STRUCTURED MATRICES AND GOOD BASES

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Abstract

So much of mathematics is involved with the representation of functions. A central example in pure and applied mathematics is the Fourier series. Its discrete version is computed by the Fast Fourier Transform, which is the most important algorithm of the last century (and we report briefly on its approximation by a "shift and add" binary transform). The Fourier basis is terrific – but imperfect. The basis functions are global instead of local, and they give poor approximation at a discontinuity (Gibbs phenomenon). New functions are being developed for interpolation and approximation and compression and many other applications.

Four properties we want are: local basis, easily refined, fast to compute, good approximation by a few terms. Splines and finite elements achieve the first three, but they don't allow compression; if we remove terms in a spline expansion this leaves blank intervals. So we turn to the wavelet construction to permit compression of data – which is needed in so many modern applications where the volume of data is overwhelming. Wavelets have two types of basis functions, one for averages and the other (the wavelets themselves) for details at all scales. When the details are not necessary they can be compressed away to leave a smoothed signal. That construction has entered the new IEEE standards for signal processing. We will explain how the construction is achieved with two filters (where one filter would fail). In matrix terms we get a banded matrix with a banded inverse.

Finally we discuss the localized eigenvectors that appear when a few entries are changed in a familiar tridiagonal Toeplitz matrix.

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