THE PRESCRIBED p -MEAN CURVATURE EQUATION OF LOW REGULARITY IN THE HEISENBERG GROUP

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ABSTRACT. I will report on recent study about regularity and the singular set of a C^1 smooth surface with prescribed p (or H)-mean curvature in the 3-dimensional Heisenberg group. As a differential equation, this is a degenerate hyperbolic and elliptic P.D.E. of second order, arising from the study of CR geometry. Assuming only the p -mean curvature $H \in C^0$, we showed that any characteristic curve is C^2 smooth and its (line) curvature equals $-H$. By introducing special coordinates and invoking the jump formulas along characteristic curves, we proved that the Legendrian (horizontal) normal gains one more derivative. Therefore the seed curves are C^2 smooth. We also obtained the uniqueness of characteristic and seed curves passing through a common point under some mild conditions, respectively. In an on-going project, we showed that the p -area element is in fact C^2 smooth along any characteristic curve and satisfies a certain ordinary differential equation of second order. Moreover, we analyzed this O.D.E. to study the singular set.

1. REGULARITY AND CURVATURE OF CHARACTERISTIC CURVES

A pseudohermitian manifold (M, J, Θ) is an odd dimensional (C^∞ smooth) manifold M (say, of dimension $2n+1$) together with a contact form Θ , i.e. a 1-form satisfying $\Theta \wedge (d\Theta)^n \neq 0$ everywhere, and an almost complex structure J , i.e. an endomorphism from the contact bundle $\xi \equiv \ker \Theta$ onto itself satisfying $J^2 = -Identity$ (usually J is also required to satisfy a certain integrability condition while, for $n = 1$, this condition always holds) (see, e.g., [10]). The natural examples arise as the boundaries of strongly pseudoconvex domains in C^{n+1} . Let D be a strongly pseudoconvex domain in C^{n+1} . Let $M = \partial D$. We take $\Theta = -i\partial\varphi$ ($i = \sqrt{-1}$) restricted to M , where φ is a defining function for M . It follows that the contact bundle ξ coincides with $TM \cap J_{C^{n+1}} TM$ where $J_{C^{n+1}}$ denotes the almost complex structure of C^{n+1} . We then take J to be the restriction of $J_{C^{n+1}}$ on ξ . We understand D through the study of ∂D viewed as a pseudohermitian manifold as above. On the other hand, we view the Heisenberg group H^n as a (flat) pseudohermitian manifold $(R^{2n+1}, \hat{J}, \hat{\Theta})$. Here $\hat{\Theta} \equiv dz + \sum_{j=1}^n (x_j dy_j - y_j dx_j)$ and $\hat{J}\hat{e}_j = \hat{e}_{j'}$, $\hat{J}\hat{e}_{j'} = -\hat{e}_j$ where $\hat{e}_j \equiv \partial_{x_j} + y_j \partial_z$, $\hat{e}_{j'} \equiv \partial_{y_j} - x_j \partial_z \in \ker \hat{\Theta}$.

Now consider a (say, C^∞ smooth) hypersurface $\Sigma \subset (M, J, \Theta)$. There is a dichotomy for points of Σ . We call a point p singular if $\xi(\equiv \ker \Theta)$ coincides with $T\Sigma$ at p . Otherwise, we call p a nonsingular point where $\xi(\equiv \ker \Theta)$ is transverse to $T\Sigma$. At a nonsingular point in Σ , we can talk about the so called (unit) horizontal (or Legendrian) normal $\nu \in \xi$ such that ν is of unit length and perpendicular to

1991 *Mathematics Subject Classification*. Primary: 35L80; Secondary: 35J70, 32V20, 53A10, 49Q10.

Key words and phrases. Heisenberg group, p -minimal surface, Bernstein-type theorem.

$\xi \cap T\Sigma$ with respect to the Levi metric $G \equiv \frac{1}{2}d\Theta(\cdot, J(\cdot))$. Consider a domain $\tilde{\Omega} \subset M$ with $\Sigma = \partial\tilde{\Omega}$. By computing the first variation of the volume (natural volume element $= \Theta \wedge (d\Theta)^n$ which is the Euclidean volume for H^n) of $\tilde{\Omega}$ in the direction ν , we obtain a notion of area for Σ , called p -area. We then define the p -mean curvature H as the first variation of the p -area in the direction ν (see [1] or [4] for more details). Here we simply write down H for a graph $\Sigma \equiv \{(x_1, y_1, \dots, x_n, y_n, u(x_1, \dots, y_n))\}$ in the Heisenberg group H^n and hence the equation of prescribed p -mean curvature H reads

$$(1.1) \quad \operatorname{div} \frac{\nabla u + \vec{F}}{|\nabla u + \vec{F}|} = H \quad \text{in } \Omega \subset R^{2n}$$

where $u : \Omega \subset R^{2n} \rightarrow R$ and $\vec{F} \equiv (-y_1, x_1, -y_2, x_2, \dots, -y_n, x_n)$ (also for general $\vec{F}$). We observed that (1.1) is a degenerate (hyperbolic and elliptic) PDE in dimension 2 ($n = 1$). We call a point where $\nabla u + \vec{F} = 0$ in Ω singular, which is the projection of a singular point in Σ onto R^{2n} . In higher dimensions ($n \geq 2$), we observed that (1.1) is subelliptic (square sum of $2n - 1$ vector fields satisfying Hörmander's condition) in the nonsingular domain $\{p \in \Omega \mid \nabla u + \vec{F} \neq 0 \text{ at } p\}$. We have been working on this equation (also for general $\vec{F}$) in recent years ([4], [5], [6], [7], [2], [3], and the survey paper [1]).

For $n = 1$, we can write

$$(1.2) \quad \frac{\nabla u + \vec{F}}{|\nabla u + \vec{F}|} = (\cos \theta, \sin \theta)$$

locally for some angle function θ . By (1.2) we can write equation (1.1) (replacing x_1, y_1 by x, y) as follows:

$$(1.3) \quad \operatorname{div}(\cos \theta, \sin \theta) \equiv (\cos \theta)_x + (\sin \theta)_y = H.$$

For $u \in C^1(\Omega)$, $\vec{F} \in C^0(\Omega)$ in a nonsingular plane domain Ω , we have $\theta \in C^0$ locally. We may consider $\theta \in C^0$ as an independent variable. Define $N \equiv (\cos \theta, \sin \theta)$ such that N and $N^\perp \equiv (\sin \theta, -\cos \theta)$ are C^0 vector fields. We also call the integral curves of $N^\perp$ characteristic curves.

Theorem 1.1 (Theorem A in [7]). *Let Ω be a domain of R^2 and $H \in C^0(\Omega)$. Let $\theta \in C^0(\Omega)$ satisfy equation (1.3) in the weak sense, i.e.*

$$(1.4) \quad \int_{\Omega} (\cos \theta, \sin \theta) \cdot \nabla \varphi + \int_{\Omega} H \varphi = 0$$

for all $\varphi \in C_0^\infty(\Omega)$. Let $\Gamma \subset \Omega$ be a (C^1 smooth) characteristic curve with σ being the unit-speed parameter. Then Γ is C^2 smooth and the curvature of Γ (along $N^\perp$ direction) equals $-H$, that is, $\frac{d\theta}{d\sigma} = -H$.

We remark that in Theorem A of [13], Pauls proved the case of $H = 0$ in which Γ is a straight line under the extra condition that components of the horizontal Gauss map (i.e., $\frac{\nabla u + \vec{F}}{|\nabla u + \vec{F}|}$ in our notation with $\vec{F} = (-y, x)$) are $W^{1,1}(\Omega)$. In Theorem 1.1 above, if θ satisfies (1.4), we proved that Γ is a minimizer for the following energy functional:

$$L_H(\Gamma) \equiv |\Gamma| - \int_{\Omega_\Gamma} H dx dy$$

where $|\Gamma|$ denotes the length of Γ and Ω_Γ is the domain surrounded by Γ and some line segments (see Section 2 in [7]). So the basic calculus of variation tells us that the curvature of Γ (along $N^\perp$ direction) equals $-H$ without invoking extra regularity assumption. Also H is only required to be C^0 . In [11] Monti and Rickly considered the case of $H = \text{constant}$ for a convex isoperimetric set. We do not need convexity in Theorem 1.1.

2. UNIQUENESS OF CHARACTERISTIC AND SEED CURVES

For $\theta \in C^0$, N ($N^\perp$, resp.) $\equiv (\cos \theta, \sin \theta)$ ($\equiv (\sin \theta, -\cos \theta)$, resp.) is a C^0 vector field. Then for any $p \in \Omega$, there exists at least one integral curve of N ($N^\perp$, resp.), called seed curve (characteristic curve, resp.) passing through p . The uniqueness of integral curves for a C^0 vector field does not hold true in general (see page 18 in [9]). Let $p \in \Omega$ and $B_r(p) \equiv \{q \in \Omega \mid |q - p| < r\}$. Define $H_M(r) \equiv \max_{q \in \partial B_r(p)} |H(q)|$.

Theorem 2.1 (Theorem B in [7]). *(a) Let $\theta \in C^0(\Omega)$ and $H \in L^1_{loc}(\Omega)$ satisfy (1.4). Let $p \in \Omega$ and suppose there is $r_0 > 0$ such that $B_{r_0}(p) \subset \subset \Omega$ and*

$$(2.1) \quad \int_0^{r_0} H_M(r) dr < \infty.$$

Then there is r_1 , $0 < r_1 \leq r_0$, such that there exists a unique seed curve passing through p in $B_{r_1}(p)$.

(b) Let $\theta \in C^0(\Omega)$ and $H \in C^{0,1}(\Omega)$ (Lipschitzian) satisfy (1.4). Then for any point $p \in \Omega$, we can find $r_1 > 0$ such that there exists a unique characteristic curve passing through p in $B_{r_1}(p)$.

In Theorem 2.1 (b), if H is only continuous, we gave an example for the nonuniqueness of characteristic curves (see Example 3.2 in [7]). Note that u is not involved in Theorems 1.1 and 2.1. Now we consider u . Let $u \in C^1$ and $\vec{F} = (F_1, F_2) \in C^1$. Recall that a point $p \in \Omega \subset \mathbb{R}^2$ is called singular (nonsingular, resp.) if $\nabla u + \vec{F} = 0$ ($\neq 0$, resp.) at p . At a nonsingular point, we set $N \equiv N^u \equiv \frac{\nabla u + \vec{F}}{|\nabla u + \vec{F}|}$. We call Ω nonsingular if every point of Ω is not singular. We had another uniqueness theorem for characteristic curves.

Theorem 2.2 (Theorem B' in [7]). *Let $u : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a C^1 smooth function such that Ω is a nonsingular domain with $\vec{F} \in C^1(\Omega)$. Then for any point $p \in \Omega$, we can find $r_1 > 0$ such that $B_{r_1}(p) \subset \Omega$ and there exists a unique characteristic curve passing through p in $B_{r_1}(p)$.*

In Theorem 2.2, we only assume $u, \vec{F} \in C^1$, and do not use any property of H , which is different from Theorem 2.1 (b). Even for the case $H = 0$, seed curves may only be C^1 smooth, but not C^2 smooth (see the remark after the proof of Theorem D in Section 5 of [7]). However if $N \equiv (\cos \theta, \sin \theta)$ arises from u (i.e., $N = N^u$), Pauls ([13]) proved that when $u \in C^1$, $\theta \in C^0 \cap W^{1,1}$, and $H = 0$, then the seed curves are C^2 smooth. In Theorem 3.4 below, we proved the same conclusion under the condition that $u \in C^1$ ($\theta \in C^0$ follows) and $H \in C^1$ (in fact, that $H \in C^0$ and only C^1 in the N direction is enough).

3. REGULARITY OF θ AND CHARACTERISTIC AND SEED CURVES

Suppose that θ arises from u , i.e., $(\cos \theta, \sin \theta) = N^u$. Let $S(u)$ denote the set of singular points. Suppose that $u \in C^1(\Omega)$ satisfies

$$(3.1) \quad \int_{\Omega} N^u \cdot \nabla \varphi + \int_{\Omega} H \varphi = 0$$

for all $\varphi \in C_0^\infty(\Omega)$ with $\vec{F} \in C^1(\Omega)$ and $S(u)$ being empty in Ω . Can we have higher order regularity for u , say, $u \in C^2$? This is impossible as shown by the following example. Let $u_g \equiv xy + g(y)$ where $g \in C^1 \setminus C^2$. Then u_g satisfies (3.1) with $H = 0$, $\vec{F} = (-y, x)$ on any nonsingular domain Ω for u_g . On the other hand, the characteristic and seed curves associated to u_g are all the same for different g 's. That is, the differentiability of g determines the differentiability of u_g , but does not affect the shape of characteristic and seed curves. On the other hand, we proved that θ is in fact C^1 smooth (hence $N \in C^1$, but not $u \in C^2$) (see Theorem 3.4 below). Before doing this we need to introduce some kind of special coordinates. These coordinates play an important role in understanding the local structure.

Definition 3.1. Let N be a C^0 vector field with $|N| \equiv 1$ on a domain $\Omega \subset \mathbb{R}^2$. A system of C^1 smooth local coordinates s, t is called a system of characteristic coordinates if s and t have the property that $\nabla s \parallel N^\perp$ and $\nabla t \parallel N$, i.e., ∇s and ∇t are parallel to $N^\perp$ and N , resp.. It follows that $\{t = \text{constants}\}$ are characteristic curves while $\{s = \text{constants}\}$ are seed curves.

Let $D \equiv |\nabla u + \vec{F}|$. For a plane C^1 vector field $\vec{F} = (F_1, F_2)$, we define

$$\text{rot} \vec{F} \equiv (F_2)_x - (F_1)_y.$$

Theorem 3.2 (existence of characteristic coordinates [7]). *Let $u : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a C^1 smooth solution to (3.1) (Ω being nonsingular) with $\vec{F} \in C^1(\Omega)$ and $H \in C^0(\Omega)$. Then for any point $p_0 \in \Omega$ there exist a neighborhood $\Omega' \subset \Omega$ and real functions $s, t \in C^1(\Omega')$ such that $\{t = \text{constants}\}$ and $\{s = \text{constants}\}$ are characteristic curves and seed curves, respectively. Moreover, there are positive functions $f, g \in C^0(\Omega')$ such that*

$$\nabla s = f N^\perp, \nabla t = g D N.$$

Also Nf and $N^\perp g$ exist and are continuous in Ω' . In fact, f and g satisfy the following equations

$$Nf + fH = 0, N^\perp g + \frac{(\text{rot} \vec{F})g}{D} = 0.$$

For a perhaps smaller neighborhood $\Omega'' \subset \Omega'$ of p_0 , the map $\Psi : (x, y) \in \Omega'' \rightarrow (s, t) \in \Psi(\Omega'')$ is a C^1 diffeomorphism such that

$$\Psi^* \left(\frac{ds^2}{f^2} + \frac{dt^2}{g^2 D^2} \right) = dx^2 + dy^2.$$

We remark that the existence of C^1 smooth s can be proved for N satisfying (1.4) (i.e., not defined by u) instead of (3.1) (see Theorem 4.1 in [7]).

Corollary 3.3 (Corollary C.1 in [7]). *Suppose we are in the situation of Theorem 3.2. Then θ is C^1 smooth in s and there holds*

$$(3.2) \quad \theta_s \equiv \frac{\partial \theta}{\partial s} = -\frac{H}{f}.$$

Since f is C^1 smooth in t , θ_s is also C^1 smooth in t if we assume that H has the same property according to (3.2). In fact, we proved that θ is C^1 smooth in t too, and hence $\theta \in C^1$. That is, θ gains one derivative.

Theorem 3.4 (regularity of θ [7]). *Let $u : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a C^1 smooth (weak) solution to $\operatorname{div} N^u = H$ in Ω (Ω being nonsingular) with $\vec{F} \in C^1(\Omega)$ and $H \in C^0(\Omega)$. Suppose $N(H)$ exists and is continuous. Then $\theta \in C^1$ and the characteristic and seed curves are C^2 smooth. Moreover, $N^\perp D$ exists and is continuous in Ω . In (s, t) coordinates near a given point as in Theorem 3.2, we have*

$$(3.3) \quad \theta_t \equiv \frac{\partial \theta}{\partial t} = \frac{\operatorname{rot} \vec{F}}{gD^2} - \frac{N^\perp(\log D)}{gD} = \frac{1}{gD^2}(\operatorname{rot} \vec{F} - N^\perp D).$$

In case $H = 0$ or a constant, we can prove Theorem 3.4 directly from the precise parametric expression of x or y . The situation $H = \text{a nonzero constant}$ arises from considering the boundary of an isoperimetric set in the 3-dimensional Heisenberg group H^1 . It has been conjectured (e.g., [12]) that an isoperimetric set in H^1 is congruent with a certain type of sphere. Based on the analysis of the singular set and characteristic curves ([4]) for a C^2 smooth surface, Ritoré and Rosales ([14]) proved such a conjecture for isoperimetric sets of class C^2 without any symmetry assumption. Later Monti and Rickly ([11]) obtained the same result for convex isoperimetric sets without regularity assumption.

According to (3.2), we have good control for θ along the characteristic curves, i.e. the s -direction. If the control for θ fails along a seed curve, i.e. t -direction, say, at some s_0 , then we showed that it fails also for s near s_0 . That is, the jump of a certain concerned quantity is kept in short " s -time" along the characteristic curves. This ends up to reach a contradiction. We borrowed the idea of conveying information along the characteristic curves from the study of hyperbolic P.D.E. (e.g., [8]).

4. REGULARITY OF THE p -AREA ELEMENT AND THE SINGULAR SET

Recall that $Ddx \wedge dy \equiv |\nabla u + \vec{F}| dx \wedge dy$ is the p (or H)-area element for a graph defined by u . For $u \in C^1$, we have $D \in C^0$. Suppose that $u \in C^1(\Omega)$ satisfies (3.1) with $\vec{F} \in C^1(\Omega)$ and $H \in C^1(\Omega)$, say (Ω being nonsingular). It follows from Theorem 3.4 that $N^\perp D$ exists and is continuous in Ω . Observe that $\frac{\partial}{\partial t} = \frac{1}{gD} N$ by Theorem 3.2. It follows from (3.2) and Theorem 3.2 that $(\theta_s)_t$ exists and is continuous. Then by a basic fact in calculus (see Lemma 5.4 in [7]) we conclude that $(\theta_t)_s$ exists and equals $(\theta_s)_t$ (in particular, $N^\perp N^\perp D$ exists and is continuous). Computing from (3.2) and (3.3), we also obtain an O.D.E. of second order for D (see below). Denote $N^\perp D$ and $N^\perp N^\perp D$ by D' and D'' , resp..

Theorem 4.1 (regularity of D and equation for D [5]). *Suppose we are in the situation of Theorem 3.4. Then D'' exists and is continuous in Ω . Moreover, D satisfies the following equation:*

$$(4.1) \quad \begin{aligned} DD'' &= 2(D' - \frac{\text{rot}\vec{F}}{2})(D' - \text{rot}\vec{F}) \\ &\quad + (N^\perp \text{rot}\vec{F})D + (NH + H^2)D^2. \end{aligned}$$

For $\vec{F} = (-y, x)$ (p -mean curvature case), we have $\text{rot}\vec{F} = 2$. In this case, (4.1) is reduced to

$$(4.2) \quad DD'' = 2(D' - 1)(D' - 2) + (NH + H^2)D^2.$$

If, furthermore, $H = 0$ in (4.2), the equation $DD'' = 2(D' - 1)(D' - 2)$ is integrable. Namely, we observed that

$$(4.3) \quad \begin{aligned} \frac{2D'}{D} &= \frac{D'D''}{(D' - 1)(D' - 2)} \\ &= \frac{-D''}{D' - 1} + \frac{2D''}{D' - 2} \\ &= -\frac{(D' - 1)'}{D' - 1} + \frac{2(D' - 2)'}{D' - 2} \end{aligned}$$

at points of a characteristic curve (line), where $D' \neq 1, 2$. Integrating (4.3), we obtained

$$|D' - 2|^2 = c|D' - 1|D^2$$

for some constant $c > 0$. It is not hard to see that when a nonsingular point tends to a singular point along a characteristic curve (line), either D' goes to 2 or D' goes to 1 ($D' \equiv 1$ in fact in this case). More generally, we had the following result.

Theorem 4.2 (behavior of D' near a singular point [5]). *Suppose we are in the situation of Theorem 3.4. Suppose $NH + H^2$ is bounded on a characteristic curve Γ approaching a singular point p . Then either D' converges to 1 or D' converges to 2 along Γ as points tend to p (where $D = 0$).*

Applying equation (4.2) and Theorem 4.2 to analyze the singular set of a C^1 smooth solution to (3.1) with $\vec{F} = (-y, x)$ is one of strategies in an on-going project. For instance, we can show the impossibility of connecting two distinct singular points (where $D = 0$) by a characteristic curve under a mild condition on H . For if possible, there exists a point on this characteristic curve with maximum D -value, and hence $D' = 0$. Evaluating equation (4.2) at this point gives

$$0 \geq D'' = \frac{4}{D} + (NH + H^2)D > 0$$

if $NH \geq 0$ (including the $H = \text{constant}$ case), a contradiction.

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