the authors in [27], by using the Krasnoselskii fixed point theorem and the Leggett-Williams theorem, obtained results of the existence of solutions of Riemann-Liouville fractional equations involving the p-Laplacian, which is given by:

$$\begin{cases} D^{\gamma}_{0^+}\Big(\phi_p\big(D^{\alpha}_{0^+}u(t)\big)\Big) = f(t,u), & 0<t<1 \text{ and } 1<\alpha,\gamma\le 2,\\ D^{\alpha}_{0^+}u(0)=u(0)=0, & \\ u'(1)=au(\xi),\ D^{\alpha}_{0^+}u(1)=bD^{\alpha}_{0^+}u(\eta), & 0\le a,b\le 1 \text{ and } 0<\xi,\eta<1. \end{cases}$$

In [28], the authors used fixed point methods to study the existence of positive solutions for Hadamard fractional integral boundary value problem given by

$$\begin{cases} D^{\beta}\Big(\phi_p\big(D^{\alpha}u(t)\big)\Big) = f(t,u), & 1<t<e,\\ u(1)=D^{\alpha}u(1)=u'(1)=u'(e)=0, & \\ \phi_p(D^{\alpha}u(e)) = \mu\int_1^e \phi_p(D^{\alpha}u(t))\frac{dt}{t}. & \end{cases}$$

We note that no such survey on the p-Laplacian operator in the frame of fractional differential equation with delay with singularity is known in the literature. Therefore, inspired by the aforementioned works, we examine the following proposition for the existence of a positive solution and stability analysis of the following problem:

$$\begin{cases} D^{\sigma}(\phi_p(D^{\nu}x))(t)+a(t)f(t,x)=0, & \\ x(1)=\phi_p(D^{\nu}x)(1)=0, & \\ A_1I^{\gamma_1}x(\eta_1)+B_1x(e)=c_1, & 0<\gamma_1,\\ A_2I^{\gamma_2}(\phi_p(D^{\nu}x))(\eta_2)+B_2\phi_p(D^{\nu}x)(e)=c_2, & 0<\gamma_2, \end{cases} \tag{1.1}$$

where ν and σ are in $(1,2]$, η_1 and η_2 are in $(1,e)$, A_1, A_2, B_1, B_2, c_1 and c_2 are fixed real numbers.

In this paper, we study the existence of solutions for the Hadamard fractional p-Laplacian given in Equation (1.1). For this purpose, we give the essential definitions and results useful for this paper in Section 2. Note that in Section 3, we give firstly the solution. Then, using the Schauder's fixed-point theorem, we obtain the existence of unique solution for the problem (1.1). And we establish the stability of the solution in Section 4. Finally, in Section 5, we give an illustrative example.

2. Preliminary

In this section, we give some basic notions like definitions, properties and theorems on fractional differential equations essentially on the Hadamard fractional calculus, which we will use in this paper. For more details, we refer the reader to [18].

Definition 2.1. Let ν be a positive number upper than 1 and $g:[1,\infty)\to\mathbb{R}$.

- The Hadamard fractional integral of order ν for a function g is defined as

$$I^{\nu}g(t)=\frac{1}{\Gamma(\nu)}\int_1^t\big(\log\frac{t}{s}\big)^{\nu-1}\frac{g(s)}{s}ds, \tag{2.1}$$

$$=\frac{1}{\Gamma(\nu)}\left\{\big(\log(t)\big)^{\nu}\int_0^1\big(1-s\big)^{\nu-1}g(e^{s\log(t)})ds\right\}, \tag{2.2}$$

where $\log(\cdot)=\log_e(\cdot)$.

- The Hadamard derivative of fractional order ν for a function g is given by

$$D^{\nu}g(t)=\big(t\frac{d}{dt}\big)^{n}I^{n-\nu}, \tag{2.3}$$

where $n=[\nu]+1$ and $[\nu]$ denotes the integer part of the real number ν.

Lemma 2.1 ([18]). *For any $t \in [1,e]$, $c \in \mathbb{R}$ and any constants ν, σ in $[1,2]$, we have*

$$I^{\nu}(c)(t) = cI^{\nu}1(t) = c\frac{1}{\Gamma(\nu)}\int_1^t (\log\frac{t}{s})^{\nu-1}\frac{ds}{s} = \frac{c(\log t)^{\nu}}{\Gamma(\nu+1)}, \tag{2.4}$$

and

$$I^{\nu}[(\log(\cdot))^{\sigma}](t) = \frac{1}{\Gamma(\nu)}\int_1^t (\log\frac{t}{s})^{\nu-1}\frac{[\log(s)]^{\sigma}ds}{s} = \frac{(\log t)^{\nu+\sigma}\Gamma(\sigma+1)}{\Gamma(\nu+\sigma+1)}. \tag{2.5}$$

Lemma 2.2 ([18]). *Let $\nu > 0$ and $x \in C[1,\infty) \cap L^1[1,\infty)$. Then the Hadamard fractional differential equation $D^{\nu}x(t) = 0$ has the solution*

$$x(t) = \sum_{i=1}^{n} c_i(\log t)^{\nu-i},$$

and the following formula holds:

$$I^{\nu}D^{\nu}x(t) = x(t) + \sum_{i=1}^{n} c_i(\log t)^{\nu-i},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ and $n = [\nu] + 1$.

The following elementary relations are useful:

1. If $q > 2$, and $max(|x|, |y|) \leq R$, then

$$|\phi_q(x) - \phi_q(y)| \leq (q-1)R^{q-2}|x-y|. \tag{2.6}$$

2. If $k > 1$, then for all positive numbers a and b, we have

$$(a+b)^k \leq (2^k - 1)(a^k + b^k). \tag{2.7}$$

In the sequel we will make use of the following Schauder's fixed-point theorem.

Theorem 2.1 ([13]). *Let E be a Banach space and F be a nonempty bounded convex and closed subset of E and $Q : F \to F$ is a compact, and continuous map. Then Q has at least one fixed point in F.*

By $C([1,e],\mathbb{R})$, we denote the Banach space of all continuous functions from $[1,e]$ to $\mathbb{R}$ endowed with the norm

$$\|x\| = \max_{t\in[1,e]} |x(t)|.$$

The following elementary relations are useful:
Let us introduce the following assumptions needed to prove the main theorems:

(H_1) For each $x \in C([1,e],\mathbb{R})$, the function $t \mapsto f(t,x)$ is measurable on $[1,e]$ and the function $x \mapsto f(t,x)$ is continuous on $C([1,e],\mathbb{R})$ for a.e $t \in [1,e]$.

(H_2) There exist nonnegative functions $p(t) \in C([1,e],\mathbb{R})$ and a bounded function $g : \mathbb{R} \mapsto \mathbb{R}$, such that

$$|f(t,x)| \le p(t)g(|x|), \quad \text{for each} \quad t \in [1,e] \quad \text{and} \quad x \in \mathbb{R},$$

and g admit a maximum denoted by $M = \max_{x\in\mathbb{R}} |g(|x|)|$.

(H_3) There exists a positive constant δ such that for all $x, y \in C([1,e],\mathbb{R})$, we have

$$|f(t,x) - f(t,y)| \le \delta |x(t) - y(t)|.$$

3. Existence Results

Theorem 3.1. *The following fractional differential equation involving the p-Laplacian*

$$\begin{cases} D^{\sigma}(\phi_p(D^{\nu}x(t))) + a(t)f(t,x) = 0, \\ x(1) = \phi_p(D^{\nu}x(1)) = 0, \\ A_1 I^{\gamma_1} x(\eta_1) + B_1 x(e) = c_1, \\ A_2 I^{\gamma_2}(\phi_p(D^{\nu}x))(\eta_2) + B_2\phi_p(D^{\nu}x)(e) = c_2, \end{cases} \tag{3.1}$$

has a solution given by

$$\begin{aligned} x(t) =& c_4^x (\log t)^{\nu-1} \\ &- \frac{1}{\Gamma(\nu)\Gamma(\sigma)} \int_1^t (\log(\frac{t}{s}))^{\nu-1} \phi_q\Big[\int_1^s (\log(\frac{s}{u}))^{\sigma-1} a(u) f(u,x) \frac{du}{u} \end{aligned}$$

$$+ c_3^x(\log(s))^{\sigma-1}\Big]\frac{ds}{s}, \tag{3.2}$$

where c_3^x and c_4^x are given by

$$c_3^x = \frac{c_2 + A_2 I^{\sigma+\gamma_2}(a(\eta_2)f(\eta_2,x)) + B_2 I^{\sigma}(a(e)f(e,x))}{\Delta_1}, \tag{3.3}$$

and

$$c_4^x = \{c_1 + A_1 I^{\nu+\gamma_1}\phi_q[I^{\sigma}(a(\eta_1)f(\eta_1,x)) + c_3^x(\log\eta_1)^{\sigma-1}] + B_1 I^{\nu}(I^{\sigma}(a(e)f(e,x)) + c_3^x)\}/\Delta_2, \tag{3.4}$$

where Δ_1 and Δ_2 are given respectively by

$$\Delta_1 = B_2 + \frac{A_2\Gamma(\sigma)}{\Gamma(\gamma_2+\sigma)}(\log\eta_2)^{\gamma_2+\sigma-1},$$

and

$$\Delta_2 = B_1 + \frac{A_1\Gamma(\nu)}{\Gamma(\gamma_1+\nu)}(\log\eta_1)^{\gamma_1+\nu-1}.$$

Proof. As argued in [18], the Hadamard differential equation in (3.1) can be written as

$$\phi_p(D^{\nu}x(t)) = -I^{\sigma}(a(t)f(t,x)) + \alpha(\log t)^{\sigma-1} + \beta(\log t)^{\sigma-2}.$$

Since $\phi_p(D^{\nu}x(1)) = 0$, then $\beta = 0$. So, we get

$$\phi_p(D^{\nu}x(t)) = -I^{\sigma}(a(t)f(t,x)) + \alpha(\log t)^{\sigma-1}. \tag{3.5}$$

By applying I^{γ_2} on both sides of (3.5) for $t = \eta_2$ and using the property (2.5), we obtain

$$I^{\gamma_2}(\phi_p(D^{\nu}x(\eta_2))) = -I^{\gamma_2+\sigma}[a(\eta_2)f(\eta_2,x)] + \alpha\frac{\Gamma(\sigma)}{\Gamma(\eta_2+\sigma)}(\log\eta_2)^{\gamma_2+\sigma-1}. \tag{3.6}$$

On the other hand, putting $t = e$ in Equation (3.5), we obtain

$$\phi_p(D^{\nu}x(e)) = -I^{\sigma}[a(e)f(e,x)] + \alpha. \tag{3.7}$$

By combining Equations (3.6), (3.7) and the second boundary condition in

(3.1), we obtain:

$$c_2 = \alpha\Big[A_2\frac{\Gamma(\sigma)}{\Gamma(\gamma_2+\sigma)}(\log\eta_2)^{\eta_2+\sigma-1} + B_2\Big] - A_2 I^{\gamma_2+\sigma}\Big[a(\eta_2)f(\eta_2,x)\Big] - B_2 I^{\sigma}\Big[a(e)f(e,x)\Big].$$

Therefore,

$$\alpha = \frac{c_2 + A_2 I^{\gamma_2+\sigma}[a(\eta_2)f(\eta_2,x)] + B_2 I^{\sigma}[a(e)f(e,x)]}{\Delta_1} =: c_3^x. \tag{3.8}$$

Then, the solution can be written as follows

$$x(t) = -I^{\nu}\{\phi_q[I^{\sigma}(a(t)f(t,x)) + c_3^x(\log(t))^{\sigma-1}]\} + \alpha'(\log t)^{\nu-1} + \beta'(\log t)^{\nu-2}.$$

Since $x(1)=0$, then $\beta'=0$, and we get

$$x(t) = -I^{\nu}\{\phi_q[I^{\sigma}(a(t)f(t,x)) + c_3^x(\log t)^{\sigma-1}]\} + \alpha'(\log t)^{\nu-1}. \tag{3.9}$$

Now, if we apply I^{γ_1} to (3.9) and replace t by η_1, then, using the property (2.5), we obtain

$$I^{\gamma_1}(x(\eta_1)) = -I^{\gamma_1+\nu}[\phi_q\{I^{\sigma}(a(\eta_1)f(\eta_1,x)) + c_3^x(\log\eta_1)^{\sigma-1}\}] + \alpha'\frac{\Gamma(\nu)}{\Gamma(\eta_1+\nu)}(\log\eta_1)^{\gamma_1+\nu-1}. \tag{3.10}$$

On the other hand, Equation (3.9) with $t=e$, yields

$$x(e) = -I^{\nu}\{\phi_q[I^{\sigma}(a(e)f(e,x)) + c_3^x]\} + \alpha'. \tag{3.11}$$

Finally, by combining Equations (3.10), (3.11) with the first boundary condition c_1, we obtain:

$$c_1 = -A_1 I^{\gamma_1+\nu}[\phi_q(I^{\sigma}(a(\eta_1)f(\eta_1,x)) + c_3^x(\log\eta_1)^{\sigma-1})] - B_1 I^{\nu}[\phi_q(I^{\sigma}(a(e)f(e,x)) + c_3^x)] + \alpha'\Big[A_1\frac{\Gamma(\nu)}{\Gamma(\gamma_1+\nu)}(\log\eta_1)^{\eta_1+\nu-1} + B_1\Big].$$

It follows that

$$\alpha' = \frac{\Delta_2'}{\Delta_2} =: c_4^x, \tag{3.12}$$

where

$$\begin{aligned}\Delta_2' = &c_1 + A_1 I^{\gamma_1+\nu}[\phi_q\{I^\sigma(a(\eta_1)f(\eta_1,x)) + c_3^x(\log\eta_1)^{\sigma-1}\}]\\ &+ B_1 I^\nu[\phi_q\{I^\sigma(a(e)f(e,x)) + c_3^x\}]\end{aligned}$$

Substituting the values of c_3^x and c_4^x in (3.9), we obtain (3.2). This completes the proof. □

Theorem 3.2. *Let $q \geq 2$. If $f \in C([1,e]\times\mathbb{R},\mathbb{R})$ is such that hypotheses (H_1) and (H_2) are satisfied, then the Hadamard fractional boundary value problem* (1.1) *has at least one solution.*

Proof. Let us define the operator $Q : C([1,e],\mathbb{R}) \to C([1,e],\mathbb{R})$ by

$$\begin{aligned}Qx(t) = &c_4^x(\log t)^{\nu-1}\\ &- \frac{1}{\Gamma(\nu)\Gamma(\sigma)}\int_1^t(\log(\frac{t}{s}))^{\nu-1}\phi_q[\int_1^s(\log(\frac{s}{u}))^{\sigma-1}a(u)f(u,x)\frac{du}{u}\\ &+ c_3^x(\log(s))^{\sigma-1}]\frac{ds}{s},\end{aligned}\tag{3.13}$$

where c_3^x and c_4^x are given respectively by Equations (3.3) and (3.4). To prove the existence of solutions for the problem (1.1), we will show the existence of fixed points of the operator Q. For this purpose, we prove that there exists a ball of $C([1,e],\mathbb{R})$ which is invariant by the map Q. Let $R > 0$, $t \in [1,e]$ and $x \in B_R$. Then, using the hypothesis (H_1) and (H_2) and the inequality (2.7), we get:

$$\begin{aligned}&|Q(x)(t)|\\ &\leq \max_{t\in[1,e]}\left\{\frac{1}{\Gamma(\nu)}\int_1^t(\log(\frac{t}{s}))^{\nu-1}\left|\phi_q\Big[\frac{1}{\Gamma(\sigma)}\int_1^s(\log(\frac{s}{u}))^{\sigma-1}a(u)f(u,x)\frac{du}{u}\right.\right.\\ &\qquad\left.\left. + c_3^x(\log(s))^{\sigma-1}\Big]\right|\frac{ds}{s} + |c_4^x|\right\}\\ &\leq \max_{t\in[1,e]}\left\{\frac{1}{\Gamma(\nu)}\int_1^t(\log(\frac{t}{s}))^{\nu-1}\left|\phi_q\Big[\frac{M\|a\|_\infty\|p\|_\infty}{\Gamma(\sigma)}\int_1^s(\log(\frac{s}{u}))^{\sigma-1}\frac{du}{u}\right.\right.\\ &\qquad\left.\left. + c_3^x(\log(s))^{\sigma-1}\Big]\right|\frac{ds}{s} + |c_4^x|\right\}\\ &\leq \max_{t\in[1,e]}\left\{\frac{1}{\Gamma(\nu)}\int_1^t(\log(\frac{t}{s}))^{\nu-1}\left|\phi_q\Big[\frac{M\|a\|_\infty\|p\|_\infty(\log s)^\sigma}{\Gamma(\sigma+1)} + c_3^x(\log s)^{\sigma-1}\Big]\right|\frac{ds}{s}\right.\end{aligned}$$

$$
\begin{aligned}
&\quad + |c_4^x|\Bigg\} \\
&\le \max_{t\in[1,e]} \Bigg\{ \frac{1}{\Gamma(\nu)} \int_1^t (\log(\frac{t}{s}))^{\nu-1} \Bigg| \phi_q \Big[\Big(\frac{M\|a\|_\infty \|p\|_\infty}{\Gamma(\sigma+1)} + c_3^x \Big) (\log s)^\sigma \Big] \Bigg| \frac{ds}{s} + |c_4^x| \Bigg\} \\
&\le \max_{t\in[1,e]} \Bigg| \Bigg\{ \frac{1}{\Gamma(\alpha)} \int_1^t (\log(\frac{t}{s}))^{\nu-1} \Big(\frac{M\|a\|_\infty \|p\|_\infty}{\Gamma(\sigma+1)} + c_3^x \Big)^{q-1} (\log s)^{(q-1)\sigma} \Big] \frac{ds}{s} \Bigg| + |c_4^x| \Bigg\} \\
&\le \max_{t\in[1,e]} \Bigg\{ \frac{\Gamma(\sigma(q-1)+1)}{\Gamma(\nu+\sigma(q-1)+1)} \Big(\frac{M\|a\|_\infty \|p\|_\infty}{\Gamma(\sigma+1)} + |c_3^x| \Big)^{q-1} (\log t)^{\nu+\sigma(q-1)} + |c_4^x| \Bigg\} \\
&\le \frac{\Gamma(\sigma(q-1)+1)}{\Gamma(\nu+\sigma(q-1)+1)} \Big(\frac{M\|a\|_\infty \|p\|_\infty}{\Gamma(\sigma+1)} + |c_3^x| \Big)^{q-1} + |c_4^x| \\
&\le (2^{q-1}-1) \frac{\Gamma(\sigma(q-1)+1)}{\Gamma(\nu+\sigma(q-1)+1)} \Bigg[\Big(\frac{M\|a\|_\infty \|p\|_\infty}{\Gamma(\sigma+1)} \Big)^{q-1} + |c_3^x|^{q-1} \Bigg] + |c_4^x|. \quad (3.14)
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
|c_3^x| &\le \frac{|c_2| + |A_2 I^{\sigma+\gamma_2} f(x,\eta_2)| + |B_2 I^\sigma f(x,e)|}{|\Delta_1|} \\
&\le \frac{|c_2| + |A_2| \frac{M\|a\|_\infty \|p\|_\infty}{\Gamma(\sigma+\gamma_2+1)} (\log \eta_2)^{\sigma+\gamma_2} + |B_2| \frac{M\|a\|_\infty \|p\|_\infty}{\Gamma(\sigma+1)}}{|\Delta_1|} \\
&\le \frac{|c_2| + M\|a\|_\infty \|p\|_\infty \big(\frac{|A_2|}{\Gamma(\sigma+\gamma_2+1)} (\log \eta_2)^{\sigma+\gamma_2} + \frac{|B_2|}{\Gamma(\sigma+1)} \big)}{|\Delta_1|} \\
&\le \frac{|c_2| + M\|a\|_\infty \|p\|_\infty \big(\frac{|A_2|}{\Gamma(\sigma+\gamma_2+1)} + \frac{|B_2|}{\Gamma(\sigma+1)} \big)}{|\Delta_1|} \\
&\le \frac{|c_2| + M\|a\|_\infty \|p\|_\infty \big(|A_2| + |B_2| \big)}{\Gamma(\sigma+1)|\Delta_1|}, \quad (3.15) \\
|c_3^x|^{q-1} &\le (2^{q-1}-1) \frac{|c_2|^{q-1} + \big(M\|a\|_\infty \|p\|_\infty \big)^{q-1} \big(|A_2| + |B_2| \big)^{q-1}}{(\Gamma(\sigma+1)|\Delta_1|)^{q-1}}, \quad (3.16)
\end{aligned}
$$

and

$$
\begin{aligned}
|c_4^x| &= \Bigg| \frac{c_1 - A_1 I^{\nu+\gamma_1} \phi_q [I^\sigma F(\eta_1, x) + c_3^x (\log \eta_1)^{\sigma-1}] - B_1 I^\nu (I^\sigma f(e,x) + c_3^x)}{\Delta_2} \Bigg| \\
&\le \Bigg\{ |c_1| + |A_1| \Bigg| I^{\nu+\gamma_1} \phi_q \Big[\frac{M\|a\|_\infty \|p\|_\infty (\log \eta_1)^\sigma}{\Gamma(\sigma+1)} + c_3^x (\log \eta_1)^{\sigma-1} \Big] \Bigg|
\end{aligned}
$$

$$
\begin{aligned}
&+|B_1|\left|I^{\nu}\big(\frac{M\|a\|_\infty\|p\|_\infty(\log(\cdot))^{\sigma}}{\Gamma(\sigma+1)}+c_3^x\big)(e)\right|\Bigg\}\Big/|\Delta_2| \\
\le&\left\{|c_1|+|A_1|\left|I^{\nu+\gamma_1}\Big[\frac{M\|a\|_\infty\|p\|_\infty}{\Gamma(\sigma+1)}+c_3^x\Big]^{q-1}(\log\eta_1)^{(q-1)\sigma}\right|\right. \\
&\left.+|B_1|\left(\frac{M\|a\|_\infty\|p\|_\infty\Gamma(\sigma+1)}{\Gamma(\nu+\sigma+1)}+\frac{c_3^x}{\Gamma(\nu+1)}\right)\right\}\Big/|\Delta_2| \\
\le&\left\{|c_1|+|A_1|\Big[\frac{M\|a\|_\infty\|p\|_\infty}{\Gamma(\sigma+1)}+c_3^x\Big]^{q-1}\frac{\Gamma(\sigma(q-1)+1)(\log\eta_1)^{\nu+\eta_1+\sigma(q-1)}}{\Gamma(\nu+\gamma_1+\sigma(q-1)+1)}\right. \\
&\left.+|B_1|\left(\frac{M\|a\|_\infty\|p\|_\infty\Gamma(\sigma+1)}{\Gamma(\nu+\sigma+1)}+\frac{c_3^x}{\Gamma(\nu+1)}\right)\right\}\Big/|\Delta_2| \\
\le&\left\{|c_1|+(2^{q-1}-1)|A_1|\Big[\frac{(M\|a\|_\infty\|p\|)_\infty^{q-1}}{\Gamma^{q-1}(\sigma+1)}+(c_3^x)^{q-1}\Big]\frac{\Gamma(\sigma(q-1)+1)}{\Gamma(\nu+\sigma(q-1)+1)}\right. \\
&\left.+|B_1|\left(\frac{M\|a\|_\infty\|p\|_\infty\Gamma(\sigma+1)}{\Gamma(\nu+\sigma+1)}+\frac{c_3^x}{\Gamma(\nu+1)}\right)\right\}\Big/|\Delta_2|. \qquad (3.17)
\end{aligned}
$$

It follows, from inequalities (3.15), (3.16) and (3.17), that the inequality (3.14) becomes

$$\|Q(x)\|\le w_1(M\|a\|_\infty\|p\|)_\infty^{q-1}+w_2M\|a\|_\infty\|p\|_\infty+w_3,$$

where

$$w_1=\frac{(2^{q-1}-1)\Gamma(\sigma(q-1)+1)}{\Gamma^{q-1}(\sigma+1)\Gamma(\nu+\sigma(q-1)+1)}\Big[1+\frac{|A_1|}{|\Delta_2|}\Big]\Big[1+(2^{q-1}-1)\big(\frac{|A_2|+|B_2|}{|\Delta_1|}\big)^{q-1}\Big],$$

$$w_2=\frac{|B_1|}{|\Delta_2|}\left[\frac{\Gamma(\sigma+1)}{\Gamma(\nu+\sigma+1)}+\frac{|A_2|+|B_2|}{\Gamma(\sigma+1)\Gamma(\nu+1)|\Delta_1|}\right],$$

and

$$
\begin{aligned}
w_3=&\frac{|c_1|}{|\Delta_2|}+\frac{|c_2||B_1|}{|\Delta_1\Delta_2|\Gamma(\nu+1)\Gamma(\sigma+1)} \\
&+\big(\frac{|c_2|}{|\Delta_1|\Gamma(\sigma+1)}\big)^{q-1}\frac{(2^{q-1}-1)^2\Gamma(\sigma(q-1)+1)}{\Gamma(\nu+(\sigma-1)(q-1)+1)}\left(1+\frac{|A_1|}{|\Delta_2|}\right).
\end{aligned}
$$

Consequently, we have

$$\|Q(x)\|\le R.$$

This proves that Q transforms all ball $B_R:=\{u\in:\|u\|\le R\}$ into itself.

From an application of Schauder's Theorem 2.1, we deduce that Q has at least a fixed point x which is the solution of the problem (3.1). □

4. Hyers-Ulam Stability

Let $C([1,e],\mathbb{R})$ be the space of all continuous functions from [1,e] to $\mathbb{R}$. Let $\mathfrak{B} = PC([1,e],\mathbb{R})$ represent the space of piecewise continuous functions. Obviously, $\mathfrak{B} = PC([1,e],\mathbb{R})$ is a Banach space with the norm

$$\|y\|_{\mathfrak{B}} = \sup_{t\in[1,e]} \{|y(t)|\}.$$

Now, we introduce the concept of Ulam-type stability for problem (1.1).

Let $x \in \mathfrak{B}$ and $\epsilon > 0$. Let us consider the following set of inequalities:

$$|D^{\sigma}(\phi_p(D^{\nu}x))(t) - f(t,x)| \leq \epsilon; \quad t \in [1,e]. \tag{4.1}$$

Definition 4.1. Problem (1.1) is Ulam-Hyers stable if there exists a real constant $c_f > 0$ such that, for given $\xi > 0$ and for each solution $x \in \mathfrak{B}$ of inequality (4.1), there exists a solution $y \in \mathfrak{B}$ of problem (1.1) with

$$|x(t) - y(t)| \leq \epsilon c_f; \quad t \in [1,e]. \tag{4.2}$$

Theorem 4.1. *With the assumptions* (H_1)*,* (H_2)*, and* (H_3)*, the fractional differential equation* (3.1) *is Hyers-Ulam stable.*

Proof. Let $x(t)$ be the fact solution of (3.2) and $y(t)$ be an approximate solution satisfying (4.1). Then, we have for each $t \in [1,e]$

$$\begin{aligned}
&|y(t) - x(t)| \\
&\leq |y(t) + I^{\mu}(\phi_q(I^{\sigma}(f(t,y)) + c_3^y(\log t)^{\sigma-1})) - c_4^y(\log t)^{\mu-1}| \\
&\quad + |I^{\mu}(\phi_q(I^{\sigma}(f(t,x)) + c_3^x(\log t)^{\sigma-1})) - \phi_q(I^{\sigma}(f(t,y)) + c_3^y(\log t)^{\sigma-1})))| \\
&\quad + |(c_4^y - c_4^x)(\log t)^{\mu-1}| \\
&\leq I^{\mu}(\phi_q(I^{\sigma}(\xi)))(t) + (q-1)R^{q-2}I^{\mu}(I^{\sigma}(|f(t,x)-f(t,y)|) + |c_3^x - c_3^y|) + |c_4^y - c_4^x| \\
&\leq \frac{\Gamma(\sigma(q-1)+1)(\xi)^{q-1}}{\Gamma(\mu+\sigma(q-1)+1)(\Gamma(\sigma+1))^{q-1}} + (q-1)R^{q-2}I^{\mu}\Big(I^{\sigma}(|f(t,x) - f(t,y)|) \\
&\quad + |c_3^x - c_3^y|\Big) + |c_4^y - c_4^x|.
\end{aligned} \tag{4.3}$$

On the other hand, using (H_2), (3.3) and (3.4), we obtain respectively

$$I^{\mu}(I^{\sigma}(|f(t,x)-f(t,y)|)\leq |g(x)-g(y)|I^{\mu+\sigma}|p|(t), \tag{4.4}$$

$$\begin{aligned}|c_3^x-c_3^y| &\leq \frac{|A_2|I^{\sigma+\gamma_2}|f(\eta_2,x)-f(\eta_2,y)|+|B_2|I^{\sigma}|f(e,x)-f(e,y)|}{|\Delta_1|}\\ &\leq |g(x)-g(y)|\frac{|A_2|I^{\sigma+\gamma_2}|p|(\eta_2)+|B_2|I^{\sigma}|p|(e)}{|\Delta_1|},\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}&|\Delta_2||c_4^x-c_4^y|\\ &\leq |A_1|\Big\{I^{\mu+\gamma_1}\phi_q I^{\sigma}|f(\cdot,x)+c_3^x(\log\cdot)^{\sigma-1}|(\eta_1)-I^{\mu+\gamma_1}\phi_q I^{\sigma}|f(\cdot,x)\\ &\qquad + c_3^x(\log\cdot)^{\sigma-1}|(\eta_1)\Big\}+|B_1|I^{\mu}\big(I^{\sigma}|f(\cdot,x)-f(\cdot,y)|+|c_3^x-c_3^y|\big)(e)\\ &\leq |A_1||g(x)-g(y)|(q-1)R^{q-1}\Bigg\{I^{\mu+\gamma_1}\phi_q I^{\sigma}|p|(\eta_1)\\ &\quad+\frac{|A_2|I^{\mu+\gamma_1+\sigma+\gamma_2}|p|(\eta_2)+|B_2|I^{\mu+\gamma_1+\sigma}|p|(e)}{|\Delta_1|}\Bigg\}+|g(x)-g(y)||B_1|\\ &\quad\times\Big[I^{\mu+\sigma}|p|(e)+\frac{|A_2|I^{\mu+\gamma_1+\sigma+\gamma_2}|p|(\eta_2)+|B_2|I^{\mu+\gamma_1+\sigma}|p|(e)}{|\Delta_1|}\Big].\end{aligned} \tag{4.6}$$

Using the hypothesis (H_3), (4.3), (4.4), (4.5) and (4.6), we obtain that

$$|y(t)-x(t)|\leq \epsilon c_f;\quad t\in[1,e]. \tag{4.7}$$

Hence (4.7) is Hyers-Ulam stable. Consequently, the singular fractional DE with delay and operator Φ_p (1.1) is Hyers-Ulam stable. □

5. Illustrative Examples

In this section, an application of the results which have proved in Sections 3 and 4, is provided.

We consider the following Hadamard differential equation involving the

$p-$Laplacian of the form

$$\begin{cases} D^{1/2}(\phi_4(D^{1/2}x))(t) + e^{-t}[(e^{-|x(t)|+\frac{1}{1+|x(t)|}})\frac{\sin(t)}{t}] = 0, \\ x(1) = \phi_4(D^{1/2}x)(1) = 0, \\ I^{1/2}x(3/2) + x(e) = 0, \\ I^{5/2}(\phi_4(D^{1/2}x))(5/2) - \phi_4(D^{1/2}x)(e) = 1. \end{cases}$$

Clearly, the function f satisfies the hypothesis (H_1), (H_2) and (H_3) with $M = 2$. Then, the existence and the stability of solution is assured and we have

$$\Delta_1 \simeq 0.62 \text{ and } \Delta_2 \simeq 2.77.$$

6. Conclusion

In this paper, we have utilized the Schauder's fixed point theorem to establish existence and uniqueness criteria for the solution of the nonlinear fractional differential equation given in (1.1). Furthermore, under some particular assumptions and conditions, we have proved stability results in the sense of Ulam for the solutions of the said problem. We claim that the approach used to prove the main results is powerful, effectual, and suitable for investigating different qualitative properties of the solutions of nonlinear fractional differential equations.

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