

On the chromatic index of graphs whose core has maximum degree two

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Abstract

Let G be a connected graph. The core of G , denoted by G_Δ , is the subgraph of G induced by the vertices of maximum degree. Hilton and Zhao [*On the edge-colouring of graphs whose core has maximum degree two*, JCMCC 21 (1996), 97-108] conjectured that, if $\Delta(G_\Delta) \leq 2$, then G is Class 2 if and only if G is overfull, with the sole exception of the Petersen graph with one vertex deleted. In this paper we prove this conjecture for all graphs G of even order such that $|V(G_\Delta)| > \max\{\frac{1}{2}|V(G)|, |V(G)| - 2\Delta(G) + 5\}$.

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1 Introduction

All graphs considered in this paper are finite and simple. The vertex and edge set of a graph G will be denoted by $V(G)$ and $E(G)$, respectively. If G is a graph and $S \subseteq V(G)$, by $N(S)$ we denote the set of vertices of G which are adjacent to at least one vertex in S . If H is a subgraph of G we denote this by $H \subseteq G$.

The *core* of G , denoted by G_Δ , is the subgraph of G induced by the set of vertices of maximum degree. For graph theoretic terminology not explicitly defined here, we refer the reader to [4].

An edge colouring of a graph G is a map $\varphi : E(G) \rightarrow \mathcal{C}$, where \mathcal{C} is a set, called the *colour set*, and $\varphi(e_1) \neq \varphi(e_2)$ for any pair (e_1, e_2) of distinct mutually incident edges of G . The minimum cardinality of the colour set in an edge colouring of G is called the *chromatic index* of G and denoted by $\chi'(G)$.

Vizing [13] proved that, for any graph G , $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. Accordingly, we say that G is *Class 1* if $\chi'(G) = \Delta(G)$ and *Class 2* if $\chi'(G) = \Delta(G) + 1$. G is called *critical* if it is Class 2, connected and, for every edge $e \in E(G)$, $\chi'(G - e) < \chi'(G)$.

A graph G is *overfull* if $|E(G)| > \Delta(G) \cdot \lfloor \frac{|V(G)|}{2} \rfloor$. It is easy to see that every overfull graph is Class 2. However the converse of this statement is not true, and it is very difficult in general to determine whether a given graph is Class 2 (or Class 1).

Fournier [5] proved that, if the core of G contains no cycles, then G is Class 1. It is natural to ask what can be said about G if G_Δ is indeed (isomorphic to) a cycle or, more generally, if it consists of vertex disjoint cycles and paths.

Let P^* denote the Petersen graph with one vertex deleted. Then P^* provides an example of a Class 2 graph whose core is a 6-cycle. Thus Fournier's result does not extend to graphs whose core is a cycle.

However Hilton and Zhao [9] have posed the following conjecture, which attributes to P^* an exceptional property among all connected graphs whose core has maximum degree at most two.

Conjecture 1 *Let G be a connected graph such that $\Delta(G_\Delta) \leq 2$. Then G is Class 2 if and only if G is overfull, unless $G \approx P^*$.*

Hilton and Zhao [7] proved the above conjecture for all graphs G such that $\Delta(G) \geq \frac{1}{2}(|V(G)| + 3)$. This bound has been recently improved to $\Delta(G) \geq \frac{1}{2}(|V(G)| - 1)$ by Koh and Song [10]. Hilton and Zhao [8] also proved Conjecture 1 for all graphs G such that $\Delta(G) \geq |V(G)| - |V(G_\Delta)|$. Further progress was made by Hilton and Zhao in [9], where the conjecture was proved for all graphs G such that $|V(G)| \geq 2k^2 + \frac{32}{3}k + \frac{47}{3}$ if $|V(G)|$ is even and $|V(G)| \geq 3k^2 + 12k + 16$ if $|V(G)|$ is odd, where $\Delta(G) = |V(G)| - |V(G_\Delta)| - k \geq k + 5$. All these results concerned graphs with relatively large maximum degree and, until recently, very little was known about graphs with small maximum degree (apart from the obvious fact that graphs with maximum degree at most two satisfy Conjecture 1). A breakthrough came in 2003, when D. Cariolaro and G.

Cariolaro [1] settled the case $\Delta(G) = 3$ of Conjecture 1 using a colour-exchange technique. Unfortunately, their method does not seem to be easily extendable to graphs of larger maximum degree.

From the classification of all critical graphs with at most five vertices of maximum degree due to Chetwynd and Hilton [2, 3], Song [11] and Song and Yap [12], and Lemma 1 below, it also follows that Conjecture 1 holds for all graphs with $|V(G_\Delta)| \leq 5$.

The purpose of the present paper is to prove the following result:

Theorem 1 *Let G be a connected graph of even order such that $\Delta(G_\Delta) \leq 2$ and $|V(G_\Delta)| > \max\{\frac{1}{2}|V(G)|, |V(G)| - 2\Delta(G) + 5\}$. Then G is Class 2 if and only if G is overfull.*

This improves, for graphs of even order such that $|V(G_\Delta)| > \frac{1}{2}|V(G)|$, the Hilton and Zhao [8] and Koh and Song [10] bounds by almost a factor of 2.

2 Preliminary lemmas

The following two important lemmas were established by Hilton and Zhao in [7].

Lemma 1 *Let G be a connected Class 2 graph with $\Delta(G_\Delta) \leq 2$. Then:*

1. G is critical;
2. $\delta(G_\Delta) = 2$;
3. $\delta(G) = \Delta(G) - 1$, unless G is an odd cycle;
4. $N(G_\Delta) = V(G)$.

Lemma 2 *Let G be a connected overfull graph, which is not an odd cycle, such that $\Delta(G_\Delta) \leq 2$. Then*

$$\Delta(G) \geq \frac{1}{2}(|V(G)| + 3).$$

By Lemma 2 and the fact that Conjecture 1 has been settled for $\Delta(G) \geq \frac{1}{2}(|V(G)| - 1)$ and for $\Delta(G) \leq 3$, Conjecture 1 is reduced to the following conjecture.

Conjecture 2 *Let G be a connected graph such that $\Delta(G_\Delta) \leq 2$ and $3 < \Delta(G) < \frac{1}{2}(|V(G)| - 1)$. Then G is Class 1.*

We shall make use, in the proof of Theorem 1, of the following well known result of P. Hall [6]. Let G be a bipartite graph with bipartition (V_1, V_2) . A matching M from V_1 to V_2 will be called *complete* if each vertex of V_1 is incident with an edge in M .

Lemma 3 *A bipartite graph with bipartition (V_1, V_2) contains a complete matching from V_1 to V_2 if and only if*

$$|N(S)| \geq |S| \text{ for every } S \subseteq V_1. \quad (1)$$

We shall refer to the condition (1) above as to *Hall's Condition*.

Finally, we shall need the following consequence of the well known Vizing's Adjacency Lemma [14].

Lemma 4 *Let G be a critical graph and let $u \in V(G)$. Then u is adjacent to at least two vertices of maximum degree of G .*

3 Proof of Theorem 1

We now prove Theorem 1.

Proof of Theorem 1: Let $\Delta = \Delta(G)$, let $p = |V(G_\Delta)|$ and let $q = |V(G)| - |V(G_\Delta)|$. Since Conjecture 1 has been reduced to Conjecture 2, we may assume that

$$4 \leq \Delta \leq \frac{1}{2}(q + p - 2). \quad (2)$$

We argue by contradiction, so suppose that G is Class 2. We shall show that G has a 1-factor F , and then derive a contradiction. Notice that $p + q = |V(G)|$ is even by assumption, so that

$$q \equiv p \pmod{2}. \quad (3)$$

From the hypothesis that $|V(G_\Delta)| > \max\{\frac{1}{2}|V(G)|, |V(G)| - 2\Delta(G) + 5\}$ it follows that

$$p > q \quad (4)$$

and

$$\Delta \geq \frac{1}{2}(q + 6). \quad (5)$$

Let $\partial(G_\Delta)$ denote the set of edges of G with exactly one end in G_Δ . By Lemma 1, G_Δ is 2-regular, so that

$$|\partial(G_\Delta)| = (\Delta - 2)p. \quad (6)$$

Moreover, by Lemma 1, every non-core vertex has degree $\Delta - 1$, so that

$$|\partial(G_\Delta)| \leq (\Delta - 1)q, \quad (7)$$

and comparing (7) with (6), we see that

$$q(\Delta - 1) \geq p(\Delta - 2). \quad (8)$$

Let $\beta_1(G_\Delta)$ denote the *edge-independence number* of G_Δ , i.e. the maximum number of independent edges in G_Δ . We show that $\beta_1(G_\Delta) \geq \frac{1}{2}(p - q)$.

By (2), $\Delta \geq 4$. Hence $\Delta \geq 5/2$, so that

$$3 \geq \frac{\Delta - 1}{\Delta - 2}. \quad (9)$$

By (8) and (9), we have

$$3q \geq p. \quad (10)$$

Hence

$$\frac{1}{2}q \geq \frac{1}{6}p.$$

Therefore,

$$\frac{1}{3}p = \frac{1}{2}p - \frac{1}{6}p \geq \frac{1}{2}(p - q). \quad (11)$$

Since G_Δ consists of disjoint cycles, and since any cycle of length k has at least $k/3$ independent edges, we have

$$\beta_1(G_\Delta) \geq \frac{1}{3}p. \quad (12)$$

By (11) and (12), we have

$$\beta_1(G_\Delta) \geq \frac{1}{2}(p - q). \quad (13)$$

Let S be a set of exactly $\frac{1}{2}(p - q)$ independent edges in G_Δ , which exists by (13). We say that a vertex is *missed* by S if it is not incident with any edge in S . There are obviously exactly q core and q non-core vertices which are missed by S . Let $W = \{w_1, w_2, \dots, w_q\}$ and $X = \{v_1, v_2, \dots, v_q\}$ be, respectively, the core and non-core vertices missed by S . We show, applying Hall's Theorem, that there exists a complete matching¹ from W to X . Let $\Gamma(w_i)$, for $1 \leq i \leq q$, be the set of non-core neighbours of the vertex $w_i \in W$. Notice that, by the 2-regularity of the core,

$$|\Gamma(w_i)| = \Delta - 2 \text{ for all } i = 1, 2, \dots, q. \quad (14)$$

Let $A \subseteq W$ and let $\Gamma = \bigcup_{w_i \in A} \Gamma(w_i)$. By (14), we may assume, in verifying Hall's condition, that $|A| \geq \Delta - 1$. Suppose that $|A| = \Delta - 1$. Without loss of generality, assume $A = \{w_1, w_2, \dots, w_{\Delta-1}\}$. If Hall's condition is not satisfied, then $|\Gamma| = \Delta - 2$. In that case all vertices of A are adjacent to all vertices in Γ . By (14), there are exactly $(\Delta - 2)(\Delta - 1) = \Delta^2 - 3\Delta + 2$ edges from A to Γ . By Lemma 1, each non-core vertex has degree $\Delta - 1$, so that there are certainly no more than $(\Delta - 2)(\Delta - 1)$ edges incident with vertices of Γ in G . Therefore all the edges incident with vertices of Γ in G join Γ to A . However, since $p > q$ by (4), we have $V(G_\Delta) \setminus A \neq \emptyset$.

Let $v \in V(G_\Delta) \setminus A$. Since there are $\Delta - 2$ edges joining v to X , this implies that there are at least $\Delta - 2$ vertices in $X \setminus \Gamma$, implying

$$q \geq 2(\Delta - 2),$$

¹Notice that we are using the same idea used by Hilton and Zhao in [7], except that Hall's condition is applied to W instead of X .

which contradicts (5).

Hence we are left only with the case that $|A| = \Delta + t$, where t is a nonnegative integer. Arguing by contradiction, assume that $|\Gamma| < |A|$. Let j be the positive integer defined by the equation

$$|\Gamma| = \Delta + t - j.$$

There are exactly

$$(\Delta + t)(\Delta - 2) = \Delta^2 + (t - 2)\Delta - 2t \quad (15)$$

edges joining A to Γ . But, by summing the degrees of the vertices of Γ , we see that the number of edges of G incident with the vertices of Γ cannot exceed

$$(\Delta + t - j)(\Delta - 1) = \Delta^2 + (t - 1 - j)\Delta + j - t. \quad (16)$$

Therefore it is obvious that the quantity in (15) cannot be larger than the quantity in (16), i.e. that

$$\Delta^2 + (t - 1 - j)\Delta + j - t \geq \Delta^2 + (t - 2)\Delta - 2t.$$

Cancelling out and simplifying, we obtain

$$j(\Delta - 1) \leq \Delta + t. \quad (17)$$

We cannot have $\Delta - 2 \leq t$, otherwise

$$\Delta + t \geq 2\Delta - 2 \geq q + 4,$$

contradicting the fact that $\Delta + t = |A| \leq q$. Thus

$$t < \Delta - 2, \quad (18)$$

and hence

$$\Delta + t < 2\Delta - 2 = 2(\Delta - 1). \quad (19)$$

Comparing (17) and (19), and recalling that j is positive, we conclude that $j = 1$, so that

$$|\Gamma| = \Delta + t - 1. \quad (20)$$

By (16) and the fact that $j = 1$, there are no more than

$$\Delta^2 + (t - 2)\Delta + 1 - t \quad (21)$$

edges incident with Γ in G . Subtracting (15) from (21), we conclude that there are at most $t + 1$ edges joining Γ to $V(G_\Delta) \setminus A$. Let w^* be a vertex in $V(G_\Delta) \setminus A$, which exists because $p > q$ by (4).

The vertex w^* is adjacent to exactly $\Delta - 2$ non-core vertices, at most $t + 1$ of which are in Γ . Therefore w^* is adjacent to at least

$$\Delta - 2 - (t + 1) = \Delta - t - 3 \quad (22)$$

non-core vertices, none of which is in Γ . From (20) and (22) it follows that

$$q \geq (\Delta + t - 1) + (\Delta - t - 3) = 2\Delta - 4,$$

contradicting inequality (5). Therefore Hall's condition is satisfied. By Hall's Theorem, there exists a complete matching from W to X . Adding S to the edges of this matching, we obtain the desired 1-factor F of G .

We now prove that $G - F$ satisfies all the conditions of Lemma 1. Since G is Class 2, $G - F$ is Class 2, too. Notice that, trivially,

$$(G - F)_\Delta \subseteq G_\Delta, \tag{23}$$

so that $\Delta((G - F)_\Delta) \leq 2$. Notice also that the inclusion in (23) is strict, since some edges of F (namely those in S) were specifically chosen to be in $E(G_\Delta)$.

We now prove that $G - F$ is connected. Notice that

$$V((G - F)_\Delta) = V(G_\Delta). \tag{24}$$

Recall that W is the set of core vertices missed by S . Let $v_1 \in V(G_\Delta) \setminus W$, and suppose that there exists $v_2 \in V(G_\Delta)$ such that v_2 lies in a different connected component of $G - F$ than v_1 . Since $v_1 \notin W$ and by the identity (24), v_1 has exactly $(\Delta - 2)$ non-core neighbours in $G - F$. By (24) and since v_2 is in the core of $G - F$, v_2 has at least $\Delta - 3$ non-core neighbours in $G - F$. Since v_1 and v_2 are in distinct connected components of $G - F$, their corresponding sets of non-core neighbours must be disjoint. By (24), this implies that

$$(\Delta - 2) + (\Delta - 3) \leq q,$$

contradicting inequality (5). Hence all core-vertices are in the same connected component of $G - F$. By Lemma 1, G is critical. By Lemma 4, every vertex of G is joined by at least two edges to G_Δ , and since at most one of these edges is in F , we conclude that every vertex in $G - F$ is joined by at least one edge to $(G - F)_\Delta$. This, added to the fact that all vertices in the core of $G - F$ are in the same connected component of $G - F$, proves that $G - F$ is connected. Thus $G - F$ satisfies all the hypotheses of Lemma 1. By Lemma 1, the core of $G - F$ is 2-regular. But this contradicts the fact that the inclusion (23) is strict, as observed above, which excludes the core of $G - F$ from being 2-regular. Hence we have a contradiction, and this contradiction proves that G is Class 1. \square

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