The Fire Index

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(Dedicated to the memory of the 343 Firefighters who lost their lives in New York City on the 11th of September 2001)

Abstract

We introduce a new graph parameter $f^*(G)$, which may be defined as the largest order of all the fan digraphs associated to the edge-deleted colorings of the critical subgraphs of G. We then show its fundamental importance in edge coloring. In particular, we give generalizations of Vizing's Theorem, Shannon's Theorem and Vizing's Adjacency Lemma, and an extension to multigraphs of the simple graph version of Vizing's Theorem which is obtained by proving that the chromatic index of an arbitrary multigraph must assume one of only two possible values. We call f^* the *Fire Index*.

1 Introduction

A *multigraph* G is, for the purposes of this paper, an ordered triple (V, E, ψ), where V and E are two disjoint finite sets called, respectively, the set of *vertices* and the set of *edges* of G, and $\psi : E \to V^{(2)}$ is a function, called *incidence function*. Here and elsewhere in this paper, the symbol $X^{(2)}$, where X is a set, is used to denote the set of *unordered pairs* of distinct elements of X. Thus the incidence function associates to each edge e of G an unordered pair $\{u, v\}$ of distinct vertices of G, which e is said to *join*. The vertices u and v are also called the *endpoints* of e. Two edges are *adjacent* if they are distinct and have at least one common endpoint. The set of edges joining two vertices u and v is denoted by uv. The cardinality of uv is denoted by $\mu(uv)$ and called (with a

*Fire Rescue Development Program (FRDP), N.G.O. and Professional Association of Firefighters, http://www.frdp.org slight abuse of terminology) the *multiplicity of the edge uv*. The quantity

$$\mu(G) = \max_{\{u,v\} \in V^{(2)}} \{\mu(uv)\}$$

is called the *maximum multiplicity* of G.

If u is a vertex of G, the quantity $\sum_{v \neq u} \mu(uv)$, i.e. the number of edges incident with u in G, is called the *degree* of u and denoted by $\deg(u)$. The *maximum degree* of G, denoted by $\Delta(G)$, is defined as the maximum of the degrees of the vertices of G.

We shall adopt the convention that, whenever appropriate, the name of the multigraph to which a certain symbol or quantity refers shall be attached to that symbol or quantity in any convenient manner, such as by means of a subscript, or superscript, etc. This will prove to be particularly useful when more than one graph is being discussed.

Let $\mathcal{C}$ be a set, whose elements we conventionally call *colors*. An *edge coloring* of a multigraph $G = (V, E, \psi)$ is a function $\varphi : E \to \mathcal{C}$ such that $\varphi(e) \neq \varphi(f)$ for every pair $\{e, f\}$ of adjacent edges of G. The *chromatic index* of G, denoted by $\chi'(G)$, is defined as

$$\chi'(G) = \min\{|\mathcal{C}|\},$$

where $\mathcal{C}$ ranges over the color sets in all the edge colorings of G. An edge coloring of G is called *optimal* if its color set $\mathcal{C}$ satisfies the condition $|\mathcal{C}| = \chi'(G)$.

It is easy to see that $\chi'(G) \geq \Delta(G)$ for any multigraph G. If $\chi'(G) = \Delta(G)$, we say that G is *Class 1*, and otherwise we say that G is *Class 2*. Virtually nothing was known about the chromatic index of arbitrary multigraphs until 1949, when C.E. Shannon [1] proved the following theorem.

Theorem 1 ***(Shannon, 1949)*** *For any multigraph G, $\chi'(G) \leq \lfloor \frac{3\Delta(G)}{2} \rfloor$.*

Shannon showed that the upper bound in his theorem is attained by an infinite family of graphs. Several years later, V.G. Vizing [2] determined another formidable upper bound on the chromatic index of multigraphs, namely

Theorem 2 ***(Vizing, 1964)*** *For any multigraph G, $\chi'(G) \leq \Delta(G) + \mu(G)$.*

Vizing's result is particularly striking when $\mu(G) = 1$, i.e. when G is a *simple graph*, because it restricts the range of the values of the chromatic index to two possible (consecutive) integers only. It should be noticed, however, that neither Theorem 1 is an improvement of Theorem 2 nor Theorem 2 is an improvement of Theorem 1.

The main objective of this paper is to prove a formula that generalizes both Theorem 1 and Theorem 2, and from which most theorems in edge coloring can be derived. Possibly this formula could prove to be of some use in an attack on the foremost unsolved problem on edge coloring of multigraphs, which is the following conjecture of Goldberg [3] and Seymour [4].

Conjecture 1 ***(Goldberg-Seymour Conjecture)*** *Let G be a Class 2 multigraph such that $\chi'(G) > \Delta(G) + 1$. Then*

$$\chi'(G) = \max\lceil |E(H)|/\lfloor |V(H)|/2\rfloor\rceil,$$

where the maximum is extended to all submultigraphs H of G of order at least two.

2 Edge coloring preliminaries

This paper is a natural continuation of [5], whose notation, terminology and results will be assumed. However, for the reader's convenience, we shall reiterate here a number of definitions which were given in [5]. An edge e of a multigraph G is called *critical* if $\chi'(G-e) < \chi'(G)$. The multigraph G itself is called *critical* if it is Class 2, has no isolated vertices, and all its edges are critical. An *e-tense coloring* ϕ of G is a partial edge coloring of G which assigns no color to e and whose restriction to $E(G-e)$ is an optimal coloring of $G-e$. The color set of ϕ is defined to be the color set of its restriction to $G-e$. Given an e-tense coloring ϕ of G with color set $\mathcal{C}$ and a vertex $w \in V(G)$, we say that a color $\alpha \in \mathcal{C}$ is *missing* at w (or that w is missing the color α) if there is no edge, having w as an endpoint, which is assigned the color α by ϕ. The set of colors missing at w is denoted by $\mathcal{C}_w$ and its cardinality is called the *color-deficiency* of w and denoted by $cdef(w)$.

Let $u \in V(G)$. A *fan* at u with respect to ϕ is a sequence of edges of the form

$$F = [e_0, e_1, e_2, \ldots, e_{k-1}, e_k],$$

where $e_0 = e$, $e_i \in uv_i$, and where the vertex v_i is missing the color of the edge e_{i+1}, for every $i = 0, 1, \ldots, k-1$. An edge f is called a *fan edge* at u if it appears in at least one fan at u. A vertex w is called a *fan vertex* at u if it is joined to u by at least one fan edge. The set of fan vertices is denoted by $V(\mathcal{F})$. A color $\alpha \in \mathcal{C}$ is called a *fan color* if it is the color of a fan edge. The set of fan colors is denoted by $\mathcal{C}_{\mathcal{F}}$. If w is a fan vertex at u, we denote by $\mu^*(uw)$ the number of *fan edges* joining u and w, and call $\mu^*(uw)$ the *fan multiplicity of the edge uw*. The main contribution of [5] was the introduction of a new concept in edge coloring, the *Fan Digraph*, which we now define.

Let G be a Class 2 multigraph and let $e \in uv$ be a critical edge. The *e-Fan Digraph* at u with respect to ϕ is the directed multigraph $\mathcal{F} = (V(\mathcal{F}), A(\mathcal{F}), \psi_{\mathcal{F}})$, where

1. $V(\mathcal{F}) = \{w \mid w \text{ is a fan vertex at } u\}$;
2. $A(\mathcal{F}) = \mathcal{C}_{\mathcal{F}} = \{\alpha \mid \alpha \text{ is a fan color at } u\}$;
3. $\psi_{\mathcal{F}} : A(\mathcal{F}) \to V(\mathcal{F}) \times V(\mathcal{F})$
   $\alpha \longmapsto (w_\alpha, z_\alpha)$,

where w_α is the unique fan vertex at u missing color α and z_α is the unique fan vertex at u joined to u by an edge colored α.

Notice that the existence of the vertices w_α and z_α is far from being obvious, and follows from deep results of edge coloring (see [5, Lemma 1 and Lemma 2]).

It follows immediately from the definitions (see [5]) that, under the above hypotheses, we have

$$|V(\mathcal{F})| \geq 2 \tag{1}$$

and, for each $w \in V(G)$,

$$cdef(w) = \begin{cases} \chi'(G) - \deg_G(w) & \text{if } w = u \text{ or } w = v \\ \\ \chi'(G) - 1 - \deg_G(w) & \text{if } w \neq u, v. \end{cases} \tag{2}$$

The following theorem expresses a central property of tense colorings and was obtained in [5] by counting the number of arcs of the fan digraph in three different ways.

Theorem 3 ***(Fan Theorem)*** *Let G be a Class 2 multigraph and let $e \in E(G)$ be a critical edge, where $e \in uv$. Let ϕ be an e-tense coloring of G and let $\mathcal{F}$ be the corresponding fan digraph. Then*

$$\sum_{w \in V(\mathcal{F})} cdef(w) = \sum_{w \in V(\mathcal{F})} \mu^*(uw) - 1 = |\mathcal{C}_{\mathcal{F}}|.$$

3 A formula for the chromatic index

We shall now deduce from Theorem 3 the formula for the chromatic index which was mentioned in the Introduction. We use the same assumptions and notations of Theorem 3. Substituting (2) in the first identity of Theorem 3, we obtain

$$\sum_{w \in V(\mathcal{F})} (\chi'(G) - 1 - \deg_G(w)) + 1 = \sum_{w \in V(\mathcal{F})} \mu^*(uw) - 1,$$

where we have used the fact that $v \in V(\mathcal{F})$ and $u \notin V(\mathcal{F})$.

Rearranging the terms, we obtain the expression

$$|V(\mathcal{F})| \cdot \chi'(G) = \sum_{w \in V(\mathcal{F})} (\deg_G(w) + \mu^*(uw)) + |V(\mathcal{F})| - 2,$$

from which the following formula follows:

$$\chi'(G) = \frac{1}{|V(\mathcal{F})|} \cdot \sum_{w \in V(\mathcal{F})} (\deg_G(w) + \mu^*(uw)) + \frac{|V(\mathcal{F})| - 2}{|V(\mathcal{F})|}\ . \tag{3}$$

Notice that this is an exact expression for the chromatic index of a multigraph, and hence, by its own nature, is superior to all known upper and lower bounds on the chromatic index. However, a disadvantage of formula (3) is that it contains several quantities which may not be immediately computable. We shall comment briefly on this. First, one should notice that the right-hand side of (3) can be evaluated only if we have a tense coloring ϕ of G (or, equivalently, a fan digraph) which can be obtained only if one knows the chromatic index of G (since $\chi'(G)$ is just the number of colors used by ϕ plus one). In other words, the knowledge of the right-hand side of (3) presupposes knowledge of a tense coloring of G, which in turns implies knowledge *a priori* of the chromatic index of the multigraph. Despite this being true, in practical situations, we are most often faced with an entirely different problem, i.e. we have only imprecise or partial information about the graph at hand and its chromatic properties, and we wish to test our information for accuracy. In this sense, the formula (3) may prove to be not only a powerful theoretical tool, but also one useful in applications. Suppose, for instance, that we have constructed an e-tense coloring of G, but we do not know whether e is a critical edge, or suppose, conversely, that we know that e is a critical edge of G, but we are unable to tell whether the coloring ϕ is an e-tense coloring. In both circumstances we may use formula (3) on the assumption that ϕ is an e-tense coloring and e is a critical edge, as a test which (at least in some cases) may allow us to decide if our assumption is false.

An expression which is even more compact than (3) may be obtained from (3) by taking (1) into account, and is the following:

$$\chi'(G) = \lceil \frac{1}{|V(\mathcal{F})|} \cdot \sum_{w \in V(\mathcal{F})} (\deg_G(w) + \mu^*(uw)) \rceil \ . \quad (4)$$

From (4) it is easily seen that (3) generalizes Vizing's Theorem, since obviously $\deg_G(w) \leq \Delta(G)$ and $\mu^*(uw) \leq \mu(G)$ for any fan vertex w. In a similar way several other classical upper bounds on the chromatic index of multigraphs can be obtained (see [6]). The interesting feature of (4) is that it emphasizes that fact that the chromatic index of G can be expressed as the ceiling of the "average" of a certain quantity associated to each fan vertex w. This quantity depends, however, on the particular tense coloring (or fan digraph) chosen. From this point of view (3) is much more interesting, since it is obvious that the quantity at the left-hand side (and, consequently, at the right-hand side) of (3) is independent on the tense coloring chosen.

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For different reasons, some of which will be indicated below, it is interesting to choose in (3) a tense coloring for which the quantity $|V(\mathcal{F})|$, i.e. the order of the corresponding fan digraph (or, equivalently, the number of fan vertices) is *maximum*. As a first justification for this choice, we now obtain a generalization of the well known Vizing's Adjacency Lemma [7]. Suppose that u is a vertex of a critical simple graph G and v is a neighbor of u. Let $e \in uv$. Let $\mathcal{F}$ be an e-fan digraph at u of maximum order. Then, from Theorem 3, we have

$$\sum_{w \in V(\mathcal{F})} cdef(w) = |V(\mathcal{F})| - 1, \tag{5}$$

where we have used the fact that all fan multiplicities are equal to 1. Since G is simple and Class 2, we have $\chi'(G) = \Delta(G) + 1$ by Vizing's Theorem. Then from (2) we obtain

$$cdef(w) = \Delta(G) - \deg_G(w) \quad \text{if } w \in V(\mathcal{F}) \setminus \{v\} \tag{6}$$

and

$$cdef(v) = \Delta(G) + 1 - \deg_G(w). \tag{7}$$

Let the *deficiency* of vertex x be denoted and defined by

$$def(x) = \Delta(G) - \deg_G(x).$$

Then, from (5), (6) and (7), we have

$$\sum_{w \in V(\mathcal{F})} def(w) = |V(\mathcal{F})| - 2.$$

Since $v \in V(\mathcal{F})$, we have

$$def(v) + \sum_{w \in V(\mathcal{F}) \setminus \{v\}} def(w) = |V(\mathcal{F})| - 2,$$

and hence

$$\sum_{w \in V(\mathcal{F}) \setminus \{v\}} def(w) = |V(\mathcal{F})| - 2 - def(v) = |V(\mathcal{F})| - 2 - \Delta(G) + \deg_G(v). \tag{8}$$

This statement is more informative than Vizing's Adjacency Lemma, which only gives information about the vertices of maximum degree adjacent to u. Indeed Vizing's Adjacency Lemma can be easily obtained by (8) by simply noticing that the number of vertices of positive deficiency in $V(\mathcal{F}) \setminus \{v\}$ cannot exceed the right-hand side of (8), and hence the number of vertices of zero deficiency (i.e. of maximum degree) in $V(\mathcal{F}) \setminus \{v\}$ must be at least

$$(|V(\mathcal{F})| - 1) - (V(\mathcal{F})| - 2 - \Delta(G) + \deg_G(v)) = \Delta(G) + 1 - \deg_G(v).$$

These are all vertices adjacent to u and distinct from v, whence Vizing's Adjacency Lemma follows. It should be apparent now that, the larger $V(\mathcal{F})$, the larger the number of neighbors of u for which we have some information. This example shows that it is of some value the idea of considering the *largest* possible fan digraph based at a given vertex. We shall now make one further step in this direction, and consider the largest possible fan digraph based at any vertex of G. In order to define this as a graph parameter, we shall assume to start with an arbitrary Class 2 multigraph M. Let $\mathcal{G}$ be the class of submultigraphs of M which have the same chromatic index of M (and hence are Class 2) and have at least one critical edge. Notice that $\mathcal{G}$ is non-empty since there always exists a *critical* submultigraph of M with the same chromatic index as M [7]. Consider then, for each $G \in \mathcal{G}$, all the critical edges e of G and, for each of these, all the e-tense colorings ϕ of G. For each such tense coloring ϕ, consider the corresponding fan digraph $\mathcal{F}$. We define $f^*(M)$ to be the *largest* order of all the fan digraphs $\mathcal{F}$ described above. Thus

$$f^*(M) = \max\{|V(\mathcal{F})|\}, \tag{9}$$

where the maximum ranges over all the e-fan digraphs of G, G ranges over all the multigraphs in $\mathcal{G}$ and e ranges over all the critical edges of G. If M is a Class 1 multigraph, we simply define $f^*(M)$ to be ∞. We have thus defined the parameter f^* for all multigraphs. We propose to call f^* the *Fire Index*.

We shall now offer some additional evidence of the importance that this parameter has in edge coloring. A more comprehensive study will form the subject of other publications of the author.

We first notice that we are now able to provide an extension to multigraphs of the *simple* graph version of Vizing's Theorem, as follows.

Theorem 4 *Let M be a multigraph. Then either $\chi'(M) = \Delta(M)$ or there exists a submultigraph G of M which admits a fan digraph $\mathcal{F}$ such that*

$$\chi'(M) = \frac{1}{f^*(M)} \cdot \sum_{w \in V(\mathcal{F})} (\deg_G(w) + \mu^*_G(uw)) + \frac{f^*(M) - 2}{f^*(M)}\ .$$

The importance, theoretical and practical, of the above statement cannot be underestimated, since it restricts to two integers only the possible value of the chromatic index of an arbitrary multigraph.

It is not too difficult to see (and it will be proved elsewhere) that, in order to evaluate the fire index, it suffices to evaluate it for critical multigraphs only. More precisely, we have the following.

Lemma 1 *Let M be a Class 2 multigraph. Then*

$$f^*(M) = \max\{f^*(H)\ :\ H \subseteq M, H \text{ critical}, \chi'(H) = \chi'(M)\}.$$

Notice that, if H is a critical multigraph, $f^*(H)$ is simply the largest order of a fan digraph of H. Thus Lemma 1 provides a significant conceptual simplification to the definition of the fire index.

Assume now that M is a Class 2 multigraph. Using (1) and the definition of fire index, it is easily seen that

$$2 \le f^*(M) \le \Delta(M). \tag{10}$$

Hence, using Theorem 4 and the obvious inequalities

$$\deg_G(w) \le \Delta(G) \le \Delta(M)$$

and

$$\sum_{w \in V(\mathcal{F})} \mu^*_G(uw) \le \sum_{w \in V(\mathcal{F})} \mu_G(uw) \le \deg_G(u) \le \Delta(G) \le \Delta(M),$$

we obtain

$$\chi'(M) \le \Delta(M) + \frac{\Delta(M) + f^*(M) - 2}{f^*(M)} . \tag{11}$$

Since the left-hand side of (11) is an integer, (11) implies

$$\chi'(M) \le \Delta(M) + \lfloor \frac{\Delta(M) + f^*(M) - 2}{f^*(M)} \rfloor. \tag{12}$$

The above inequality provides (because $f^*(M)$ is in the range (10)), a generalization of Shannon's Theorem, which we state below (clearly, for Class 1 multigraphs, the right-hand side of (12) must be interpreted in the limit sense, i.e. as $\Delta(M) + 1$).

Theorem 5 *For any multigraph M,*

$$\chi'(M) \le \Delta(M) + \lfloor \frac{\Delta(M) + f^*(M) - 2}{f^*(M)} \rfloor.$$

It is now clear from the monotonicity of the function $(\Delta + f^* - 2)/f^*$ with respect to f^*, that, the larger the value of $f^*(M)$, the better is the bound on the chromatic index provided by Theorem 5. This provides a further motivation to the introduction of the fire index.

Finally we notice that, if we evaluate the right-hand side of the inequality of Theorem 5 by letting the fire index take successive integer even values $2, 4, 6, 8, 10, 12$, we obtain the quantities

$$\lfloor 3\Delta/2 \rfloor, \lfloor (5\Delta+2)/4 \rfloor, \lfloor (7\Delta+4)/6 \rfloor, \lfloor (9\Delta+6)/8 \rfloor, \lfloor (11\Delta+8)/10 \rfloor, \lfloor (13\Delta+10)/12 \rfloor.$$

Curiously, these are precisely the quantities which appear in connection to the partial proofs of the Goldberg-Seymour Conjecture obtained, respectively, by Vizing [8], Goldberg [9], Andersen [10], Goldberg [11], Nishizeki and Kashiwagi [12] and Favrholdt et al. [13]. This clearly indicates a strong connection between the fire index and the most elusive unsolved question on the edge coloring of multigraphs, which we shall try to clarify in the future.

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References

[1] C.E. Shannon, *A theorem on coloring the lines of a network,* J. Math. Phys., **28** (1949), 148-151.

[2] V.G. Vizing, *On an estimate of the chromatic class of a p-graph,* Diskret. Analiz., **3** (1964), 25-30 (in Russian).

[3] M.K. Goldberg, *On multigraphs of almost maximal chromatic class,* Diskret. Analiz., **23** (1973), 3-7 (in Russian).

[4] P.D. Seymour, *On multicolorings of cubic graphs and conjectures of Fulkerson and Tutte,* Proc. Lond. Math. Soc., **33** (1979), 423-460.

[5] D. Cariolaro, *On fans in multigraphs,* J. Graph Theory, **51** (2006), 301-318.

[6] D. Cariolaro, *Some consequences of a theorem on fans,* manuscript.

[7] V.G. Vizing, *Critical graphs with a given chromatic class,* Diskret. Analiz, **5** (1965), 6-17 (in Russian).

[8] V.G. Vizing, *The chromatic class of a multigraph,* Cybernetics, **3** (1965), 32-41.

[9] M.K. Goldberg, *Remark on the chromatic class of a multigraph,* Vycisl. Mat. i Vycisl. Tech. (Kharkov), **5** (1975), 128-130 (in Russian).

[10] L.D. Andersen, *On edge-colourings of graphs,* Math. Scand., **40** (1977), 161-175.

[11] M.K. Goldberg, *Edge-coloring of multigraphs: recoloring technique,* J. Graph Theory, **8** (1984), 123-137.

[12] T. Nishizeki and K. Kashiwagi, *On the 1.1 edge-coloring of multigraphs,* SIAM J. Discr. Math., **3** (1990), 391-410.

[13] L.M. Favrholdt, M. Stiebitz, B. Toft, *Graph edge coloring: Vizing's Theorem and Goldberg's Conjecture,* manuscript.