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# On the measure of a fuzzy set based on continuous t-conorms

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**Abstract**

The original aim of this research was the introduction of a compositive measure of information (see e.g. [5]) on a class (an algebra) of fuzzy subsets of an universe $\Omega$. However, there is a bijection between the class of compositive informations and the class of decomposable measures (see e.g. [2]). Therefore, we restated our problem as the one of the construction of a decomposable measure over the space $(\Omega, \tilde{\mathscr{A}})$, starting from a crisp measure defined on $(\Omega, \mathscr{A})$. Here $\mathscr{A}$ and $\tilde{\mathscr{A}}$ are suitable algebras of, respectively, crisp and fuzzy subsets of $\Omega$ (the measurable ones).

This problem has been analyzed by many authors in some special cases; in particular if the crisp measure is a possibility (Sugeno; see e.g. [6]) or an archimedean decomposable measure [7]. Here we present an approach which permits us to construct such a fuzzy measure in the case where the crisp one has an arbitrary continuous composition law. The main results and the detailed proofs are contained in [4].

Finally, the original information problem can be solved by a symmetrization of the obtained results. © 1997 Elsevier Science B.V.

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## 1. Preliminaries

In this section we recall some definitions and some notations regarding fuzzy sets (Section 1.1) and decomposable measures (Section 1.2).

### *1.1. Fuzzy sets*

In the following definition we identify a fuzzy set with its *membership function*. We remark that this identification will not cause any problem.

**Definition 1.1.** A fuzzy subset of a space $\Omega$ is a map

$\tilde{A}: \Omega \to [0,1]$,

where, $\forall \omega \in \Omega$, $\tilde{A}(\omega)$ represents the membership degree of $\omega$ with respect to the subset $\tilde{A}$.

**Definition 1.2.** Union, intersection and complementation are the classical ones:

$$(\tilde{A} \cup \tilde{B})(\omega) = \max[\tilde{A}(\omega), \tilde{B}(\omega)],$$

$$(\tilde{A} \cap \tilde{B})(\omega) = \min[\tilde{A}(\omega), \tilde{B}(\omega)],$$

$$(C\tilde{A})(\omega) = 1 - \tilde{A}(\omega).$$

Many crisp sets are classically associated to the fuzzy ones, such as the $\alpha$-cut, the support, the

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nearest crisp and so on. In this paper we need the notion of $\alpha$-level.

**Definition 1.3.** For each $\alpha\in[0,1]$, the $\alpha$-level of a fuzzy set $\tilde{A}$ is the crisp subset defined by

$$A^{\alpha} = \{\omega\in\Omega \mid \tilde{A}(\omega) = \alpha\}.$$

It is evident that the $\alpha$-level operation commutes with union and intersection, but the same property does not hold with respect to complementation.

We will use later the following notation: given a countable family $\mathscr{E} = \{B_j \mid j\in J \subset \mathbb{N}\}$ of disjoint crisp subsets of $\Omega$, and a subset $\{\beta_j \mid j\in J\}$ of non-null numerical values in $[0,1]$, we define the fuzzy subset $\bigcup \beta_j B_j$ by

$$\Big(\bigcup \beta_j B_j\Big)(\omega) = \begin{cases} \beta_j & \text{if } \omega\in B_j, \\ 0 & \text{if } \omega\notin \bigcup B_j. \end{cases}$$

Note that, if a fuzzy subset $\tilde{A}$ of $\Omega$ takes up only a finite number of values $\alpha_i$, $i = 1, \dots, n$, then we can write $\tilde{A}$ by means of the canonical decomposition

$$\tilde{A} = \bigcup_{i=1}^{n} \alpha_i A^i \quad (\text{where } A^i = A^{\alpha_i}), \tag{1.1}$$

### *1.2. Decomposable measures*

In this paper we deal with a particular class of Sugeno's measures, namely the decomposable ones, which has been studied by several authors, as, for example, E.P. Klement, S. Weber, and many others.

**Definition 1.4.** A Sugeno measure over a measurable space $(\Omega, \mathscr{A})$ is a map

$$m : \mathscr{A} \to [0,1]$$

with the following properties:

$$m(\emptyset) = 0,$$

$$m(\Omega) = 1,$$

$$A \subset B \Rightarrow m(A) \leqslant m(B).$$

Among the class of Sugeno's measures, the decomposable ones have the property that the measure of the union of two disjoint subsets $A$ and $B$ is determined by $m(A)$ and $m(B)$. More precisely

**Definition 1.5.** A Sugeno's measure is said to be decomposable if there exists a composition law

$$\top : [0,1]\times[0,1] \to [0,1]$$

such that

$$m(A\cup B) = m(A) \top m(B) \quad \forall A, B\in\mathscr{A},\ A\cap B = \emptyset.$$

**Definition 1.6.** The structure $(\Omega, \mathscr{A}, m, \top)$ will be denoted as "decomposable measure space", or as "decomposable space" if no equivocation arises.

It is easy to recognize that the composition law $\top$ has to be a t-conorm. In this paper we will consider only continuous t-conorms, which have the following standard representation (see e.g. [1]).

**Theorem 1.1.** *Let* $\mathscr{D} = \{]a_i, b_i[ \mid i\in N \subset \mathbb{N}\}$ *be a countable family of open disjoint subintervals of* $[0,1]$, *and let* $\mathscr{F} = \{f_i : [a_i, b_i] \to \mathbb{R} \mid i\in N\}$ *be a family of continuous and strictly increasing functions with* $f_i(a_i) = 0$. *The map* $\top$ *defined on* $[0,1]\times[0,1] = [0,1]^2$ *by*

$$x \top y = \begin{cases} f_i^{(-1)}[f_i(x) + f_i(y)] & \text{if } (x,y)\in[a_i,b_i]^2, \\ \max(x,y) & \text{otherwise} \end{cases} \tag{1.2}$$

(*where* $f_i^{(-1)}(\xi) = f_i^{-1}[\min\{\xi, f_i(b_i)\}]$ *is the pseudo-inverse of* $f_i$) *is a t-conorm, and, conversely, any continuous t-conorm has the form* (1.2), *with a suitable choice of the families* $\mathscr{D}$ *and* $\mathscr{F}$. *The function* $f_i$ *is called the additive generator of the t-conorm, with respect to interval* $[a_i, b_i]$.

Obviously, the subset $\Delta = [0,1] - \bigcup_{i\in N} ]a_i, b_i[$ is closed, and completely determines the family $\mathscr{D}$. Its elements are called $\top$-*idempotent*, or simply *idempotent* when no equivocation exists. It is easy to recognize (see e.g. [1]) that the following properties hold:

$$x \top y \geqslant \max(x,y),$$

$$x\in\Delta \Leftrightarrow x \top x = x,$$

$$x\in\Delta \Rightarrow x \top y = \max(x,y),$$

$$(x,y)\in\, ]a_i, b_i[^2 \Leftrightarrow x \top y > \max(x,y),$$

$$x\in[a_i,b_i],\ y\in[a_j,b_j],\ j\neq i \Rightarrow x \top y = \max(x,y).$$

## 2. The problem

Let $\Sigma = (\Omega, \mathscr{A}, m, \top)$ be a decomposable space. The problem we solve in this paper is, roughly speaking, that of the construction of a decomposable measure on a suitable family $\tilde{\mathscr{A}}$ of fuzzy subsets of $\Omega$, which is compatible with the space $\Sigma$.

In a previous research ([2], see also [3]) we considered the case where the fuzzy subset takes up only finitely many non-null values $(\alpha_1, \alpha_2, \ldots, \alpha_n)$. Since in this case the representation (1.1) holds, any measure $\bar{m}$ we associate to the fuzzy set $\tilde{A}$ will be a function of its $\alpha$-levels, as well as of the values $\alpha_i$. Since we supposed that $\bar{m}$ depends on the $A^i$ via their crisp measures $m_i = m(A^i)$, we have

$$\bar{m}(\tilde{A}) = G_n(\alpha_1, \ldots, \alpha_n; m_1, \ldots, m_n).$$

We reserve a particular attention to the restriction of the measure $\bar{m}$ to the family $\mathscr{U}$ of the one-level sets (case $n = 1$), that is to fuzzy subsets of the type $\alpha A$ defined by

$$(\alpha A)(\omega) = \begin{cases} \alpha & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A. \end{cases}$$

In this case we have

$$\bar{m}(\alpha A) = G[\alpha, m(A)]. \tag{2.1}$$

**Definition 2.1.** We will designate $G$ as the “structure function” associated to the measures $\bar{m}$ and $m$.

The meaning of the structure function (2.1) imposes to $G$ the following conditions:

$$G(0, x) = 0,$$

$$G(1, x) = x,$$

$$\alpha' < \alpha'' \Rightarrow G(\alpha', x) \leqslant G(\alpha'', x),$$

$$x' < x'' \Rightarrow G(\alpha, x') \leqslant G(\alpha, x'').$$

Our aim is to construct a decomposable fuzzy measure, that is we have to impose that $\bar{m}(\tilde{A} \cup \tilde{B}) = \bar{m}(\tilde{A}) \bar{\top} \bar{m}(\tilde{B})$, for a suitable *fuzzy composition law* $\bar{\top}$. The properties of fuzzy union impose that $\bar{\top}$ must be a t-conorm. Since we are dealing with continuous conorms, $\bar{\top}$ has the same kind of representation of the law $\top$, that is the form (1.2). It is evident that the characteristic elements of $\bar{\top}$ (namely the idempotent's set $\Delta_{\bar{\top}}$ and the increasing functions $\bar{f}_j$) will differ, in general, from the ones which determine $\top$ ($\Delta$ and $f_i$). Nevertheless, it has been proved that $\bar{\top}$ is strongly related to the crisp law $\top$ and with the function $G_n$. The relationship is characterized as follows: if $\tilde{A} = \bigcup_{i=1}^{n} \alpha_i A^i$, $\tilde{B} = \bigcup_{i=1}^{n} \alpha_i B^i$ are two disjoint fuzzy subsets of $\Omega$ then $\bar{m}(\tilde{A} \cup \tilde{B})$ can be obtained either directly by applying the function $G_n$ to the $\alpha$-levels of the set $\tilde{A} \cup \tilde{B}$, or indirectly, by using the expression $\bar{m}(\tilde{A}) \bar{\top} \bar{m}(\tilde{B})$. By letting $m_i' = m(A^i)$, $m_i'' = m(B^i)$, we obtain

$$\begin{aligned} &G_n(\alpha_1, \ldots, \alpha_n \colon m_1' \top m_1'', \ldots, m_n' \top m_n'') \\ &\quad = G_n(\alpha_1, \ldots, \alpha_n \colon m_1', \ldots, m_n') \bar{\top} G_n(\alpha_1, \ldots, \alpha_n \colon \\ &\qquad m_1'', \ldots, m_n''). \end{aligned} \tag{2.2}$$

The restriction of Eq. (2.2) to the family $\mathscr{U}$ takes the form

$$G(\alpha, m' \top m'') = G(\alpha, m') \bar{\top} G(\alpha, m''). \tag{2.3}$$

It has been proved (see [3]) that (2.2) is an immediate consequence of (2.3). Therefore we give the following.

**Definition 2.2.** We refer to (2.3) as to the compatibility equation between the two measures $\bar{m}$ and $m$, or, more precisely between the structure function $G$ and the two composition laws $\bar{\top}$ and $\top$. Because of its meaning, Eq. (2.3) holds for all pairs $(m', m'')$ belonging to the topological closure of the set $\{(x, y) \in [0, 1]^2 \mid x \top y < 1\}$.

**Theorem 2.1.** (Finite case, Bertoluzza [2] and Bodini [3]).
- *$G(\alpha, \Delta) \subseteq \bar{\Delta}$, $\forall \alpha$, and consequently $\bigcup ]\bar{a}_j, \bar{b}_j[ \subseteq \bigcup G(\alpha, ]a_i, b_i[)$;*
- *let $\bar{M} = \max\{G(\alpha_i, m_i = m(A_i)) \mid i = 1, \ldots, n\}$. Then*
- *If $\bar{M} \in \bar{\Delta}$, then $\bar{m}(\tilde{A}) = \bar{M}$.*
- *If $\bar{M} \notin \Delta$, then $\bar{M}$ belongs to some $]\bar{a}, \bar{b}[ \subseteq G(\alpha, ]a, b[)$, with $\alpha \in \{\alpha_1 \ldots \alpha_n\}$, $]a, b[ \in \{]a_1, b_1[ \ldots ]a_n, b_n[\}$, and we have*

$$\bar{m}(\tilde{A}) = \bar{f}^{(-1)}[\textstyle\sum \bar{f} \circ G(\alpha_r, m_r)]$$

*where the sum is extended to all the values $r$ for which $G(\alpha_r, m_r) \in ]\bar{a}, \bar{b}[$, and $\bar{f}$ is the function which determines $\bar{\top}$ in $]\bar{a}, \bar{b}[$. Moreover, if $f(x)$ is the function which determines $\top$ in $]a, b[$, then we have*

$$G(\alpha, x) = \bar{f}^{(-1)}[k(\alpha) \cdot f(x)] \tag{2.4}$$

*for a suitable non-decreasing positive function $k(\alpha)$.*

This result can be extended, without any difficulty, to the case where $\operatorname{Ran}(\tilde{A})$ is denumerable, provided that the measures $\bar{m}$, $m$ satisfy some additional conditions, namely the $\sigma$-decomposability with respect to the laws $\bar{\top}$ and $\top$.

The purpose of this paper is to extend the results just obtained to a more general class of fuzzy subsets (which will be specified later) and to obtain, in this case, an explicit form for the measure $\bar{m}(\tilde{A})$.

## 3. Extension of a discrete fuzzy measure

**Definition 3.1.** A fuzzy set $\tilde{A}$ is measurable if, $\forall \alpha, \beta \in [0, 1]$, the following property holds:

$$A_\alpha^\beta = \{\omega \in \Omega \mid \alpha \leqslant \tilde{A}(\omega) < \beta\} \in \mathscr{A}.$$

It has been recognized [4] that the class of the measurable subsets of the space $\Omega$ (according to the above definition) satisfies the axioms of a $\sigma$-algebra with respect to the operations of union and complementation introduced by Definition 1.2. Thus, $(\Omega, \tilde{\mathscr{A}})$ is a measurable space.

**Definition 3.2.** Let $\tilde{A}$ be a measurable fuzzy set. A standard approximation of $\tilde{A}$ is a discrete valued fuzzy set of the kind

$$\tilde{P}(\tilde{A}; \alpha_1, \ldots, \alpha_n) = \bigcup_{i=1}^{n} \alpha_i A_i,$$

where $0 \leqslant \alpha_1 < \cdots < \alpha_n \leqslant 1$ and $A_i = \{\omega \mid \alpha_{i-1} \leqslant \tilde{A}(\omega) < \alpha_i\}$.

**Definition 3.3.** Let $\mathscr{P}(\tilde{A})$ be the family of the standard approximations of $\tilde{A}$. The fuzzy measure of $\tilde{A}$ is defined as follows:

$$\bar{m}(\tilde{A}) = \sup\{\bar{m}(\tilde{P}) \mid \tilde{P} \in \mathscr{P}(\tilde{A})\}. \tag{3.1}$$

**Proposition 3.1.** *The measure defined by (3.2) is a decomposable measure on the class of the fuzzy measurable subsets.*

**Proof.** The only problem is to show that $\bar{m}$ is decomposable. Let $\tilde{A}$, $\tilde{B}$ be two disjoint fuzzy subsets and $\tilde{P}$ a standard approximation of $\tilde{A} \cup \tilde{B}$. Define

$$\tilde{P}_A(\omega) = \begin{cases} \tilde{P}(\omega) & \text{if } \omega \in \Omega_A, \\ 0 & \text{otherwise,} \end{cases}$$

$$\tilde{P}_B(\omega) = \begin{cases} \tilde{P}(\omega) & \text{if } \omega \in \Omega_B, \\ 0 & \text{otherwise,} \end{cases}$$

where $\Omega_A = \{\omega \mid \tilde{A}(\omega) > 0\}$, $\Omega_B = \{\omega \mid \tilde{B}(\omega) > 0\}$. Clearly, $\tilde{P} = \tilde{P}_A \cup \tilde{P}_B$, where the latter is a disjoint union, and $\tilde{P}_A$, $\tilde{P}_B$ are standard approximations of $\tilde{A}$ and $\tilde{B}$, respectively. By taking the supremum over the standard approximations of $\tilde{A} \cup \tilde{B}$, we find that $\bar{m}(\tilde{A} \cup \tilde{B}) \leqslant \bar{m}(\tilde{A}) \bar{\top} \bar{m}(\tilde{B})$. At the same time $\bar{m}(\tilde{P}_A) \bar{\top} \bar{m}(\tilde{P}_B) = \bar{m}(\tilde{P})$, and since $\bar{\top}$ is continuous, we conclude that $\bar{m}(\tilde{A}) \bar{\top} \bar{m}(\tilde{B}) \leqslant \bar{m}(\tilde{A} \cup \tilde{B})$. Thus, the decomposability is proved. □

In the next section we shall make use of Proposition 3.1 to generalize an already known result, namely the expression of $\bar{m}(\tilde{A})$ in the case where the composition law $\top$ is the type-sup one.

## 4. Possibility measures

The possibility is a particular continuous decomposable measure. Its extension to the fuzzy subsets can be obtained as a particular case of the general representation theorem which we will present in the next section. Nevertheless, because of the simplicity and significance of the proof, we prefer to present it in a separate section.

Let $(\Omega, \mathscr{A}, m, \top)$ be a type-sup decomposable space, and let $G$ be a structure function associated with this space. This means that $m$ is a possibility measure, and $\bar{m}$ is generated by $G$, that is

$$m(A \cup B) = \max[m(A), m(B)],$$

$$\bar{m}(\alpha A) = G[\alpha, m(A)].$$

In such a case, Theorem 2.1 assures that $\bar{m}$ is also a possibility measure, and in particular we have

$$\bar{m}(\alpha A \cup \beta B) = \max[\bar{m}(A), \bar{m}(B)]$$

no matter whether $A$ and $B$ are disjoint or not.

**Definition 4.1.** The discriminant element of a measurable fuzzy subset $\tilde{A}$ of $\Omega$ is the number

$$\Gamma(\tilde{A}) = \sup\{\bar{m}(\alpha A) \mid \alpha A \subset \tilde{A}\}. \tag{4.1}$$

The definition of discriminant element, given here for the possibility measures, also holds, in the same form, for the general case which we will analyse in the next section.

**Theorem 4.1.** $\bar{m}(\tilde{A}) = \Gamma(\tilde{A})$.

**Proof.** If $\alpha A \subset \tilde{A}$, then $\alpha A \in \mathscr{P}(\tilde{A})$. Therefore, by the definition (3.1) of the fuzzy measure we have $\Gamma(\tilde{A}) \leqslant \bar{m}(\tilde{A})$. To show the inverse inequality, let $\tilde{P} = \tilde{P}(\tilde{A}; \alpha_1, \dots, \alpha_n)$ be a standard approximation of $\tilde{A}$. Then, by definition, $\tilde{P} = \bigcup_{i=1}^{n} \alpha_i A_i$. Because $\bar{m}$ is a possibility measure we have $\bar{m}(\tilde{P}) = \sup_{i=1}^{n} \bar{m}(\alpha_i A_i) \leqslant \Gamma(\tilde{A})$, and inequality $\bar{m}(\tilde{A}) \leqslant \Gamma(\tilde{A})$ follows when $\tilde{P}$ varies among all the standard approximations of $\tilde{A}$. □

Note that, in general, the computation of $\Gamma(\tilde{A})$ is much easier than that of $\bar{m}(\tilde{A})$. In fact, let $A^{\alpha} = \{\omega \mid \tilde{A}(\omega) \geqslant \alpha\}$, and $\varphi(\alpha) = m(A^{\alpha})$, for each $\alpha \in [0, 1]$.

**Theorem 4.2.** $\Gamma(\tilde{A}) = \sup\{G(\alpha, \varphi(\alpha) \mid \alpha \in [0, 1]\}$.

**Proof.** It suffices to note that, if $\alpha A \subset \tilde{A}$, then $A \subset A^{\alpha}$, so that $\bar{m}(\alpha A) \leqslant G[\alpha, \varphi(\alpha)]$. □

Thus, the evaluation of $\Gamma(\tilde{A})$, for a given fuzzy subset $\tilde{A}$, is reduced to the maximization of a real-valued positive function on the unit interval.

## 5. The general case

Assume now that the laws $\top$ and $\bar{\top}$ are arbitrary continuous t-conorms on $[0, 1]$ and let $G$ be a structure function, satisfying the compatibility equation (2.3).

Let $\Delta^{c}_{\top}$ and $\Delta^{c}_{\bar{\top}}$ be the sets of non-idempotents for the laws $\top$ and $\bar{\top}$, respectively. Then, according to the continuity of $\top$ and $\bar{\top}$, we may write

$$\Delta^{c}_{\top} = \bigcup_{i \in I} ]a_i, b_i[, \qquad \Delta^{c}_{\bar{\top}} = \bigcup_{j \in J} ]\bar{a}_j, \bar{b}_j[, \tag{5.1}$$

where $I, J \subset \mathbb{N}$, and the intervals which appear in the expression of $\Delta^{c}_{\top}$ given by (5.1) (as well as those appearing in the expression of $\Delta^{c}_{\bar{\top}}$) are pairwise disjoint. They are referred here as non-idempotent intervals.

Let $\tilde{A}$ be a measurable fuzzy subset of $\Omega$, and let $\Gamma = \Gamma(\tilde{A})$ be the discriminant element of $\tilde{A}$ (see Definition 4.1).

**Theorem 5.1.** *If $\Gamma(\tilde{A})$ is an idempotent element for the law $\bar{\top}$, then $\bar{m}(\tilde{A}) = \Gamma(\tilde{A})$.*

**Proof.** In fact, if $\tilde{P} = \tilde{P}(\alpha_1, \dots, \alpha_n)$ is a standard approximation of $\tilde{A}$, then

$$\bar{m}(\tilde{P}) = \mathop{\bar{\top}}_{i=1}^{n} \bar{m}(\alpha_i A_i) \leqslant \Gamma \,\bar{\top}\, \Gamma \,\top \dots \bar{\top}\, \Gamma = \Gamma,$$

and consequently $\bar{m}(\tilde{A}) \leqslant \Gamma$. On the other hand, $\sup\{\bar{m}(\tilde{P}) \mid \tilde{P} \in \mathscr{P}(\tilde{A})\} \geqslant \Gamma$, because all the $\alpha A$'s appearing in the definition of $\Gamma$, belong to $\mathscr{P}(\tilde{A})$. Therefore, $\bar{m}(\tilde{A}) \geqslant \Gamma$ and the theorem is proved. □

From now on we shall assume that $\Gamma$ is non-idempotent with respect to $\bar{\top}$, and that $]c, d[$ is the non-idempotent interval containing $\Gamma$, i.e. $c < \Gamma < d$. It is easy to recognize that the following rough approximation for the measure $\bar{m}(\tilde{A})$ holds:

$$c < \Gamma \leqslant \bar{m}(\tilde{A}) \leqslant d. \tag{5.2}$$

In fact, on the one hand, $\bar{m}(\tilde{A}) \geqslant \bar{m}(\alpha A) \geqslant \Gamma$, while on the other, since $\bar{m}(\alpha_i A_i) \leqslant d$, expression (1.2) assures that $\bar{m}(\tilde{P}) = \bar{\top} \bar{m}(\alpha_i A_i) \leqslant d$, and therefore the right-hand side inequality of (5.2) holds.

Let us write, for simplicity,

$$\varphi_{\alpha}(x) = G(\alpha, x), \quad \forall \alpha \in [0, 1], \ \forall x \in [0, 1].$$

Then, when $\alpha$ is fixed, $\varphi_{\alpha}$ is a continuous monotonic function of $x$. Let

$$T_0 = \{\alpha \mid \varphi_{\alpha}(1) \leqslant c\}.$$

Because of the monotonicity of the function $G$, $T_0$ is a subinterval of $[0,1]$ of the kind $[0,\alpha_0]$. For each $\alpha > \alpha_0$, let

$$\xi(\alpha) = \sup\{x \in [0,1] \mid \varphi_\alpha(x) \leqslant c\}.$$

**Theorem 5.2.** *For each $\alpha > \alpha_0$, there exists a non-idempotent interval $]a,b[$ of the law $\top$, such that $\xi(\alpha) = a$. In other words, for each $\alpha > \alpha_0$ there exists a $\top$-non-idempotent interval $]a,b[$, such that $\varphi_\alpha(a) = c$ and $\varphi_\alpha(x) > c$ for all $x > a$.*

**Proof.** Let $\alpha > \alpha_0$, and let $k = \sup\{x \in [0,1] \mid x \top x < 1\}$. (by continuity of $\top$, $k \top k = 1$). We start by showing that $\xi = \xi(\alpha) < k$. Assume that $\xi \geqslant k$. Then $c = \varphi(\xi) \geqslant \varphi_\alpha(k)$. We may apply the compatibility equation to the pair $(k,k)$, obtaining

$$\varphi_\alpha(1) = \varphi_\alpha(k \top k) = \varphi_\alpha(k)\, \bar{\top}\, \varphi_\alpha(k) \leqslant c\, \bar{\top}\, c = c,$$

which is in contradiction with the assumption that $\alpha > \alpha_0$. Hence $\xi < k$, and we may write the compatibility equation for the pair $(\xi, \xi)$ in the form

$$c = \varphi_\alpha(\xi)\, \bar{\top}\, \varphi_\alpha(\xi) = \varphi_\alpha(\xi \top \xi).$$

This means, by remembering the definition of $\xi$, that $\xi \top \xi \leqslant \xi$. But $\xi \top \xi \geqslant \xi$, since $\top$ is a t-conorm, hence $\xi \top \xi = \xi$ and $\xi$ is idempotent with respect to the law $\top$.

Now let $x_1$ be an element of $[0,1]$ such that $\xi < x_1 < k$ and $\varphi_\alpha(x_1) < d$. If $x_1$ was $\top$-idempotent, then $\varphi_\alpha(x_1)$ would be $\bar{\top}$-idempotent, which is impossible because, in our hypotheses, $\varphi_\alpha(x_1) \in ]c,d[$. Hence, $x_1$ is non-idempotent (with respect to $\top$) and the same argument works for all $x \in [\xi, x_1]$, thus concluding our proof.

With reference to the decomposition $\Delta_\top^c$ introduced in (5.1), let us pose, for each $i \in I$,

$$T_i = \{\alpha \in [0,1] \mid \xi(\alpha) = a_i\}.$$

The subsets $T_i$ are pairwise-disjoint intervals, and because of Theorem 5.2 $T_0 \cup \{\bigcup_{i\in I} T_i\} = [0,1]$. For each $i \in I$, let

$$T_i' = \left\{\alpha \in T_i \,\middle|\, \lim_{\varepsilon \to 0} m(A_\alpha^{\alpha+\varepsilon}) < a_i\right\},$$

$$J_i = T_i - T_i',$$

$$\Omega_i' = \{\omega \in \Omega \mid \tilde{A}(\omega) \in T_i'\},$$

$$\Omega_i = \{\omega \in \Omega \mid \tilde{A}(\omega) \in J_i\},$$

$$\tilde{A}_i' = \begin{cases} \tilde{A}(\omega) & \text{if } \omega \in \Omega_i', \\ 0 & \text{otherwise,} \end{cases}$$

$$\tilde{A}_i = \begin{cases} \tilde{A}(\omega) & \text{if } \omega \in \Omega_i, \\ 0 & \text{otherwise,} \end{cases}$$

where $A_\alpha^{\alpha+\varepsilon} = \{\omega \in \Omega \mid \alpha \leqslant \tilde{A}(\omega) < \alpha + \varepsilon\}$. □

**Theorem 5.3.** *$T_i'$ is right open in $T_i$.*

**Proof.** Let $\alpha \in T_i'$ and $\varepsilon_0 > 0$ be such that $m(A_\alpha^{\alpha+\varepsilon_0}) < a_i$. For any $\beta$ such that $\alpha < \beta < \alpha + \varepsilon_0$, we have $\lim_{\varepsilon\to 0} m(A_\beta^{\beta+\varepsilon}) < a_i$, so that $\beta \in T_i'$ and the proof is concluded. □

Note that $T_i'$ is a Borel subset of $T_i$, which is also a Borel set, and it is evident that $J_i$ is also a Borel subset of $T_i$.

Now we introduce an interesting subdivision of $\Omega$ related to the given fuzzy set $\tilde{A}$. Let $R = \bigcup_{i\in I} J_i$, $N = T_0 \cup \{\bigcup_{i\in I} T_i'\}$, so that $R \cup N = [0,1]$. Moreover, let

$$\Omega_R = \{\omega \in \Omega \mid \tilde{A}(\omega) \in R\},$$

$$\Omega_N = \{\omega \in \Omega \mid \tilde{A}(\omega) \in N\}.$$

The above quoted subdivision of $\tilde{A}$ (related to the partition $\{\Omega_R, \Omega_N\}$), is the following: subsets $\tilde{A}_R$ and $\tilde{A}_N$ defined by

$$\tilde{A}_R(\omega) = \begin{cases} \tilde{A}(\omega) & \text{if } \omega \in \Omega_R, \\ 0 & \text{otherwise,} \end{cases}$$

$$\tilde{A}_N(\omega) = \begin{cases} \tilde{A}(\omega) & \text{if } \omega \in \Omega_N, \\ 0 & \text{otherwise.} \end{cases}$$

It is evident that $\tilde{A}_R$ and $\tilde{A}_N$ are disjoint and $\tilde{A}_R \cup \tilde{A}_N = \tilde{A}$.

**Definition 5.1.** We refer to $\tilde{A}_R$ and to $\tilde{A}_N$, respectively, as to the "relevant part" and to the "null part" of $\tilde{A}$.

**Theorem 5.4.** *Let $\tilde{A}_R$ and $\tilde{A}_N$ be the relevant and the null part of a measurable fuzzy set $\tilde{A}$. Then*

(i) *$\tilde{A}_R$, $\tilde{A}_N$ are measurable fuzzy subsets,*

(ii) *$\bar{m}(\tilde{A}) = \bar{m}(\tilde{A}_R)$ (the measure of $\tilde{A}$ coincides with that of its relevant part),*

(iii) *if $\tilde{P}$ is a standard approximation of $\tilde{A}_R$, then $\bar{m}(\tilde{P}) \geqslant c$.*

**Proof.** (i) $\tilde{A}_R$ and $\tilde{A}_N$ are measurable since $R$ and $N$ are Borel sets.

(ii) It will suffice to show that $\bar{m}(\tilde{A}_N) \leqslant c$, because $\bar{m}(\tilde{A}) = \bar{m}(\tilde{A}_R) \bar{\top} \bar{m}(\tilde{A}_N)$ and $\bar{m}(\tilde{A}) > c$. Let $\{[\alpha_l, \beta_l] \mid l \in L\}$ be a countable covering of $T_i'$, such that, for each $l \in L$, $m(A_{\beta_l}^{\alpha_l} < a_i)$. By the idempotency of $a_i$ we have

$$m\left(\bigcup_{l \in L} A_{\beta_l}^{\alpha_l}\right) = \mathop{\top}_{l \in L} m(A_{\beta_l}^{\alpha_l}) \leqslant a_i.$$

Now, if $\alpha \in T_i'$ and $A \subseteq \Omega_i'$, then $\bar{m}(\alpha A) \leqslant c$ by the definition of $T_i$. Hence, any standard approximation $\tilde{P}$ of $\tilde{A}_i'$ satisfies $\bar{m}(\tilde{P}) \leqslant c$, so that $\bar{m}(\tilde{A}_i') \leqslant c$, and by taking the union over index $i$ we obtain $\bar{m}(\tilde{A}_N) \leqslant c$.

(iii) Let now $\tilde{P} = \tilde{P}(\alpha_1 \dots \alpha_n)$ be a standard approximation of $\tilde{A}_R$. Because $\tilde{P}$ approximates $\tilde{A}_R$ and $\bar{\top}$ is a t-conorm, it is evident that $\bar{m}(\tilde{P}) \geqslant \bar{m}(\tilde{P}^*)$, where $\tilde{P}^*$ is a suitable approximation of the kind $\alpha A$, with $A = A_\alpha^\varepsilon$ and $\alpha \in J_i$ for a certain $i$. It follows now from the definition of $J_i$, that $\bar{m}(\alpha A) \geqslant c$, so that $\bar{m}(\tilde{P}) \geqslant c$, and the proof is concluded. □

**Remark.** Definition 3.3 can be applied to all the fuzzy subset in $\mathscr{P}(\Omega)$: it is the inner measure. The results stated to this point hold for the subsets of $\mathscr{P}(\Omega)$ for which the measures of the standard approximations exist. On the contrary, Theorems 5.5 and 5.6 below hold only for the measurable subsets of Definition 3.1; they represent a relation between the abstract measure of Definition 3.3 and Weber's integral, when the discriminant element is non-idempotent.

**Theorem 5.5.** *Let $\bar{f}$ be the additive generator of the restriction of $\bar{\top}$ to the interval $]c, d[$, and let $f_i$ be the additive generator of the restriction of $\top$ to the interval $]a_i, b_i[$. Then we have*

$$\bar{m}(\tilde{A}_i) = \bar{f}^{(-1)}\left[\int_{\Omega_i} k(\tilde{A}_i)\, d(f_i \circ m)\right]$$

*if $f_i \circ m$ is additive,* (5.3)

$$\bar{m}(w\tilde{A}_i) = \bar{f}^{(-1)}\left[\sum_r \int_{\Omega_i \cap \Omega^r} k(\tilde{A}_i)\, d(f_i \circ m)\right]$$

*if $f_i \circ m$ is pseudo-additive,* (5.4)

*where the notion of pseudo-additivity is the one given in* [2], *where $\{\Omega^r\}$ is a m-achievable partition of $\Omega$, that is a collection of disjoint subsets such that $m(\Omega^r) < m(\Omega)$, $\sum m(\Omega^r) > m(\Omega)$, and k is the function of formula* (2.4).

**Proof.** Let us suppose firstly that the set function $\bar{f} \circ \bar{m}$ is an additive measure as in [7], pag. 126, and let $\tilde{P}_i = \bigcup_j \alpha_j A_{i,j}$ be a standard approximation of $\tilde{A}_i$ such that $\bar{m}(\alpha_j A_{i,j}) \in ]c, d[$. Now we can use the restricted law $\bar{\top}_{|[c,d]}$ and the representation (2.4), to obtain

$$\begin{aligned}\bar{m}(\tilde{P}_i) &= \bar{f}^{(-1)}\left[\sum_j \bar{f}\{\bar{m}(\alpha_j A_{i,j})\}\right]\\ &= \bar{f}^{(-1)}\left[\sum_j \bar{f} \circ G\{\alpha_j, m(A_{i,j})\}\right]\\ &= \bar{f}^{(-1)}\left[\sum k(\alpha_j) \cdot f_i \circ m(A_{i,j})\right].\end{aligned}$$

The measure of $\tilde{A}_i$ is obtained by taking the supremum over all the standard approximations. Function $\bar{f}$ is continuous and $A_i$ is measurable (the proof is quite similar to Theorem 5.4 (ii)), and therefore the supremum takes the integral form of (5.3).

If we choose in (2.4) $k(\alpha) = \alpha$, then we can obtain the same result by computing Weber's integral (see [7, formula (A) p. 129]) of the membership function $\tilde{A}_i$, since $(\Omega_i, \mathscr{A} \cap \Omega_i, \bar{m}, \bar{\top})$ is an archimedean decomposable space.

Now suppose that $\bar{f} \circ \bar{m}$ is pseudo-additive, and let $\{\Omega^r\}$ be a *m-achievable* partition of $\Omega$. For each value $r$, we can use the above procedure to express $\bar{m}(\tilde{A}_i \cap \Omega^r)$ by means of an integral over $\Omega_i \cap \Omega^r$; then we compose these measures by means of $\bar{\top}_{|[c,d]}$, thus obtaining (5.4).

As in the previous case the same result can be obtained, when $k(\alpha) = \alpha$, by using the form (P) [7, p. 129] of Weber's integral.

**Corollary** (Main result). *With the notations introduced above, we obtain*

$$\bar{m}(\tilde{A}) = \bar{f}^{(-1)}\left[\sum_{i\in I} \bar{f}\circ m(\tilde{A}_i)\right]. \tag{5.5}$$

**Proof.** Since $\bar{m}(\tilde{A}_i) \in ]c,d[$, it suffices to apply recursively $|I|$ times the restriction of $\bar{\top}$ to this interval. □

## 6. The archimedean case

The case of the archimedean measures has been analyzed by several authors. A very complete treatment can be found, e.g. in [7]. Following our approach it is a particular case of the main result. In fact, if $(\Omega, \mathscr{A}, m, \top)$ is archimedean, then $(\Omega, \tilde{\mathscr{A}}, \bar{m}, \bar{\top})$ is of the same kind (the proof of this statement has been given in [2]). This means that

$$\Delta = \bar{\Delta} = \{0, 1\},$$

$$x \top y = f^{(-1)}[f(x) + f(y)],$$

$$x \bar{\top} y = \bar{f}^{(-1)}[\bar{f}(x) + \bar{f}(y)].$$

Hence, with reference to the theorems of the previous section, $[c,d] = [0,1]$ and the family of the intervals $]a_i, b_i[$ reduces to the unique interval $]0,1[$. Now, by applying Theorem 5.5 we obtain

$$\bar{m}(\tilde{A}) = \bar{f}^{(-1)}\left[\int_{\Omega} \tilde{A}\, d(f\circ m)\right]$$

or

$$\bar{m}(\tilde{A}) = \bar{f}^{(-1)}\left[\sum_{r} \int_{\Omega\cap\Omega^r} \tilde{A}\, d(f\circ m)\right]$$

(according to the cases of additivity or pseudo-additivity) which coincide with Weber's results (see [7, p. 129, formulas, A, P]).

**Remark.** There exists a formal resemblance between our main result (5.5) and the formula (P) of p. 129 of the above quoted paper, but it is only formal, because the meaning of the two expressions are completely different. In fact, Weber's result regards only archimedean spaces and the sum in (P) refers to those collection of pairwise disjoint subsets for which the measure of the union does not coincide with the sum of the measures of the "components". On the contrary, in our case, the decomposable measure space in non-archimedean, and the sum in (5.5) refers to subsets which have measure in the intervals $]a_i, b_i[$ and image by function $G$ in $]c,d[$, that is a sum with a completely different meaning.

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