On cross hedging, BSDE and Malliavin’s calculus*

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Chapter 1

Cross hedging: basic concepts, martingale optimality and BSDE


1.1 Basis risk: definition and examples

Basis = price of **hedged asset** - price of **hedging instrument**

Problem of basis risk: uncertainties of processes describing the evolution of prices of **asset** and **hedging instrument** not identical, only **highly correlated**

**Example 1:** weather derivatives

**hedged asset:** heating oil sales,  **hedging instrument:** HDD derivative

HDD derivative: contract paying a premium in case HDD above a critical threshold

**Example 2:** commodity markets

**hedged asset:** power spot price,  **hedging instrument:** power futures

Futures: contract to deliver amount of commodity at pre-fixed price

Hedge spot price fluctuations on time slots not coinciding with futures delivery dates
1.2 a toy example

**Aim:** show problems with hedging basis risk, given very high correlation

Diagram indicates high correlation between jet fuel spot price and heating oil spot price.
1.3 Cross hedging principle: correlation

simplest caricature of hedging problem:

static situation: \( Y \) hedged asset, \( X \) hedging instrument, both standard Gaussian, possibly strongly correlated

\[ \rho = E(XY) \quad \text{(correlation of } X \text{ and } Y) \]

decomposition of \( Y \) into part parallel to \( X \) and independent standard Gaussian part \( Z \):

\[ Z = \frac{1}{\sqrt{1 - \rho^2}}[Y - \rho X] \]

then

\[ \sqrt{1 - \rho^2}E(XZ) = E(XY) - \rho E(X^2) = 0, \]

hence \( Z \) independent of \( X \), and

\[ Y = \rho X + Y - \rho X = \rho X + \sqrt{1 - \rho^2}Z. \]
1.4 Cross hedging principle: mean variance

What quantity $a$ of position $X$ would agent hold to optimally hedge position $Y$? The quality of hedging: minimize quadratic error

$$E((Y-aX)^2) = E(((\rho-a)X+\sqrt{1-\rho^2}Z)^2) = (\rho-a)^2 + (1 - \rho^2)$$ minimal,

i.e.

$$a = \rho, \quad \text{Hedging error: } \sqrt{1 - \rho^2}Z$$

<table>
<thead>
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<th>$\rho$</th>
<th>$\sqrt{1 - \rho^2}$</th>
<th>% uncertainty hedged</th>
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<td>0.9</td>
<td>0.44</td>
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1.5 Mean variance hedging of jet fuel by heating oil

simple model for price processes (better: geometric BM) (J. C. Hull 2008)

jet fuel

\[ J_t = J_0 + \mu t + \sigma Y_t, \quad t \geq 0 \]

heating oil

\[ H_t = H_0 + \nu t + \beta X_t, \quad t \geq 0 \]

\[ \mu, \nu, \sigma, \beta \in \mathbb{R}, \quad X \text{ and } Y \text{ correlated BM, to be estimated from data;} \]

for \( \sigma \) and \( \beta \): sample standard deviation for \( x_1, \ldots, x_n \):

\[ \hat{s} = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \frac{1}{n} \sum_{j=1}^{n} x_j)^2}. \]

yields ML estimates \( \hat{\sigma} \approx 3,998, \hat{\beta} \approx 3,835; \)

For correlation between \( X \) and \( Y \): sample (Pearson) correlation coefficient for \( x_1, \ldots, x_n \):

\[ \hat{\rho} = \frac{n \sum_i x_i y_i - \sum_i x_i \sum_i y_i}{\sqrt{n \sum_i x_i^2 - (\sum_i x_i)^2} \sqrt{n \sum_i y_i^2 - (\sum_i y_i)^2}} \]

yields ML estimator for correlation \( \hat{\rho} = 0.897 \)
1.5 Mean variance hedging of jet fuel by heating oil

decomposition of the jet fuel price

\[ J_t = J_0 + \mu t + 0.897 \hat{\sigma} X_t + 0.443 \hat{\sigma} Z_t, \]

\( Z \) BM independent of \( X \).

Airline aims at hedging increasing fuel prices by buying heating oil futures; suppose \( K = E[H_1] = H_0 + \nu \) is price of heating oil futures at time 0; quantity of futures \( a \) the airline has to hold to minimize quadratic error determined by

\[ E((J_1 - J_0) - a(H_1 - K))^2 = \mu^2 + (0.897 \hat{\sigma} - a \hat{\beta})^2 + 0.321 \hat{\sigma}^2, \]

i.e. \( a = \frac{0.897 \hat{\sigma}}{\hat{\beta}} \).

Hedging error at time 1

\[ I_1 = 0.443 \hat{\sigma} Z_1. \]

Correlation between spot prices almost 0.9; only 56% of standard deviation of price change can be hedged!
1.6 Conclusions: mean variance hedging

- Even if correlation very high, hedging error large!

- correlation high: small change in correlation entails big change in percentage of basis risk relative to total risk

- correlation low: small change in correlation entails essentially no change in percentage of basis risk relative to total risk

- downside part of basis risk has to be properly respected
1.7 Our approach of utility based hedging

Aims:

- present a **purely probabilistic** approach, combining martingale optimality and BSDE
- determine **utility indifference price**
- determine **explicit optimal cross hedging strategy**, using Malliavin’s calculus
- clarify role of **correlation in hedging**
- describe reduction of risk by **cross hedging**
1.8 The financial market model

**Index process**, e.g. temperature, spot price

\[ dR_t = \sigma(t, R_t) dW_t + b(t, R_t) dt, \]

\( W \) \( d \)-dimensional Brownian motion, \( b : [0, T] \times \mathbb{R}^m \rightarrow \mathbb{R}^m \), \( \sigma : [0, T] \times \mathbb{R}^m \rightarrow \mathbb{R}^{m \times d} \) deterministic functions, globally Lipschitz and of sublinear growth, i.e.

there exists \( C \in \mathbb{R}_+ \) such that for all \( t \in [0, T] \) and \( x, x' \in \mathbb{R}^m \)

\[
\begin{align*}
|b(t, x) - b(t, x')| + |\sigma(t, x) - \sigma(t, x')| & \leq C|x - x'|, \\
|b(t, x)| + |\sigma(t, x)| & \leq C(1 + |x|).
\end{align*}
\]

**Markov process**, \( R^{t,r}_s \): start at \( t \) in \( r \)

Hedged asset: liability or derivative \( F(R_T) \), \( F : \mathbb{R}^m \rightarrow \mathbb{R} \) bounded
1.8 The financial market model

Hedging instrument: correlated financial market, \( k \) risky assets with price process:

\[
\frac{dS^i_t}{S^i_t} = \tau_i(t, R_t) dW_t + c_i(t, R_t) dt = \tau_i(t, R_t)[dW_t + \theta_t dt], \quad i = 1, \ldots, k,
\]

\[c : [0, T] \times \mathbb{R}^m \rightarrow \mathbb{R}^k, \quad \tau : [0, T] \times \mathbb{R}^m \rightarrow \mathbb{R}^{k \times d}, \quad \theta = \tau^*[\tau\tau^*]^{-1}c,\]

\( I_k \) the \( k \times k \) unit matrix

\[c \text{ is bounded,} \quad \varepsilon I_k \leq (\tau(t, r)\tau^*(t, r)) \leq K I_k \quad \text{for some } 0 < \varepsilon < K, \text{ all } (t, r) \in [0, T] \times \mathbb{R}^m.\]

To exclude arbitrage, assume \( d \geq k \). Correlation expressed by \( \tau \) and \( \sigma \).
1.9 The optimal investment problem

(N. El Karoui, R. Rouge ’00; J. Sekine ’02; J. Cvitanic, J. Karatzas ’92, Kramkov, Schachermayer ’99,...)

investment strategy $\lambda$: value of portfolio fraction invested in risky assets

wealth gain on $[0, s]$ (here $\tau_t = \tau(t, \cdot)$ etc.)

$$G_s^\lambda = v + \sum_{i=1}^{k} \int_0^s \lambda_i^u \frac{dS_i^u}{S_i^u} = v + \int_0^s \lambda_u \tau_u [dW_u + \theta_u du],$$

utility function: $U(x) = -e^{-\alpha x}$ ($0 < \alpha$ risk aversion); maximal expected utility from terminal wealth without and with derivative:

$$V^0(v) = \sup_{\lambda \in \tilde{C}} EU(G_T^\lambda), \quad V^F(v) = \sup_{\lambda \in \tilde{C}} EU(G_T^\lambda - F(R_T))$$

utility indifference $V^F(v^F) = V^0(v^0)$, $\lambda^0$ resp. $\lambda^F$ optimal strategies

utility indifference price derivative hedge

$$\Delta_v = v^F - v^0 = p = p(r) = p(t, r)$$

$$\Delta_\lambda = \lambda^F - \lambda^0$$
1.10 Optimization under non-convex constraints

Interpretation as maximization problem with convex constraints

\[ \tilde{C} \subset \mathbb{R}^k \text{ convex}, \quad \lambda \in \tilde{C} \]

\[ p_t = \lambda_t \tau_t \in C_t = \tilde{C} \tau_t \]

\[ C_t \text{ convex} \]

**Aim:** construct solution combining *martingale optimality* with BSDE, even for non-convex constraints

(N. El Karoui, R. Rouge ’00 for convex constraints)

\[ \tilde{C} \subset \mathbb{R}^k \text{ closed}, \quad \lambda \in \tilde{C} \]
1.10 Optimization under non-convex constraints

\[ F = F(R_T) \] hedged asset

**First formulation:**

Find

\[
V(v) = \sup_{\lambda \in \tilde{C}} E(U(G_T^\lambda - F)) = \sup_{\lambda \in \tilde{C}} E(U(v + \int_0^T \lambda_s \tau_s [dW_s + \theta_s ds] - F)).
\]

For simplicity:

\[
p = \lambda \tau, \\
C = \tilde{C} \tau.
\]

\[
G^p_t = v + \int_0^t p_s [dW_s + \theta_s ds], \quad t \in [0, T]
\]

**Second formulation:**

Find

\[
V(v) = \sup_{p \in C} E(U(G_T^p - F)) = \sup_{p \in C} E(-\exp(-\alpha(v + \int_0^T p_s [dW_s + \theta_s ds] - F))).
\]
1.11 Martingale optimality and BSDE

Idea: Construct family of processes $Q^{(p)}$ such that

Form 1

$Q^{(p)}_0 = \text{constant,}$  
$Q^{(p)}_T = -\exp(-\alpha (G^p_T - F)),$  
$Q^{(p)}$ supermartingale, $p \in C,$  
$Q^{(p*)}$ martingale, for (exactly) one $p^* \in C.$

Then

$E(-\exp(-\alpha [G^p_T - F])) = E(Q^{(p)}_T)$
$\leq E(Q^{(p)}_0)$
$= V(v)$
$= E(Q^{(p*)}_0)$
$= E(-\exp(-\alpha [G_T^{(p*)} - F])).$

Hence $p^*$ optimal strategy.
1.11 Martingale optimality and BSDE

BSDE

Given: $F$ (terminal variable), $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ measurable.

(BSDE) $Y_t = F - \int_t^T Z_s dW_s + \int_t^T f(s, Z_s) ds, \quad t \in [0, T].$

forward form $Y_t = Y_0 + \int_0^t Z_s dW_s - \int_0^t f(s, Z_s) ds, \quad Y_T = F, \quad t \in [0, T].$

Solution: pair $(Y, Z)$ satisfying (BSDE), describes stochastic dynamics which controls $Y$ into $F$

Simplest case: $f = 0$. Then $Y_t = E(F|\mathcal{F}_t) = E(F) + \int_0^t Z_s dW_s$ (martingale representation).

Come back to general problem of solving a BSDE later; first show what the consequences of this approach are.
1.11 Martingale optimality and BSDE

Introduction of BSDE into problem

Find generator \( f \) of BSDE

\[
Y_t = F - \int_t^T Z_s dW_s + \int_t^T f(s, Z_s) ds, \quad Y_T = F,
\]

such that with

\[
Q^{(p)}_t = -\exp(-\alpha [G^p_t - Y_t]), \quad t \in [0, T],
\]

we have

\[
Q^{(p)}_0 = -\exp(-\alpha (v - Y_0)) = \text{constant},
\]

(fulfilled)

form 2

\[
Q^{(p)}_T = -\exp(-\alpha (G^p_T - F)) \quad (\text{fulfilled})
\]

\[
Q^{(p)} \quad \text{supermartingale}, \quad p \in C,
\]

\[
Q^{(p^*)} \quad \text{martingale, for (exactly) one} \quad p^* \in C.
\]

This gives solution of valuation problem.
1.12 Construction of generator of BSDE

How to determine $f$:

Suppose $f$ generator of BSDE. Then by Ito’s formula

$$Q_t^{(p)} = -\exp(-\alpha [v + G_t^p - Y_t])$$

$$= Q_0^{(p)} + M_t^{(p)} + \int_0^t \alpha Q_s^{(p)} [-p_s\theta_s - f(s, Z_s) + \frac{\alpha}{2} (p_s - Z_s)^2] ds,$$

with a local martingale $M^{(p)}$. $Q^{(p)}$ satisfies (form 2) iff for

$$q(\cdot, p, z) = -f(\cdot, z) - p\theta + \frac{\alpha}{2} (p - z)^2, \quad p \in \mathcal{A}, z \in \mathbb{R},$$

we have

(form 3) $$q(\cdot, p, z) \geq 0, \quad p \in \mathcal{A} \quad \text{(supermartingale)}$$

$$q(\cdot, p^*, z) = 0, \quad \text{for (exactly) one } p^* \in \mathcal{A} \quad \text{(martingale)}.$$
1.12 Construction of generator of BSDE

Now

\[ q(\cdot, p, z) = -f(\cdot, z) - p\theta + \frac{\alpha}{2}(p - z)^2 \]
\[ = -f(\cdot, z) + \frac{\alpha}{2}(p - z)^2 - (p - z)\cdot\theta + \frac{1}{2\alpha}\theta^2 - z\theta - \frac{1}{2\alpha}\theta^2 \]
\[ = -f(\cdot, z) + \frac{\alpha}{2}[p - (z + \frac{1}{\alpha}\theta)]^2 - z\theta - \frac{1}{2\alpha}\theta^2. \]

Under non-convex constraint \( p \in C \):

\[ [p - (z + \frac{1}{\alpha}\theta)]^2 \geq d^2(C, z + \frac{1}{\alpha}\theta). \]

with equality for at least one possible choice of \( p^* \) due to closedness of \( C \).

Hence (form 3) is solved by the choice

form 4 \[ \frac{f(\cdot, z)}{p^*} = \frac{\alpha}{2}d^2(C, z + \frac{1}{\alpha}\theta) - z\cdot\theta - \frac{1}{2\alpha}\theta^2 \quad \text{(supermartingale)} \]

such that \( d(C, z + \frac{1}{\alpha}\theta) = d(p^*, z + \frac{1}{\alpha}\theta) \quad \text{(martingale)}. \)
1.12 Construction of generator of BSDE

Problem: Let
\[ \Pi_C(v) = \{ p \in \mathbb{R}^d : d(C, v) = d(p, v) \} \]. Find measurable selection \( p_t^* \) from \( \Pi_{C_t}(Z_t + \frac{1}{\alpha} \theta_t) \). Solved by classical measurable selection method.

Measurable selection \( p_t^* \) from \( \pi_{C_t}(z + \frac{1}{\alpha} \theta_t) \)
Chapter 2

BSDE: existence and uniqueness


2.1 Backward stochastic differential equations

Let \( T > 0 \), (time horizon), \( m \in \mathbb{N} \) (dimension), \((\Omega, \mathcal{F}, P)\) canonical \( d \)-dimensional Wiener space, \( W = (W^1, \cdots, W^d) \) canonical Wiener process, \((\mathcal{F}_t)_{t \geq 0}\) canonical filtration

\( L^2(\mathbb{R}^m) \) space of \( \mathbb{R}^m \)-valued \( \mathcal{F}_T \)-measurable random variables, normed by \( E(|X|^2)^{1/2} \);

\( H^2(\mathbb{R}^m) \) space of \((\mathcal{F}_t)_{0 \leq t \leq T}\)-adapted processes \( X : \Omega \times [0, T] \to \mathbb{R}^m \), normed by \( ||X||_2 = E(\int_0^T |X_t|^2 dt)^{1/2} \);

\( H^1(\mathbb{R}^m) \) space of \((\mathcal{F}_t)_{0 \leq t \leq T}\)-adapted processes \( X : \Omega \times [0, T] \to \mathbb{R}^m \), normed by \( ||X||_1 = E(\int_0^T |X_t|^2 dt)^{1/2} \);

for \( \beta > 0 \) and \( X \in H^2(\mathbb{R}^m) \) let

\[
||X||_{2,\beta} = E(\int_0^T e^{\beta t} |X_t|^2 dt),
\]

and \( H^{2,\beta}(\mathbb{R}^m) \) space \( H^2(\mathbb{R}^m) \) normed by \( || \cdot ||_{2,\beta} \).
2.1 Backward stochastic differential equations

describe hypotheses for parameters of BSDE

*terminal condition* $F$ belongs to $L^2(\mathbb{R}^m)$

generator depends on $(y, z)$:

$$f : \Omega \times \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^{d \times m} \rightarrow \mathbb{R}^m,$$

product measurable, adapted in time, satisfies

$$(H1) \ f(\cdot, 0, 0) \in H^2(\mathbb{R}^m),$$

$f$ uniformly Lipschitz, i.e. there is $C \in \mathbb{R}$ such that for any

$(y_1, z_1), (y_2, z_2) \in \mathbb{R}^m \times \mathbb{R}^{n \times m}, P \otimes \lambda$-a.e. $(\omega, t) \in \Omega \times \mathbb{R}_+$

$$(H2) \ |f(\omega, t, y_1, z_1) - f(\omega, t, y_2, z_2)| \leq C[|y_1 - y_2| + |z_1 - z_2|].$$

**Definition 1.** A pair of functions $(f, F)$ fulfilling, besides the mentioned measurement requirements, hypotheses $(H1), (H2)$ is said to be a standard parameter.
2.1 Backward stochastic differential equations

**Aim:** Given a standard parameter \((f, F)\), solve problem of finding a pair of \((\mathcal{F}_t)_{0 \leq t \leq T}\)-adapted processes \((Y_t, Z_t)_{0 \leq t \leq T}\) such that the **backward stochastic differential equation (BSDE)**

\[
(*) \quad dY_t = Z_t^*dW_t - f(\cdot, t, Y_t, Z_t)dt, \quad Y_T = F,
\]

holds.

Use contraction argument on suitable Banach space; this needs *a priori* inequalities.
2.2 BSDE: a priori inequalities

Lemma 1. For $i = 1, 2$ let $(f^i, F^i)$ be standard parameters, $(Y^i, Z^i) \in H^2(\mathbb{R}^m) \times H^2(\mathbb{R}^{d \times m})$ solutions of (*) for $(f^i, F^i)$, where for $z \in \mathbb{R}^{d \times m}$ we denote $|z| = (tr(zz^*))^{\frac{1}{2}}$. Let $C$ be a Lipschitz constant for $f^1$. Define for $0 \leq t \leq T$

$$\delta Y_t = Y^1_t - Y^2_t,$$

$$\delta_2 f_t = f^1(\cdot, t, Y^2_t, Z^2_t) - f^2(\cdot, t, Y^2_t, Z^2_t).$$

Then for any triple $(\lambda, \mu, \beta)$ with $\lambda > 0$, $\lambda^2 > C$, $\beta \geq C(2 + \lambda^2) + \mu^2$ we have

$$\|\delta Y\|_{2,\beta}^2 \leq T[e^{\beta T} E(|\delta Y_T|^2) + \frac{1}{\mu^2} \|\delta_2 f\|_{2,\beta}^2],$$

$$\|\delta Z\|_{2,\beta}^2 \leq \frac{\lambda^2}{\lambda^2 - C}[e^{\beta T} E(|\delta Y_T|^2) + \frac{1}{\mu^2} \|\delta_2 f\|_{2,\beta}^2].$$
2.2 BSDE: proof of a priori inequalities

Proof

1. \((Y, Z) \in H^2(\mathbb{R}^m) \times H^2(\mathbb{R}^{d \times m})\) solution of (*), standard parameters \((F, f)\), i.e. for \(0 \leq t \leq T\)

\[
(*) \quad Y_t = F - \int_t^T Z^*_s dW_s + \int_t^T f(\cdot, s, Y_s, Z_s) ds.
\]

We show:

\[
\sup_{0 \leq t \leq T} |Y_t| \in L^2(\mathbb{R}^m).
\]

By (*) we have

\[
\sup_{0 \leq t \leq T} |Y_t| \leq |F| + \int_0^T |f(\cdot, s, Y_s, Z_s)| ds + \sup_{0 \leq t \leq T} \left| \int_t^T Z^*_s dW_s \right|,
\]

hence by Doob’s inequality

\[
E(\sup_{0 \leq t \leq T} |\int_t^T Z^*_s dW_s|^2) \leq 4E(\sup_{0 \leq t \leq T} |\int_0^t Z^*_s dW_s|^2) \leq 8E(\int_0^T |Z_s|^2 ds).
\]
2.2 BSDE: proof of a priori inequalities

(H1) and (H2) guarantee $|F| + \int_0^T |f(\cdot, s, Y_s, Z_s)|ds \in L^2(\mathbb{R})$. So

$$E\left( \sup_{0 \leq t \leq T} |Y_t|^2 \right) < \infty.$$
2.2 BSDE: proof of a priori inequalities

2. Now derive auxiliary equation; apply Itô’s formula to semimartingale \((e^{\beta s}|\delta Y_s|^2)_{0 \leq s \leq T}\); obtain for \(0 \leq t \leq T\)

\[
e^{\beta T}|\delta Y_T|^2 - e^{\beta t}|\delta Y_t|^2 = \beta \int_t^T e^{\beta s}|\delta Y_s|^2 ds + 2 \int_t^T e^{\beta s} \langle \delta Y_s, f^1(\cdot, s, Y_s^1, Z_s^1) \rangle ds - f^2(\cdot, s, Y_s^2, Z_s^2) ds - \beta \int_t^T e^{\beta s} \langle \delta Y_s, \delta Z_s^* dW_s \rangle ds + \int_t^T e^{\beta s}|\delta Z_s|^2 ds.
\]

By reordering the terms in the equation we obtain

\[
e^{\beta t}|\delta Y_t|^2 + \beta \int_t^T e^{\beta s}|\delta Y_s|^2 ds + \int_t^T e^{\beta s}|\delta Z_s|^2 ds = e^{\beta T}|\delta Y_T|^2 + 2 \int_t^T e^{\beta s} \langle \delta Y_s, \delta Z_s^* dW_s \rangle ds - f^2(\cdot, s, Y_s^2, Z_s^2) ds.
\]
2.2 BSDE: proof of a priori inequalities

3. We prove for $0 \leq t \leq T$:

$$E(e^{\beta t} | \delta Y_t|^2) \leq E(e^{\beta T} | \delta Y_T|^2) + \frac{1}{\mu^2} E(\int_t^T e^{\beta s} |\delta_2 f_s|^2 ds).$$

take expectations on both sides of equation from 2.:

$$E(e^{\beta t} | \delta Y_t|^2) + \beta E(\int_t^T e^{\beta s} |\delta Y_s|^2 ds) + E(\int_t^T e^{\beta s} |\delta Z_s|^2 ds) = E(e^{\beta T} | \delta Y_T|^2)$$

$$+ 2E(\int_t^T e^{\beta s} \langle \delta Y_s, f^1(\cdot, s, Y_s^1, Z_s^1) - f^2(\cdot, s, Y_s^2, Z_s^2) \rangle ds).$$

Now by our assumptions for $0 \leq s \leq T$

$$|f^1(\cdot, s, Y_s^1, Z_s^1) - f^2(\cdot, s, Y_s^2, Z_s^2)| \leq |f^1(\cdot, s, Y_s^1, Z_s^1) - f^1(\cdot, s, Y_s^2, Z_s^2)| + |\delta_2 f_s|$$

$$\leq C[|\delta_s Y| + |\delta_s Z|] + |\delta_2 f_s|. $$
2.2 BSDE: proof of a priori inequalities

The latter implies

\[ \int_t^T E(2e^{\beta s}|\langle \delta Y_s, f^1(\cdot, s, Y^1_s, Z^1_s) - f^2(\cdot, s, Y^2_s, Z^2_s)\rangle|) ds \]

\[ \leq \int_t^T 2e^{\beta s} E(|\delta Y_s| [C(|\delta Y|) + |\delta Z|] + |\delta f_s|) ds \]

\[ = \int_t^T 2e^{\beta s} [CE(|\delta Y_s|^2) + E(|\delta Y| (C|\delta Z|) + |\delta f_s|)] ds. \]

Now for \( C, y, z, t > 0 \) with \( \mu, \lambda > 0 \)

\[ 2y(Cz + t) = 2Cyz + 2yt \]

\[ \leq C[(y\lambda)^2 + (\frac{z}{\lambda})^2] + (y\mu)^2 + (\frac{t}{\mu})^2 \]

\[ = C(\frac{z}{\lambda})^2 + (\frac{t}{\mu})^2 + y^2(\mu^2 + C\lambda^2). \]
2.2 BSDE: proof of a priori inequalities

With this estimate last term inequality further:

\[ \int_t^T 2e^{\beta s}[C E(|\delta Y_s|^2) + E(|\delta_s Y |(C|\delta_s Z|) + |\delta_2 f_s|)] ds \]

\[ \leq \int_t^T e^{\beta s}[2CE(|\delta Y_s|^2) + \frac{C}{\lambda^2} E(|\delta_s Z|^2) \]

\[ + \frac{1}{\mu} E(|\delta_2 f_s|^2) + (\mu^2 + C\lambda^2) E(|\delta_s Y|^2)] ds \]

\[ = \int_t^T e^{\beta s}[(\mu^2 + C(2 + \lambda^2)) E(|\delta Y_s|^2) \]

\[ + \frac{C}{\lambda^2} E(|\delta_s Z|^2) + \frac{1}{\mu} E(|\delta_2 f_s|^2)] ds. \]
### 2.2 BSDE: proof of a priori inequalities

Summarizing, using assumptions on parameters

\[
\begin{align*}
(**) \quad E(e^{\beta t}|\delta Y_t|^2) & \leq E\left(\int_t^T e^{\beta s}|\delta Y_s|^2 ds\right)\left[-\beta + C(2 + \lambda^2) + \mu^2\right] \\
& + E\left(\int_t^T e^{\beta s}|\delta Z_s|^2 ds\right)\left[\frac{C}{\lambda^2} - 1\right] + E(e^{\beta T}|\delta Y_T|^2) \\
& + \frac{1}{\mu^2} E\left(\int_t^T e^{\beta s}|\delta_2 f_s|^2 ds\right) \\
& \leq E(e^{\beta T}|\delta Y_T|^2) + \frac{1}{\mu^2} E\left(\int_t^T e^{\beta s}|\delta_2 f_s|^2 ds\right).
\end{align*}
\]

This is the claimed inequality.

4. To obtain the **first inequality**, integrate the inequality resulting from 3. in \( t \in [0,T] \).

5. The **second inequality follows from (**) by taking the second term from the right hand side to the left. •
2.3 BSDE: existence and uniqueness

**Theorem 1.** Let \((f, F)\) be standard parameters. Then there exists a uniquely determined pair \((Y, Z) \in H^2(\mathbb{R}^m) \times H^2(\mathbb{R}^{d \times m})\) satisfying

\[
Y_t = F - \int_t^T Z_s^* dW_s + \int_t^T f(\cdot, s, Y_s, Z_s) ds, \quad 0 \leq t \leq T.
\]

**Proof**

Consider

\[
\Gamma : H^{2,\beta}(\mathbb{R}^m) \times H^{2,\beta}(\mathbb{R}^{d \times m}) \to H^{2,\beta}(\mathbb{R}^m) \times H^{2,\beta}(\mathbb{R}^{d \times m}), (y, z) \mapsto (Y, Z),
\]

where \((Y, Z)\) is solution of BSDE

\[
(*) \quad Y_t = F - \int_t^T Z_s^* dW_s + \int_t^T f(\cdot, s, y_s, z_s) ds, \quad 0 \leq t \leq T.
\]
2.3 BSDE: proof of existence and uniqueness

1. We prove: \((Y, Z)\) is well defined. By assumptions

\[
F + \int_t^T f(\cdot, s, y_s, z_s) ds \in L^2(\Omega), \quad 0 \leq t \leq T.
\]

Therefore

\[
M_t = E(F + \int_0^T f(\cdot, s, y_s, z_s) ds | \mathcal{F}_t), \quad 0 \leq t \leq T,
\]

is a martingale. \(M\) possesses continuous version and is square integrable. Hence martingale representation provides (unique) \(Z \in H^2(\mathbb{R}^{d \times m})\) such that

\[
M_t = M_0 + \int_0^t Z_s^* dW_s, \quad 0 \leq t \leq T.
\]

Let now

\[
Y_t = M_t - \int_0^t f(\cdot, s, y_s, z_s) ds.
\]
2.3 BSDE: proof of existence and uniqueness

Then $Y$ is square integrable, and

$$Y_t = E(F + \int_t^T f(\cdot, s, y_s, z_s) ds | \mathcal{F}_t), 0 \leq t \leq T.$$ 

Hence

$$Y_T = F = M_0 + \int_0^T Z^*_s dW_s - \int_0^T f(\cdot, s, y_s, z_s) ds,$$

and thus for $0 \leq t \leq T$

$$Y_t = F - M_0 - \int_0^T Z^*_s dW_s + \int_0^T f(\cdot, s, y_s, z_s) ds + M_0 + \int_0^t Z^*_s dW_s - \int_0^t f(\cdot, s, y_s, z_s) ds$$

$$= F - \int_t^T Z^*_s dW_s + \int_t^T f(\cdot, s, Y_s, Z_s) ds.$$
2.3 BSDE: proof of existence and uniqueness

2. We prove: For $\beta > 2(1 + T)C$ the mapping $\Gamma$ is a contraction.

For this purpose, let $(y^1, z^1), (y^2, z^2) \in H^{2,\beta}(\mathbb{R}^m) \times H^{2,\beta}(\mathbb{R}^{d \times m}),$
$(Y^1, Z^1), (Y^2, Z^2)$ corresponding solutions of (*) according to 1. Apply Lemma 1 with $C = 0, \beta = \mu^2,$ and $f^i = f(\cdot, y^i, z^i).$ Then

$$||\delta Y||_{2,\beta} \leq \frac{T}{\beta} E(\int_0^T e^{\beta s}|f(\cdot, s, y^1_s, z^1_s) - f(\cdot, s, y^2_s, z^2_s)|^2 ds),$$

$$||\delta Z||_{2,\beta} \leq \frac{1}{\beta} E(\int_0^T e^{\beta s}|f(\cdot, s, y^1_s, z^1_s) - f(\cdot, s, y^2_s, z^2_s)|^2 ds).$$

Since $f$ is Lipschitz continuous, we further obtain

$$||\delta Y||_{2,\beta} \leq \frac{2TC}{\beta} ||\delta y||_{2,\beta} + ||\delta z||_{2,\beta},$$

$$||\delta Z||_{2,\beta} \leq \frac{2C}{\beta} ||\delta y||_{2,\beta} + ||\delta z||_{2,\beta}.$$
2.3 BSDE: proof of existence and uniqueness

Summarizing

\[(**): \quad ||\delta Y||_{2,\beta} + ||\delta Z||_{2,\beta} \leq \frac{2C(T+1)}{\beta} [||\delta y||_{2,\beta} + ||\delta z||_{2,\beta}].\]

By choice of \(\beta\), \(\Gamma\) is a contraction.

3. Now let \((\overline{Y}, \overline{Z})\) be the fixed point of \(\Gamma\), which exists due to 2. Let

\[Y_t = E(F + \int_t^T f(\cdot, s, \overline{Y}_s, \overline{Z}_s) ds | \mathcal{F}_t), \quad 0 \leq t \leq T.\]

Then \(Y\) is continuous and \(P\)-a.s. identical to \(\overline{Y}\). Then \((Y, \overline{Z})\) is solution of BSDE.

4. Uniqueness follows from the contraction property of \(\Gamma\) and the uniqueness of the fixed point. •
2.4 BSDE: recursion for solution

**Corollary 1.** Let $\beta > 2(1 + T)C$, $((Y^k, Z^k))_{k \geq 0}$ given by $Y^0 = Z^0 = 0$,

$$Y^{k+1}_t = F - \int_t^T (Z^{k+1}_s)^* dW_s + \int_t^T f(\cdot, s, Y^k_s, Z^k_s) ds$$

according to preceding proof. Then $((Y^k, Z^k))_{k \geq 0}$ converges in $H^{2,\beta}(\mathbb{R}^m) \times H^{2,\beta}(\mathbb{R}^{d \times m})$ to unique solution $(Y, Z)$ of (*).

**Proof**

The inequality (**) in the proof of Theorem 1 recursively yields

$$\|Y^{k+1} - Y^k\|_{2,\beta} + \|Z^{k+1} - Z^k\|_{2,\beta} \leq \epsilon^k [\|Y^1 - Y^0\|_{2,\beta} + \|Z^1 - Z^0\|_{2,\beta}],$$

with $\epsilon = \frac{2C(T+1)}{\beta} < 1$. This implies

$$\sum_{k \in \mathbb{N}} [\|Y^{k+1} - Y^k\|_{2,\beta} + \|Z^{k+1} - Z^k\|_{2,\beta}] < \infty.$$

Now a standard argument applies. •
Chapter 3

Measure solutions of BSDE


3.1 The concept of measure solution

idea:

hedging
historical measure $P$

martingale measure $Q$

BSDE
compute $(Y, Z)$ from $f, F$ martingale representation of $F$ ($f = 0$)

simplify: $m = d = 1$ $|f(s, y, z)| \leq cz^2, s \in [0, T], z \in \mathbb{R}$

with $g(s, y, z) = \frac{f(s, y, z)}{z}$ BSDE takes form

\begin{align}
(1) \quad Y_t &= F - \int_t^T Z_s [dW_s - g(s, Y_s, Z_s)ds] \\
Q &= \exp\left( \int_0^T g(s, Y_s, Z_s) dW_s - \frac{1}{2} \int_0^T g(s, Y_s, Z_s)^2 ds \right) \cdot P
\end{align}

$W^Q = W - \int_0^\cdot g(s, Y_s, Z_s)ds$ is a $Q$-Brownian motion, hence

\begin{align}
Y_t &= E^Q(F|\mathcal{F}_t) = E^Q(F) + \int_0^t Z_s dW^Q_s
\end{align}
3.1 The concept of measure solution

**Pb:** Starting from a measure $Q \sim P$, how can one state these findings without reference to a solution $(Y, Z)$?

Let

$$R = \frac{dQ}{dP}, \quad R_t = E(R|\mathcal{F}_t), \quad 0 \leq t \leq T.$$ 

Then by martingale representation

$$R = 1 + \int_0^t \eta_s dW_s$$

for some adapted $\eta$ a.s. square integrable on $[0, T]$. By positivity of $R$, we also have that $\zeta = \frac{\eta}{R}$ is a.s. square integrable on $[0, T]$. Moreover,

$$R = 1 + \int_0^t R_s \zeta_s dW_s,$$

hence

$$R = \exp(\int_0^t \zeta_s dW_s - \int_0^t |\zeta_s|^2 ds).$$
3.1 The concept of measure solution

Now let

\[ W^Q = W - \int_0^\cdot \zeta_s ds. \]

Then \( W^Q \) is \( Q \)-Brownian motion and by preservation of martingale representation property we have for square integrable \( \mathcal{F}_T \)-measurable \( F \)

\[ Y = E^Q(F|\mathcal{F}) = E^Q(F) + \int_0^\cdot Z_s dW^Q_s \]

with some unique adapted square integrable \( Z \).

**Definition 2.** Let \( g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be adapted in \((\omega, t)\), \( Q \sim P \), and \( F \in L^2(\mathcal{F}_T, Q) \), \( \zeta, Y, Z \) as above. We call \( Q \) measure solution related to \((g, F)\) if

\[ \zeta = g(\cdot, \cdot, Y, Z) \quad P \otimes \lambda - \text{a.e.} \]
3.1 The concept of measure solution

In this case

\[ Y = E^Q(F) + \int_0^\cdot Z_s dW_s^Q \]

\[ = E^Q(F) + \int_0^\cdot Z_s dW_s - \int_0^\cdot Z_s \zeta_s ds \]

\[ = E^Q(F) + \int_0^\cdot Z_s dW_s - \int_0^\cdot Z_s g(\cdot, s, Y_s, Z_s) ds. \]

So \((Y, Z)\) is solution in classical sense of BSDE with generator \((f, F)\), where

\[ f(\cdot, t, y, z) = zg(\cdot, t, y, z). \]
3.2 Strong solutions induce measure solutions

**Problem 1:** Do strong solutions induce measure solutions? Assume from now on

\[ f \text{ locally Lipschitz, } |f(s, y, z)| \leq c|z|^2 \]

Assume (1) possesses strong solution \((Y, Z)\)

Recall

\[ |g(s, y, z)| \leq c|z|, \quad s \in [0, T], z \in \mathbb{R} \]

\[ M = \int_0^\cdot g(s, Y_s, Z_s) \, dW_s, \quad \tau_n = \inf\{t \geq 0 : \langle M \rangle_t \geq n\} \]

**Crucial question:** \(V = \exp(M - \frac{1}{2} \langle M \rangle)\) martingale?

\[ Q^n = V_T|\mathcal{F}_{\tau_n} \cdot P, \quad \tilde{W}^n = W - \int_0^{\tau_n \wedge \cdot} g(s, Y_s, Z_s) \, ds \]

Criterion for martingale property of \(V\) (Liptser, Shiryaev ’77; Heyde, Wong ’04):

\[ Q^n(\tau_n < T) \to 0 \quad (n \to \infty) \]
3.2 Strong solutions induce measure solutions

Let $F$ be bounded. Recall $Y^n = E_{Q^n}^n(F|\mathcal{F}_{\tau^n} \wedge \cdot)$

\[ Q^n(\tau_n < T) \leq \frac{1}{n} E_{Q^n}^n \left( \int_0^{\tau^n} g(s, Y_s, Z_s)^2 ds \right) \]
\[ \leq c \frac{1}{n} E_{Q^n}^n \left( \int_0^{\tau^n} (Z_s)^2 ds \right) \]
\[ = c \frac{1}{n} E_{Q^n}^n \left( | \int_0^{\tau^n} Z_s d\tilde{W}_s^n |^2 \right) \]
\[ = c \frac{1}{n} E_{Q^n}^n (| Y^n_{\tau_n} - Y^n_0 |^2) \]
\[ \leq \frac{4}{n} c ||F||_{\infty}^2 \to 0 \ (n \to \infty). \]

**Theorem 2**

Assume $f$ locally Lipschitz, $|f(s, y, z)| \leq cz^2$ and $F$ is bounded. Then $(Y, Z)$ is a classical solution for $(f, F)$ if and only if there exists a probability measure $Q$, equivalent to $P$, such that $Q$ is a measure solution for $(g, F')$ of (1).
3.3 An algorithm for Lipschitz generators

**Problem 2:** How to find a measure solution without knowledge of a strong one?

Consider Lipschitz generator. For simplicity $f(0, \cdot) = 0$, $F$ bounded. Assume

$$|f(s, y, z) - f(s, y', z')| \leq C[|y - y'| + |z - z'|], \quad s \in [0, T], (y, z, y', z') \in \mathbb{R}^4.$$ 

Then $g(s, y, z) = \frac{f(s, y, z)}{z}$ is bounded by $C$.

Natural to introduce $BMO$ spaces: let $BMO(P)$ be space of progressive processes normed by

$$\|Z\|_{BMO} = \inf\{C : C > 0, E(\int_t^T |Z_s|^2 ds |\mathcal{F}_t)^{\frac{1}{2}} \leq C\}.$$

for $\beta > 0$ let $BMO_\beta(P)$ be space of progressive processes normed by

$$\|Z\|_{BMO_\beta} = \inf\{C : C > 0, E(\int_t^T e^{\beta s} |Z_s|^2 ds |\mathcal{F}_t)^{\frac{1}{2}} \leq C\}.$$

All these norms are equivalent.
3.3 An algorithm for Lipschitz generators

Algorithm:
Recursion: $Y^0 = Z^0 = 0, \zeta^1 = g(\cdot, Y^0, Z^0)$.

\[ Q^1 = \exp \left[ \int_0^T \zeta^1_s dW_s - \frac{1}{2} \int_0^T (\zeta^1_s)^2 ds \right] \cdot P, \quad W^{Q^1} = W - \int_0^T \zeta^1_s ds, \]

\[ Y^1 = E(F|\mathcal{F}_.) = E(F) + \int_0^T Z^1_s dW^{Q^1}_s. \]

Assume $Y^n, Z^n$ given, $\zeta^{n+1} = g(\cdot, Y^n, Z^n)$. Define

\[ Q^{n+1} = \exp \left[ \int_0^T \zeta^{n+1}_s dW_s - \int_0^T (\zeta^{n+1}_s)^2 ds \right] \cdot P, \quad W^{Q^{n+1}} = W - \int_0^T \zeta^{n+1}_s ds, \]

\[ Y^{n+1} = E^{Q^{n+1}}(F|\mathcal{F}_.) = E^{Q^{n+1}}(F) + \int_0^T Z^{n+1}_s dW^{Q^{n+1}}_s. \]

By boundedness of $g$, $(Q^n)_{n \in \mathbb{N}}$ well defined and measures are equivalent with $P$. 

3.3 An algorithm for Lipschitz generators

Lemma 2

There exist $\beta > 0$, $\alpha \in [0, 1]$ such that for $n \in \mathbb{N}$

$$
||Y^{n+1} - Y^n||_{BMO^\beta} + ||Z^{n+1} - Z^n||_{BMO^\beta}
\leq \alpha (||Y^n - Y^{n-1}||_{BMO^\beta} + ||Z^n - Z^{n-1}||_{BMO^\beta}).
$$

Proof:

Apply Itô’s formula to $(e^{\beta s}|Y_s|^2)_{t \leq s \leq T}$ with $Y_t = \int_t^T X_s ds + \int_t^T Z_s dW_s$, $Y_T = 0$:

$$
e^{\beta s}|Y_s|^2 = e^{\beta t}|Y_t|^2 + \int_t^s \beta e^{\beta r}|Y_r|^2 dr
- \int_t^s 2 e^{\beta r} Y_r X_r dr
- \int_t^s 2 e^{\beta r} Y_r Z_r dW_r
+ \int_t^s e^{\beta r}|Z_r|^2 dr.
$$
3.3 An algorithm for Lipschitz generators

Hence for $s = T$

$$\int_t^T \beta e^{\beta r} |Y_r|^2 ds + \int_t^T e^{\beta r} |Z_r|^2 dr \leq 2 \left[ \int_t^T e^{\beta r} Y_r X_r dr + \int_t^T e^{\beta r} Y_r Z_r dW_r \right].$$

Use Young’s inequality for $x, y \in \mathbb{R}, \lambda > 0$:

$$2xy \leq \lambda y^2 + \frac{1}{\lambda} x^2.$$

Therefore

$$\beta E(\int_t^T e^{\beta r} |Y_r|^2 ds \mid \mathcal{F}_t) + E(\int_t^T e^{\beta r} |Z_r|^2 dr \mid \mathcal{F}_t) \leq E(\int_t^T e^{\beta r} [\lambda |Y_r|^2 + \frac{1}{\lambda} |X_r|^2 dr \mid \mathcal{F}_t].$$
3.3 An algorithm for Lipschitz generators

For $\beta > \lambda$, this entails

$$(\beta - \lambda)\|Y\|_{BMO^\beta} + \|Z\|_{BMO^\beta} \leq \frac{1}{\lambda} \|X\|_{BMO^\beta}.$$

Observe

$$Y^n = F - \int_t^T Z^n_s \cdot g(\cdot, s, Y^{n-1}_s, Z^{n-1}_s) ds + \int_s^T Z^n_s dW_s,$$

$$Y^{n+1} = F - \int_t^T Z^{n+1}_s \cdot g(\cdot, s, Y^n_s, Z^n_s) ds + \int_s^T Z^{n+1}_s dW_s.$$

Then the above estimates are valid with

$$Y = Y^{n+1} - Y^n,$$

$$Z = Z^{n+1} - Z^n,$$

$$X = Z^n \cdot g(\cdot, Y^{n-1}, Z^{n-1}) - Z^{n+1} \cdot g(\cdot, Y^n, Z^n).$$
3.3 An algorithm for Lipschitz generators

We further estimate $X$ with some constant $C_1$:

$$|X| \leq |Z^n \cdot g(\cdot, Y^{n-1}, Z^{n-1}) - Z^{n-1} \cdot g(\cdot, Y^{n-1}, Z^{n-1})|$$
$$+ |Z^{n-1} \cdot g(\cdot, Y^{n-1}, Z^{n-1}) - Z^n \cdot g(\cdot, Y^n, Z^n)|$$
$$+ |Z^n \cdot g(\cdot, Y^n, Z^n) - Z^{n+1} \cdot g(\cdot, Y^n, Z^n)|$$
$$\leq C_1[|Z^n - Z^{n-1}| + |Y^n - Y^{n-1}| + |Z^n - Z^{n-1}| + |Z^{n+1} - Z^n|].$$

Hence we can write with some constant $C_2$

$$|X|^2 \leq C_2[|Z^n - Z^{n-1}|^2 + |Y^n - Y^{n-1}|^2 + |Z^{n+1} - Z^n|^2].$$

Now choose $\lambda = 4, \beta = \lambda + \frac{1}{2}$ to get

$$\frac{1}{2}||Y||_{BMO^\beta}^2 + (1 - \frac{1}{8})||Z||_{BMO^\beta}^2 \leq \frac{1}{4}[||Z^n - Z^{n-1}||_{BMO^\beta}^2 + ||Y^n - Y^{n-1}||_{BMO^\beta}^2].$$

Set $\alpha = \frac{1}{2}$. •
3.3 An algorithm for Lipschitz generators

If we define

\[ ||(Y, Z)||_\beta = \left( ||Y||^2_{BMO\beta} + ||Z||^2_{BMO\beta} \right)^{\frac{1}{2}}, \]

we thus obtain for \( n \in \mathbb{N} \)

\[ ||(Y^{n+1} - Y^n, Z^{n+1} - Z^n)||_\beta \leq \sqrt{\alpha} ||(Y^n - Y^{n-1}, Z^n - Z^{n-1})||_\beta, \]

hence the pair of processes converges in \( BMO(P) \), by Banach’s fixed point theorem, to a pair \( (Y, Z) \). Let

\[ \zeta = g(\cdot, Y, Z), Q = \exp\left( \int_0^\cdot \zeta_s dW_s - \frac{1}{2} \int_0^\cdot (\zeta_s)^2 ds \right) \cdot P. \]

Observe

\[
Y^n = F - \int_t^T Z^n_s \cdot g(\cdot, s, Y^{n-1}_s, Z^{n-1}_s) ds + \int_t^T Z^n_s dW_s \\
= F - \int_t^T f(\cdot, s, Y^{n-1}_s, Z^{n-1}_s) ds + \int_t^T (Z^{n-1}_s - Z^n_s) \cdot g(\cdot, s, Y^{n-1}_s, Z^{n-1}_s) ds \\
+ \int_t^T Z^n_s dW_s.
\]
3.3 An algorithm for Lipschitz generators

Now recall the convergence of the processes in $BMO(P)$, the Lipschitz continuity of $f$, and the boundedness of $g$. Therefore the right hand side of the equation converges to

$$F - \int_t^T Z_s \cdot g(\cdot, s, Y_s, Z_s) ds + \int_t^T Z_s dW_s.$$

Upon extracting a subsequence of $(Y^n)_{n \in \mathbb{N}}$ we obtain

$$Y = F - \int_t^T f(\cdot, s, Y_s, Z_s) ds + \int_t^T Z_s dW_s.$$

This completes the proof of

Theorem 3
Let $F$ be bounded, $f = z \cdot g$ uniformly Lipschitz continuous in $(y, z)$. The sequence $(Y^n, Z^n)_{n \in \mathbb{N}}$ converges in $BMO(P)$ to a pair of processes $(Y, Z)$ such that $\mathcal{E}(g(\cdot, Y, Z) \cdot W)_T \cdot P$ is a measure solution to the BSDE given by $(g, F)$. 
3.4 A more general result, including generators quadratic in $z$

$(Q_n)_{n \in \mathbb{N}}$ measure solutions to BSDE given by $f_n, F_n, n \in \mathbb{N}$, $f_n \to f$, $F_n \to F$, $f_n = z \cdot g_n, n \in \mathbb{N}$

$p, q$ such that $\frac{1}{p} + \frac{1}{q} = 1$

$F_n \to F$ in $L^{2q}$ and $F \in L^{q^2}$

$Y^n \to \tilde{Y}$ uniformly in $L^q$

for a.a. $(\omega, s)$, $g_n(\omega, s, \cdot) \to g(\omega, s, \cdot)$ uniformly on compacts $K$ not intersecting the space $\{(y, z) : z = 0\}$, $g(\omega, s, \cdot)$ continuous outside this space

$$\sup_{n \in \mathbb{N}} \mathbb{E}(\frac{dQ_n}{dP}^p) < \infty, \quad \sup_{n \in \mathbb{N}} \mathbb{E}(\frac{dP}{dQ_n}^p) < \infty.$$ 

**Theorem 4**

Then there exists a measure solution to the BSDE given by $(f, F)$.

The proof uses weak convergence of probability measures.
3.5 Quadratic growth generators, unbounded terminal variable

(Briand, Bao, Delbaen, Hu,...)

Assume now \( f(s, z) = \alpha z^2, z \in \mathbb{R}, s \in [0, T], \) and that \( F \) is unbounded.

**Theorem 5**

If \( F \) is either bounded above or below (by 0) and \( E(\exp(\gamma |F|)) < \infty \) for some \( \gamma \geq \alpha \), (1) possesses a measure solution. Moreover, \( Q \sim P \) and (1) possesses a strong solution.

**Unboundedness of \( F \): serious complications**

BSDE with infinite time horizon (by the transformation \( t \mapsto \frac{t}{1+t} \) we may return to horizon 1)

\[
Y_t = F - \int_t^\infty Z_s dW_s + \int_t^\infty \frac{1}{2} Z_s^2 ds.
\]

For \( a, b > 0 \), let \( \tau_b = \inf\{t \geq 0 : W_t \leq bt - 1\} \), \( F = 2a(b - a)\tau_b - 2a \).

By Laplace transform techniques: \( E(\exp(\gamma |F|)) < \infty, \quad \gamma < \frac{b^2}{4a|b-a|} \).
3.5 Quadratic growth generators, unbounded terminal variable

The first solution

\[ Y = 2aW \wedge \tau_b - 2a^2 (\tau_b \wedge \cdot), \ Z = 2a1_{[0, \tau_b]} \] solution of (2).

Measure solution property:

\[ E \left( \frac{1}{2} \int Z \, dW \right)_{\tau_b} = e^{aW_{\tau_b} - a^2 \tau_b} = e^{a(b - \frac{a}{2}) \tau_b - a}. \]

By Laplace transform techniques

\[ E(e^{a(b - \frac{a}{2}) \tau_b - a}) = e^{-b[1 - \frac{2}{b^2} a(b - \frac{a}{2}) - 1] - a} = e^{-b[1 - \frac{a}{b}] - 1] - a}, \]

and the latter equals 1 in case \( b \geq a \) and \( \exp(2(b - a)) < 1 \) in case \( a > b \).

Hence first solution measure solution if \( b \geq a \), not measure solution if \( a > b \).
3.5 Quadratic growth generators, unbounded terminal variable

The second solution
If $2a > b$

$$Y = 2b - 4a + 2(b - a)W_{\tau_b \wedge} - 2(b - a)^2(\tau_b \wedge \cdot), \quad Z = 2(b - a)1_{[0, \tau_b]}.$$ 

Measure solution property:

We have

$$\mathbb{E}\left(\frac{1}{2} \int ZdW\right)_{\tau_b} = e^{(b-a)W_{\tau_b} - \frac{1}{2}(b-a)^2\tau_b} = e^{(a-b)}e^{\frac{1}{2}(b-a)(b+a)\tau_b}.$$ 

By Laplace transform techniques

$$e^{(a-b)}\mathbb{E}e^{-\frac{1}{2}(b-a)(b+a)\tau_b} = e^{(a-b)}e^{-b(\sqrt{1-(1-a^2/b^2)}-1)} = 1.$$ 

Therefore second solution is measure solution of (3).
3.5 Quadratic growth generators, unbounded terminal variable

Summary:

\[ b \geq 2a \quad \text{one solution which is measure solution} \]

\[ 2a > b \geq a \quad \text{two solutions, both measure solutions} \]

\[ a > b \quad \text{two solutions, one measure solution, the other one not} \]
Chapter 4

Cross hedging: the explicit formula via Malliavin’s calculus


4.1 Back to cross hedging: main BSDE result

Theorem 6

\((Y, Z)\) unique solution of BSDE

\[ Y_t = F - \int_t^T Z_s dW_s - \int_t^T f(s, Z_s) ds, \quad t \in [0, T], \]

with

\[ f(t, Z_t) = -\frac{\alpha}{2} d^2(C_t, Z_t + \frac{1}{\alpha} \theta_t) + Z_t \cdot \theta_t + \frac{1}{2\alpha} \theta_t^2. \]

Then value function of utility optimization problem under constraint \(p \in C\) given by

\[ V(v) = -\exp(-\alpha[v - Y_0]). \]

There exists an (non-unique) optimal trading strategy \(p^* \in C\) such that

\[ p_t^* \in \Pi_{C_t}(Z_t + \frac{1}{\alpha} \theta_t), \quad t \in [0, T]. \]

Proof:
- existence, uniqueness for BSDE with quadratic non-linearity in \(z\) (M. Kobylanski ’00)
- measurable selection theorem for \(\Pi_{C_t}(Z_t + \frac{1}{\alpha} \theta_t)\)
- BMO properties of the martingales \(\int Z_s dW_s, \int p_s^* dW_s\)
  for uniform integrability of exponentials (regularity of coefficients)•
4.2 Calculation of derivative hedge

generalization to \([t, T]\) instead of \([0, T]\), cond. on \(R_t = r\):

\((Y^{t,r}, Z^{t,r}), p^{t,r}\) (without \(F\)) resp. \((\hat{Y}^{t,r}, \hat{Z}^{t,r}), \hat{p}^{t,r}\) (with \(F\)) instead of \((Y, Z), p\)
yields

\[
V^0(t, v, r) = -\exp(-\alpha(v - Y^{t,r}_t)), \quad V^F(t, v, r) = -\exp(-\alpha(v - \hat{Y}^{t,r}_t)),
\]

instead of \(V(v) = -\exp(v - Y_0)\).

due to linearity of \(C(t, r)\) projections unique and linear, hence

\[
p^{t,r}_s = \Pi_{C(t,r)}[Z^{t,r}_s + \frac{1}{\alpha}\theta(s, R^{t,r}_s)], \quad \hat{p}^{t,r}_s = \Pi_{C(t,r)}[\hat{Z}^{t,r}_s + \frac{1}{\alpha}\theta(s, R^{t,r}_s)],
\]

and so

\[
(\Delta_{\lambda, \tau})(s, R^{t,r}_s) = \Pi_{C(t,r)}[\hat{Z}^{t,r}_s - Z^{t,r}_s].
\]
4.3 Markov property and its consequences

Markov property of $R$ implies (Kobylanski ’00, El Karoui, Peng, Quenez ’97):

**Theorem 7**
There are measurable (deterministic) functions $u$ and $\hat{u}$ such that

$$Y_{s}^{t,r} = u(s, R_{s}^{t,r}), \quad \hat{Y}_{s}^{t,r} = \hat{u}(s, R_{s}^{t,r}).$$

There are measurable (deterministic) functions $v$ and $\hat{v}$ such that

$$Z_{s}^{t,r} = v\sigma(s, R_{s}^{t,r}), \quad \hat{Z}_{s}^{t,r} = \hat{v}\sigma(s, R_{s}^{t,r}).$$

**Corollary 1**

$$p(t, r) := Y_{t}^{t,r} - \hat{Y}_{t}^{t,r} = u(t, r) - \hat{u}(t, r)$$

is the **indifference price**, i.e. $V^{F}(t, v - p(t, r), r) = V^{0}(t, v, r)$.

$p$ depends only on $R$, not on $S$.

**Aim:** Explicit description of $\Delta_{\lambda}$
4.4 Differentiability

**Theorem 8 (Parameter Differentiability)** smoothness conditions on $F, f$
There exists a version of $(\hat{Y}^{t,r}_s, \hat{Z}^{t,r}_s)$ such that a.s.
- $\hat{Y}^{t,r}_s$ is continuous in $s$ and cont. differentiable in $r$ (classical sense)
- $\hat{Z}^{t,r}_s$ is differentiable in a weak sense (norm topology)
- $(\nabla_r \hat{Y}^{t,r}_r, \nabla_r \hat{Z}^{t,r}_r)$ solves the BSDE

\[
\nabla_r \hat{Y}^r_t = \nabla_r F(R^{t,r}_s) \nabla_r R^{t,r}_s - \int_t^T \nabla_r \hat{Z}^{t,r}_s dW_s \\
+ \int_t^T \left[ \nabla_r f(s, R^{t,r}_s, \hat{Z}^{t,r}_s) \nabla_r R^{t,r}_s \\
+ \nabla_z f(s, R^{t,r}_s, \hat{Z}^{t,r}_s) \nabla_r \hat{Z}^{t,r}_s \right] ds.
\]

Proof uses norm inequalities, and inverse Hölder inequalities, based on BMO properties of the stochastic integral processes of $\hat{Z}^{t,r}_s$

**Theorem 9 (Malliavin Differentiability)**

\[
D_\vartheta \hat{Y}^{t,r}_s = \nabla_r \hat{u}(s, R^{t,r}_s) D_\vartheta R^{t,r}_s
\]

and

\[
\hat{Z}^{t,r}_s = D_s \hat{Y}^{t,r}_s = \nabla_r \hat{u}(s, R^{t,r}_s) \sigma(s, R^{t,r}_s)
\]
4.5 Explicit description of derivative hedge

Properties of the BSDEs \[\iff\]

**Theorem 10**
The *indifference price* \[p(t, r) = Y^{t,r}_t - \hat{Y}^{t,r}_t\] is differentiable in \(r\).

**Theorem 11**
The *derivative hedge* \(\Delta_\lambda\) at time \(t\) depends only on \(R_t\), and

\[
\Delta_\lambda(t, r) \tau(t, r) = \Pi C(t, r) [\hat{Z}^{t,r}_t - Z^{t,r}_t]
\]

\[
= \Pi C(t, r) [\nabla_r (\hat{Y}^{t,r}_t - Y^{t,r}_t) \sigma(t, r)]
\]

\[
= -\Pi C(t, r) [\nabla_r p(t, r) \sigma(t, r)].
\]

**Remarks:**

- **complete case:** \(\Delta_\lambda = 'delta hedge'

- where is the risk aversion \(\alpha\)?
4.6 Example: Heating degree days

- common underlying of weather derivatives

- $T_i =$ average of the maximum and the minimum temperature on day $i$ at a specific location

- $HDD_i = \max (0, 18 - T_i)$

Cumulative heating degree days

$$cHDD_t = \sum_{i=1}^{30} HDD_{t-i}$$

Derivatives:

- Option: $(cHDD - K)^+$

- Swap: $b(cHDD - K)$
4.6 Example: Heating degree days

\( cHDD \): 

- statistical analysis shows: cHDDs are log-normally distributed (M. Davis ‘01)

- \( cHDD \) can be modeled as a geometric Brownian motion

\[
dX_t = \mu X_t dt + \nu X_t dW_t
\]

(moving average)

Other indices: cooling degree days

\[
CDD_i = \min (0, 18 - T_i)
\]
4.6 Example: Heating degree days

- \( R = \text{cHDDs (geometric Brownian Motion)} \)

- \( d = 2 \)

- 1-dim market + index: \( k = m = 1 \)

- index volatility: \( \sigma = \begin{pmatrix} c & 0 \end{pmatrix} \)

- price volatility: \( \tau = \begin{pmatrix} \tau_1 & \tau_2 \end{pmatrix} \) with \( c, \tau_1, \tau_2 \in \mathbb{R} \setminus \{0\} \)

Then

\[
\Delta \lambda(t, r) = -c \frac{\partial p(t, r)}{\partial r} \frac{\tau_1}{\tau_1^2 + \tau_2^2}.
\]
4.6 Example: Heating degree days; diversification pressure derivative hedge:

\[
\Delta_{\lambda}(t, r) = -c \frac{\partial p(t, r)}{\partial r} \frac{\tau_1}{\tau_1^2 + \tau_2^2}.
\]

Call option: \( F(R_T) = (R_T - K)^+ \)

\[\Rightarrow \frac{\partial p(t, r)}{\partial r} > 0 \]

Comparison of the optimal strategies:

- \( \tau_1 c < 0 \) (negative correlation)
  \[\Rightarrow F(R_T) \text{ diversifies portfolio} \Rightarrow \Delta_{\lambda} > 0 \]
  \[\Rightarrow \hat{p} > p \]

- \( \tau_1 \alpha > 0 \) (positive correlation)
  \[\Rightarrow F(R_T) \text{ amplifies portfolio} \Rightarrow \Delta_{\lambda} < 0 \]
  \[\Rightarrow \hat{p} < p \]
4.7 Some further ideas and results

- algorithm for measure solution: approximation by \((Q^n_n \in \mathbb{N})\) as in random Lipschitz case: tightness for generator of subcubic growth \(|f(\cdot, z)| \leq c |z|^\gamma, \gamma < 3\) (cf Bao, Delbaen, Hu)

- ideas as in the examples above: complete solution of Skorokhod embedding problem for Wiener process with linear drift (Diplomarbeit G. Heyne)

- direct approach of nonlinear Feynman-Kac formula by measure solutions in Markovian case (Diplomarbeit J. Zhang)

- measure solution intermediate notion on the way to efficient notion of weak solution (A. Fromm)