Imitation, Local Interaction, and Efficiency: Reappraisal

Hsiao-Chi Chen,∗ Yunshyong Chow, and Li-Chau Wu

Abstract

We revisit the model of Alós-Ferrer and Weidenholzer (2006) but under the assumption that risk-dominant equilibria are Pareto efficient. It is found that risk-dominant equilibria, non-risk-dominant equilibria and some non-monomorphic states can emerge in the long run when players interact with their immediate neighbors only.

Keywords: Coordination game, imitation, local interaction

JEL Classification: C72, C73, D83

Which of Nash equilibria in coordination game (hereafter CG) will emerge in the long run has been intensively studied in the literature of evolutionary games. Risk-dominant equilibria have been predicted by many works (e.g., Blume (1993, 1995), Ellison (1993), Kandori et al. (1993), Young (1993)). However, under imitation dynamics and local interaction, Alós-Ferrer and Weidenholzer (2006) show that risk-dominant equilibria survive uniquely in the long run when players interact with their immediate neighbors only. But payoff-dominant equilibria will be selected when players’ interactions are neither global nor limited to the immediate neighbors. In Alós-Ferrer and Weidenholzer (2006), it is assumed that risk-dominant equilibria are not Pareto efficient. In this note, we revisit Alós-Ferrer and Weidenholzer’s (2006) model but under the assumption that risk-dominant equilibria are Pareto efficient instead. We find that not only risk-dominant equilibria, but also non-risk-dominant equilibria and some non-monomorphic states can emerge in the long run when players interact with their immediate neighbors only. The intuition is simple. When risk-dominant equilibria are not Pareto-efficient, payoff-dominant-strategy takers can clump together and expand. Thus, it is costly for risk-dominant equilibria to jump out of their basin of attraction such that they can survive uniquely. In contrast, if risk-dominant equilibria are Pareto-efficient, the expansion force of non-risk-dominant strategy takers is weakened. Thus, it is less costly for risk-dominant equilibria to jump out of their basin of attraction so that some non-monomorphic states can survive as well.

∗E-mail address: hchen@mail.ntpu.edu.tw (H.-C. Chen), chow@math.sinica.edu.tw (Y.Chow), and wuli@nttu.edu.tw (L.-C. Wu).
Let $N = \{1, 2, \ldots, n\}$, $n \geq 5$, be the set of players. Players are assumed to sit sequentially and equally spaced around a circle. Each individual has exactly two neighbors. For $i \in N$, let $N_i = \{i - 1, i + 1\}$ be the set of player $i$’s neighbors. At each time period $t \in \{1, 2, 3, \ldots\}$, players meet with each of their two neighbors once to play $2 \times 2$ symmetric CG below.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$a, a$</td>
<td>$b, c$</td>
</tr>
<tr>
<td>B</td>
<td>$c, b$</td>
<td>$d, d$</td>
</tr>
</tbody>
</table>

where $a > c$ and $d > b$ such that both $(A, A)$ and $(B, B)$ are strict Nash equilibria. Alós-Ferrer and Weidenholzer (2006) further assume that $d > a$ and $a + b > c + d$ so that $(A, A)$ is risk dominant and $(B, B)$ is Pareto efficient. In contrast, we assume that $a \geq d$ and $a + b > c + d$ so that $(A, A)$ is both Pareto efficient and risk-dominant, and $(B, B)$ is non-risk-dominant. As in Alós-Ferrer and Weidenholzer (2006), we normalize the above game as

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$1, 1$</td>
<td>$0, \alpha$</td>
</tr>
<tr>
<td>B</td>
<td>$\alpha, 0$</td>
<td>$\beta, \beta$</td>
</tr>
</tbody>
</table>

where $\alpha = \frac{c - b}{a - b}$ and $\beta = \frac{d - b}{a - b}$. Hence,

$$\alpha < 1, \quad 0 < \beta \leq 1 \quad \text{and} \quad \alpha + \beta < 1. \quad (1)$$

Our state space $S \equiv \{A, B\}^N$ is a set containing all players’ strategy profiles. Denote $\vec{A} = (A, A, \ldots, A)$ and $\vec{B} = (B, B, \ldots, B)$ the states in which all players take strategies $A$ and $B$, respectively. At the beginning of each period, players’ actions and payoffs occurred (after revision) in the last period are observable to their neighbors. And players are assumed to imitate the strategies earning the highest total payoff among their neighbors and themselves. Given state $\vec{s} = (s_1, s_2, \ldots, s_n) \in S$, let $z_i(\vec{s})$ be player $i$’s total payoff after playing with his neighbors. Therefore,

$$z_i(\vec{s}) = \begin{cases} n_i^A(\vec{s}) & \text{if } s_i = A, \\
\alpha \cdot n_i^A(\vec{s}) + \beta \cdot (2 - n_i^A(\vec{s})) & \text{if } s_i = B, \end{cases}$$

where $n_i^A(\vec{s}) = |\{j \in N_i : s_j = A\}|$ is the number of player $i$’s neighbors taking strategy $A$. Accordingly, player $i$’s next-period rational choice, $r_i(\vec{s})$, will satisfy

$$r_i(\vec{s}) \in \arg \max_{j \in N_i \cup \{i\}} z_j(\vec{s}). \quad (2)$$

Whenever there is a tie, strategy $A$ or $B$ will be taken with strictly positive probability. Let $r(\vec{s}) = (r_1(\vec{s}), \ldots, r_n(\vec{s}))$ be any rational choice resulted from (2). At the end of each period, all players are allowed to revise their rational choices with probability
\( \epsilon > 0 \). For fixed \( \epsilon \), our dynamic system is a Markov chain on \( S \). Let \( Q_0 \) and \( Q_\epsilon \) be the transition probability matrices for the rational and revised processes respectively. Being a perturbation of \( Q_0 \), we have \( Q_\epsilon(s, \bar{u}) \approx \text{Constant} \cdot \epsilon^{U(s, \bar{u})} \) for any \( s, \bar{u} \in S \), where \( U(s, \bar{u}) = \min_{i \in \Omega} (d(\bar{r}(s), \bar{u}) \) and \( d(\bar{r}(s), \bar{u}) = \{ i \in N : r_i(s) \neq u_i \} \) counts the total number of player \( i \) who revises his rational choice \( \bar{r}_i(s) \) at state \( s \).

Because \( Q_\epsilon(s, \bar{u}) > 0 \) for all \( s, \bar{u} \in S \), revision makes our dynamic system \( \{X_t\} \) ergodic. Let \( \mu_* \) be the associated unique invariant distribution under \( Q_\epsilon \). We are interested in the limit probability distribution \( \mu_* = \lim_{\epsilon \to 0} \mu_\epsilon \) and its support \( S_* \equiv \{ s \in S : \mu_\epsilon(s) > 0 \} \). Each element in \( S_* \) is called a long run equilibrium (hereafter LRE). A non-monomorphic state consists of \( A \)-strings alternating with equal number of \( B \)-strings since all players sit around a circle as follows.

\[
\cdots A \cdots A B \cdots B A \cdots A B \cdots B \cdots B \cdots B \cdots B \cdots B \cdots \quad (3)
\]

where \( a_i, b_i \) are the lengths of its \( i \)-th \( A \)-string and \( B \)-string respectively. For positive integers \( m \) and \( p \), define \( M_{2m,p} = \{ s \in S : \text{all } a_i \geq m \text{ and } b_j = p \text{ in } (3) \} \) consisting of non-monomorphic states with all \( A \)-strings of length \( \geq m \) and all \( B \)-strings of length \( p \). The LREs of our dynamic system are given below.

**Theorem 1** Under the imitation rule (2), \( S_* = \{ \bar{A} \} \) except the following two cases:

(i) If \( \alpha > 1/2 \) then \( S_* = \{ \bar{B} \} \) if \( 5 \leq n \leq 6 \), \( S_* = \{ \bar{A}, \bar{B} \} \cup M_{2,3} \) if \( 7 \leq n \leq 12 \) and \( S_* = \{ \bar{A} \} \cup M_{3,1} \) if \( n \geq 13 \).

(ii) If \( \alpha = 1/2 \), then \( S_* = \{ \bar{A}, \bar{B} \} \) if \( 5 \leq n \leq 6 \), and \( S_* = \{ \bar{A} \} \) if \( n \geq 7 \).

Theorem 1 shows that the non-risk-dominant equilibrium \( \bar{B} \) and some non-monomorphic states can be LREs as well. For large population, risk-dominant equilibrium \( \bar{A} \) is in favor. This is no wonder. But it is not the unique LRE in \( S_* \) in Case (i) above. The reason is that in this case, strategy \( A \) is not strong as (4) and (6) below indicate a singleton \( A \)-player cannot hold under \( Q_0 \) and any \( A \)-player block of size \( \leq 2 \) will be eliminated in the next period when surrounded by singleton \( B \)-players. Yet, any \( A \)-player block of size \( \geq 3 \) can hold and expand till it is surrounded by singleton \( B \)-players. On the other hand, strategy \( B \) is not so strong as (5) below indicates any \( B \)-player block of size \( \geq 2 \) will shrink under \( Q_0 \) unless it is surrounded by singleton \( A \)-players, though singleton \( B \)-player can hold under \( Q_0 \). As a consequence, \( \{ \bar{A}, \bar{B} \} \cup M_{\geq 3,1} \) are all the ergodic states under \( Q_0 \) except \( 8|n \), in which case there are some extra ergodic states as shown in (7) below. Furthermore, (8) implies that \( \bar{A} \) and states in \( M_{\geq 3,1} \) are sort-of at equal potential as they can reach other by the minimum cost 1. Hence, states in \( M_{\geq 3,1} \) stand up with \( \bar{A} \) as LREs when the population size is \( \geq 13 \).

In conclusion, under imitation dynamics, Alós-Ferrer and Weidenholzer (2006) show that the selection of risk-dominant equilibria are sensitive to players’ interaction ways. Our results further demonstrate that the selection of risk-dominant equilibria is sensitive to games’ payoff structure as well.

**Proof of Theorem 1.** Only the case \( \alpha > 0 \) is considered, the remaining cases can
be treated similarly. Ellison’s (2000) Radius and Coradius Theorem is adopted when \(|S_0| = 1\), while Freidlin-Wentzell Method is used when \(S_r\) is complicated as in part (i). Since \(S_r \subseteq S_0\), the set of all ergodic states under \(Q_0\), the first step is to determine \(S_0\). Certainly, \(\{\hat{A}, \hat{B}\} \subseteq S_0\). Let \(M \overset{\text{def}}{=} S_0 \setminus \{\hat{A}, \hat{B}\}\) be the set of non-monomorphic ergodic states. Using \(\alpha > 0\) and (1), we have \(0 = \min\{1, 0, \alpha, \beta\} < 1 = \max\{1, 0, \alpha, \beta\}\). Figure A in the Appendix implies that with \(* = A\) or \(B\) independently,

\[
r(\ast, B, A, B, \ast) = B,
\]

which means that an isolated \(A\)-player would change to strategy \(B\) in next period under \(Q_0\). The following classifications are used to determine the other strategy-updating rules under \(Q_0\).

**Case (i)** \(\alpha > 1/2\). For \(2\alpha > 1\), we can use \(\alpha + \beta < 1\) in (1) to get \(\alpha > \beta\) and thus \(2\beta < 1\). Under \(Q_0\), we get from Figures B and BB in the Appendix that

\[
r(\ast, A, B, A, \ast) = B, \quad r(B, A, B, B, \ast) = B \quad \text{and} \quad r(A, A, B, B, \ast) = A,
\]

Certainly, \(r(\ast, B, B, A, B) = B\) and \(r(\ast, B, B, A, A) = A\) by symmetry. Eq (5) means that a \(B\)-player will keep his strategy \(B\) in next period iff he is isolated or confronted with an isolated \(A\)-player. As to a non-isolated \(A\)-player, Figure AA shows that

\[
r(A, B, A, A, B) = B \quad \text{and} \quad r(\ast, B, A, A, A) = r(B, B, A, A, B) = A.
\]

Using (4)-(6) and definition of \(S_0\), we have the following observations:

**O1** If \(\vec{s} \in M\) and \(\vec{t}\) is reachable from \(\vec{s}\) under \(Q_0\), then \(\vec{t}\) is ergodic as well.

**O2** Any \(A\)-string with length \(\geq 3\) can hold and grow until it is surrounded by singleton \(B\)-strings, which can hold under \(Q_0\). In particular, \(M_{\geq 3, 1} \subseteq M\).

**O3** A singleton \(A\)-string will be absorbed into a larger \(B\)-string under \(Q_0\) and the singleton \(A\) will not be recovered afterwards. Hence, all \(a_i \geq 2\) for any \(\vec{s} \in M\).

**O4** Since \(a_i \geq 2\) for any \(\vec{s} \in M\) by (O3), any \(B\)-string with length \(\geq 2\) in \(\vec{s} \in M\) will shrink until it disappears or becomes a singleton. Note that the length of any its neighboring \(A\)-strings does not decrease in the process. When encountered by some \(A\)-string of length 2, a singleton \(B\)-string could expand under \(Q_0\) to length 2 or 3 in next period. In view of (O1) and (O3), it will disappear in next period under \(Q_0\) in former case. In latter case it will shrink back to be singleton under \(Q_0\). Hence, \(b_i = 1\) or 3 for any \(\vec{s} \in M\).

**O5** Any \(A\)-string of length 2 will be eliminated in the next period when surrounded by singleton \(B\)-strings. In view of (O2) and (O4) above, we can deduce that if \(\vec{s} \in M\) has some \(A\)-string with length \(\geq 3\) then \(\vec{s} \in M_{\geq 3, 1}\).

**O6** Let \(M^* = M \setminus M_{\geq 3, 1}\). By (O5), all \(a_i = 2\) for \(\vec{s} \in M^*\). Using (O2) and (O4) again, two successive \(B\)-strings in \(\vec{s}\) must have length 1 and 3 respectively. Say, \(b_{i-1} = 3\) and \(b_i = 1\). Because \(a_{i-1} = a_{i+1} = 2\), the same argument shows \(b_{i-2} = 1\) and \(b_{i+1} = 3\). Repeating over and over, we conclude that if exists, any \(\vec{s} \in M^*\) must have the following periodic structure:

\[
\vec{s} = \overset{\text{repeat } \frac{3}{2}\text{times}}{\mathbf{A} \ A B B \ B \ A A B} \leftrightarrow \overset{\text{repeat } \frac{2}{3}\text{times}}{\mathbf{B} \ A A B \ A A B} = \vec{s}.
\]
Hereafter, \( \bar{u} \equiv \bar{v} \) means \( U(\bar{u}, \bar{v}) = c \) and \( \bar{u} \napp \bar{v} \) means \( U(\bar{v}, \bar{u}) = c \) as well. It follows from (7) that \( M^* \neq \emptyset \) iff \( 8|n \).

Next, we need to find \( v(s) \), the minimum cost among all spanning tree rooted at \( s \). Certainly, only \( s' \in S_0 \) needs to be considered. Write \( M_{\geq 3, 1} = \bigcup_{k \geq 1} M_k \), where \( k \) is the number of \( A \)-strings in the representation (3) for \( s' \).

**Step 1.** For convenience, define \( M_0 = \{ \bar{A} \} \). The following diagram shows any \( s' \in M_k \) with \( k \geq 1 \) can reach some state in \( M_{k-1} \) at the minimum cost 1 and vice versa:

\[
\begin{align*}
\cdots & B A \cdots A \; \overset{a_i \geq 2}{\rightarrow} \; B A \cdots A B \; \overset{1}{\rightarrow} \; \cdots \; B A \cdots A A \cdots A B \; \cdots.
\end{align*}
\]

Since \( |M_0| = 1 \), (8) implies that all states in \( M_{\geq 3, 1} \cup \{ \bar{A} \} \) can reach any fixed \( s' \in M_{\geq 3, 1} \cup \{ \bar{A} \} \) at cost 1 for each state. So the total cost is \( |M_{\geq 3, 1}| \).

**Step 2.** By (4)-(6), the most economical path for \( B \) to reach out and thus to \( M_{\geq 3, 1} \cup \{ \bar{A} \} \) is as follows. Depending on whether \( 2|n \) or not, we get as in (O4) that

\[
\begin{align*}
\bar{B} \; \overset{2}{\rightarrow} \; \cdots & \; \overset{0}{\rightarrow} \; \cdots B A A B B \; \overset{0}{\rightarrow} \; \cdots \; B A B A B \; \overset{0}{\rightarrow} \; \cdots \; \overset{0}{\rightarrow} \; \bar{A}
\end{align*}
\]

**Step 3.** When \( 8|n \), states \( s, s' \) in (7) form an irreducible class under \( Q_0 \) as \( Q_0(s, s') = Q_0(s', s) = 1 \). The following path shows an optimal way for the class to reach out:

\[
\begin{align*}
\overset{0}{\bar{s}} \; \overset{0}{\rightarrow} \; s = A A B \; \overset{1}{\rightarrow} \; B A A B B \; \overset{1}{\rightarrow} \; B A A \; \overset{0}{\rightarrow} \; \cdots \; \cdots \; \overset{0}{\rightarrow} \; \bar{A}
\end{align*}
\]

as the newly formed \( A \)-string will absorb its neighboring \( B \)'s till it reaches \( \bar{A} \). Since \( |M^*| = 8 \), all states in \( M^* \) can reach \( \bar{A} \) at a minimum total cost \( 8/2 = 4 \).

**Step 4.** In case \( 8|n \) so \( M^* \neq \emptyset \), (O2) indicates that the following path is optimal to reach \( M^* \) from \( M_{\geq 3, 1} \cup \{ \bar{A} \} \):

\[
\begin{align*}
M_4 \ni \underbrace{A A B \; \overset{0}{\rightarrow} \; A A B} \; \cdots & \; \overset{2}{\rightarrow} \; A A B \; \overset{0}{\rightarrow} \; A A B \; \cdots \; \overset{\text{repeat } 8 \text{ times}}{\rightarrow} \; \bar{w} \in M^*.
\end{align*}
\]

**Step 5.** We now find an optimal path from \( M_{\geq 3, 1} \cup \{ \bar{A} \} \) to \( \{ \bar{B} \} \). In order to avoid any \( A \)-string with length \( \geq 3 \) which can hold under \( Q_0 \) as shown in (O2), it saves to start from some \( s' \in M_{\geq 3, 1} \) which has as many \( B \)'s as possible. Moreover, it takes at least \( \ell \) revisions under \( Q_\epsilon \) to eliminate any \( A \)-string with length \( \geq 3\ell + 2 \) in \( s' \in M_{\geq 3, 1} \). Since any \( A \)-string in \( s' \in M_{\geq 3, 1} \) needs at least one revision to be eliminated under \( Q_\epsilon \), a little calculation shows that it is most economical to have the block \( B A A A A A \) duplicated in \( s' \in M_{\geq 3, 1} \) up to the maximum allowed \( \left\lfloor \frac{n}{6} \right\rfloor \) times and one revision is enough to eliminate those five \( A \)'s in any such block. As to the remained block of length \( r = n - 6 \left\lfloor \frac{n}{6} \right\rfloor \), an optimal choice for being both in \( M_{\geq 3, 1} \) and economical is \( \emptyset, A, AA, AAA, BAAAA \) and \( BAAAA \) for \( r = 0, 1, 2, 3, 4 \) and 5 respectively. Of course,
an extra mutation is needed if \( r \geq 1 \). Let \( \vec{s} \in M_{\geq 3.1} \) be such a state, an optimal path from \( M_{\geq 3.1} \cup \{\vec{A}\} \) to \( \vec{B} \) is as follows:

\[
\vec{s} \xrightarrow{\begin{bmatrix} \frac{n}{6} \end{bmatrix}} BAA \vec{B} AA \cdots (\emptyset, \vec{B}, \vec{B} A, \vec{B} AA, A \vec{B} AA, ABA \vec{B} A) \xrightarrow{0} \vec{B}.
\]

Since \( \frac{n}{4} \geq \frac{n}{6} \) for \( 8|n \), it is also an optimal path from \( M \cup \{\vec{A}\} \) to \( \{\vec{B}\} \).

All together, we have \( v(\{\vec{B}\}) = |M_{\geq 3.1}| + \left\lceil \frac{n}{6} \right\rceil + 4 \cdot \chi_{\{s|n\}} \) and \( v(\{\vec{s}\}) = |M_{\geq 3.1}| + 2 + 4 \cdot \chi_{\{s|n\}} \) for \( \vec{s} \in M_{\geq 3.1} \cup \{\vec{A}\} \). In case \( 8|n \), (9) shows that \( v(\{\vec{s}\}) = |M_{\geq 3.1}| + 2 + \left\lceil \frac{n}{4} \right\rceil + 3 \) for \( \vec{w} \in M^* \). Since \( S_* = \{\vec{s} \in S_0 : v(\vec{s}) = \min_{\vec{w} \in S_0} v(\vec{w})\} \) by Theorems 4.1 in Chen and Chow (2009), the conclusion follows by comparing \( \left\lceil \frac{n}{6} \right\rceil \) with 2. For instance, if \( n \geq 13 \) then \( \left\lceil \frac{n}{6} \right\rceil > 2 \) and thus, \( S_* = \{\vec{A}\} \cup M_{\geq 3.1} \).

**Case (ii) \( \alpha = 1/2 \).** All the updating rules in Case (i) remain valid except both the first rules in (5) and (6) are revised as follows:

\[
0 < \text{Prob}(r(A, A, B, A, *) = B) < 1 \quad \text{and} \quad 0 < \text{Prob}(r(A, B, A, A, B) = B) < 1.
\]

Consequently, we have that with positive probability under \( Q_0 \),

(O7) any \( A \)-string with length \( \geq 2 \) can hold and grow until it reaches \( \vec{A} \).

By (4) and (5), any non-monomorphic state without \( A \)-string of length \( \geq 2 \) belongs to the basin of attraction of \( \vec{B} \). Hence, \( S_0 = \{\vec{A}, \vec{B}\} \). In view of (4) and (O7), the following path shows \( v(\{\vec{A}\}) = 2 \):

\[
\vec{B} \xrightarrow{2} \cdots BB \vec{A} A B B \cdots \xrightarrow{0} \cdots BA \vec{A} A B \cdots \xrightarrow{0} \cdots \xrightarrow{0} \vec{A}.
\]

Because we still have \( \text{Prob}(r(A, A, A, A, B, *) = A) = 1 \), any \( A \)-string of length \( \geq 3 \) should be avoided in order to reach \( \vec{B} \) from \( \vec{A} \). Hence the following path is optimal:

\[
\vec{A} \xrightarrow{\begin{bmatrix} \frac{n}{3} \end{bmatrix}} AA \vec{B} \xrightarrow{\cdots (0, \vec{B}, A \vec{B})} \xrightarrow{0} \vec{B} \text{ and thus } v(\{\vec{B}\}) = \left\lceil \frac{n}{3} \right\rceil.
\]

The conclusion follows by comparing 2 with \( \left\lceil \frac{n}{3} \right\rceil \). Note that \( n \geq 5 \) by assumption.

**Case (iii) \( \alpha < 1/2 \) and \( \beta < 1/2 \).** While (4) remains valid, we have

\[
r(B, A, B, A, *) = B, \quad r(A, A, B, B, A) = A \quad \text{and} \quad r(A, A, B, B, B) = A
\]

under \( Q_0 \). Moreover, a non-isolated \( A \)-player and an isolated \( B \)-player would rationally update their strategies in next period respectively according to

\[
r(*, A, A, B, *) = A, \quad r(B, A, B, A, B) = B \quad \text{and} \quad r(A, A, B, A, *) = A.
\]
The first rule above and (11) imply that (O7) holds with probability 1 under $Q_0$. Therefore, $S_0 = \{\vec{A}, \vec{B}\}$ as the basin of attraction at $\vec{B}$ remain the same as in Case (ii). The path in (10) shows $CR(\{\vec{A}\}) = 2$. Because of (O7), any $A$-string of length $\geq 2$ has to be avoided in order to escape from $\vec{A}$. Hence, $R(\{\vec{A}\}) \geq 3 > CR(\{\vec{A}\})$. By Ellison’s Radius and Coradius Theorem, $S_* = \{\vec{A}\}$ as claimed in the theorem.

**Case (iv) $\alpha \leq 1/2 \leq \beta$.** Similar to Case (i), we have $\beta > \alpha$. All updating rules in Case (iii) remain valid, except the last rule in (11) needs to be modified. Depending on $\beta = 1/2$ or $\beta > 1/2$, we have $0 < \text{Prob}(r(A, A, B, B, B) = B) = 1 < 1$ or $\text{Prob}(r(A, A, B, B, B) = B) = 1$. As in Case (iii), we have $S_* = \{\vec{A}\}$ for $\beta = 1/2$.

In case $\beta > 1/2$, the rule $\text{Prob}(r(A, A, B, B, B) = B) = 1$ means that any $B$-string of length $\geq 3$ can hold when surrounded by $A$-strings of length $\geq 2$. In view of (O3) and (12), it is not difficult to show that $M = M_{2,3}$ implies that $M_{2,3} = \bigcup_{k \geq 1} M_k$ and $x_{n} = \{\vec{A}\}$ as in Case (i). By shrinking the $B$-strings at cost 1 each move, any $s \in M_k$ with $k \geq 1$ can move within $M_k$ and reach $M_{2,3}$ as shown below:

$$\cdots A \cdot AB \cdot B \cdots A \cdot A \cdots \rightarrow \cdots A \cdot AB \cdot BB \cdot A \cdot A \cdots \rightarrow \cdots A \cdot A \cdot \beta A \cdot A \cdot A \cdots \in M_{k-1}. $$

Then by the second rule in (11),

$$\cdots A \cdots AB \cdot B \cdot A \cdot A \cdots \rightarrow \cdots A \cdots AB \cdot BB \cdot A \cdot A \cdots \rightarrow \cdots A \cdots A \cdot A \cdot A \cdots \in M_{k-1}. $$

Together with (10), this implies $CR^*(\{\vec{A}\}) = 2$. Note that the state after $\vec{B}$ in (10) is in $M_1$. Since $R(\{\vec{A}\}) \geq 3$ by the first rule in (12), $S_* = \{\vec{A}\}$ as in Case (iii).

**Appendix A**

In Figures A, B, AA, BB, the state columns depict strategies adopted by five consecutive players $i - 2, i - 1, i, i + 1$ and $i + 2$.

**Figure A**

<table>
<thead>
<tr>
<th>State $\vec{s}$</th>
<th>Total payoffs for players $i-1$, $i$, and $i+1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cdots ABABA \cdots$</td>
<td>$z_{i-1}(\vec{s}) = 2\alpha$, $z_{i}(\vec{s}) = 0$, $z_{i+1}(\vec{s}) = 2\alpha$</td>
</tr>
<tr>
<td>$\cdots ABABB \cdots$</td>
<td>$z_{i-1}(\vec{s}) = 2\alpha$, $z_{i}(\vec{s}) = 0$, $z_{i+1}(\vec{s}) = \alpha + \beta$</td>
</tr>
<tr>
<td>$\cdots BBABB \cdots$</td>
<td>$z_{i-1}(\vec{s}) = \alpha + \beta$, $z_{i}(\vec{s}) = 0$, $z_{i+1}(\vec{s}) = \alpha + \beta$</td>
</tr>
</tbody>
</table>

**Figure B**

<table>
<thead>
<tr>
<th>State $\vec{s}$</th>
<th>Total payoffs for players $i-1$, $i$, and $i+1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cdots AABAA \cdots$</td>
<td>$z_{i-1}(\vec{s}) = 1$, $z_{i}(\vec{s}) = 2\alpha$, $z_{i+1}(\vec{s}) = 1$</td>
</tr>
<tr>
<td>$\cdots AABAB \cdots$</td>
<td>$z_{i-1}(\vec{s}) = 1$, $z_{i}(\vec{s}) = 2\alpha$, $z_{i+1}(\vec{s}) = 0$</td>
</tr>
<tr>
<td>$\cdots BABAB \cdots$</td>
<td>$z_{i-1}(\vec{s}) = 0$, $z_{i}(\vec{s}) = 2\alpha$, $z_{i+1}(\vec{s}) = 0$</td>
</tr>
</tbody>
</table>

6
**Figure AA**

<table>
<thead>
<tr>
<th>State $\vec{s}$</th>
<th>Total payoffs for players $i−1, i,$ and $i+1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ldots AAABA \ldots$</td>
<td>$z_{i−1}(\vec{s}) = 2$, $z_i(\vec{s}) = 1$, $z_{i+1}(\vec{s}) = 2\alpha$</td>
</tr>
<tr>
<td>$\ldots AAABB \ldots$</td>
<td>$z_{i−1}(\vec{s}) = 2$, $z_i(\vec{s}) = 1$, $z_{i+1}(\vec{s}) = \alpha + \beta$</td>
</tr>
<tr>
<td>$\ldots BAABA \ldots$</td>
<td>$z_{i−1}(\vec{s}) = 1$, $z_i(\vec{s}) = 1$, $z_{i+1}(\vec{s}) = 2\alpha$</td>
</tr>
<tr>
<td>$\ldots BAABB \ldots$</td>
<td>$z_{i−1}(\vec{s}) = 1$, $z_i(\vec{s}) = 1$, $z_{i+1}(\vec{s}) = \alpha + \beta$</td>
</tr>
</tbody>
</table>

**Figure BB**

<table>
<thead>
<tr>
<th>State $\vec{s}$</th>
<th>Total payoffs for players $i−1, i,$ and $i+1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ldots AABBA \ldots$</td>
<td>$z_{i−1}(\vec{s}) = 1$, $z_i(\vec{s}) = \alpha + \beta$, $z_{i+1}(\vec{s}) = \alpha + \beta$</td>
</tr>
<tr>
<td>$\ldots AABBB \ldots$</td>
<td>$z_{i−1}(\vec{s}) = 1$, $z_i(\vec{s}) = \alpha + \beta$, $z_{i+1}(\vec{s}) = \alpha + \beta$</td>
</tr>
<tr>
<td>$\ldots BABBA \ldots$</td>
<td>$z_{i−1}(\vec{s}) = 0$, $z_i(\vec{s}) = \alpha + \beta$, $z_{i+1}(\vec{s}) = \alpha + \beta$</td>
</tr>
<tr>
<td>$\ldots BABBB \ldots$</td>
<td>$z_{i−1}(\vec{s}) = 0$, $z_i(\vec{s}) = \alpha + \beta$, $z_{i+1}(\vec{s}) = 2\beta$</td>
</tr>
</tbody>
</table>

**References**


