Chromatic Polynomials of Hypergraphs

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\textsuperscript{1}Joint work with Ruixue Zhang
Thanks a lot for the invitation and financial support for attending this conference in Taipei.
The main results are from the paper below:
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Also in Arxiv http:arxiv.org/abs/1611.04245
An Asia map

Any two countries with a common boundary are assigned different colors.
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Four colour conjecture

(Francis Guthrie, 1852)

Any map can be coloured with at most 4 colours
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Any map can be coloured with at most 4 colours such that each region is assigned one colour and any two regions with a common boundary are coloured differently.
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Any loopless plane graph has a proper vertex-colouring with at most 4 colours.

Any plane triangulation has a proper vertex-colouring with at most 4 colours.
First proof of 4-color Conjecture

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Another proof

4CT was proven again in 1996 by Robertson, Sanders, Seymour and Thomas. It is also a computer-based proof. They found another set of the unavoidable configurations of size 633 which are reducible.
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They found another set of the unavoidable configurations of size 633 which are reducible.
Map colouring to Vertex colouring

Two graphs with 3-colourings:

![Graph 1](image1.png)

![Graph 2](image2.png)
Map colouring to Vertex colouring

Two graphs with 3-colourings:

Each vertex is assigned a colour
Map colouring to Vertex colouring

Two graphs with 3-colourings:

Each vertex is assigned a colour and any two adjacent vertices are assigned different colours.
A $k$-colouring of a graph $G$ is a way of assigning $k$ colours to vertices in $G$, one colour for each vertex, such that any two adjacent vertices are assigned different colours.

Any graph with maximum degree $k$ has a $(k + 1)$-colouring.

Brooks' Theorem: Any connected graph with maximum degree $k$ which is not a complete graph nor an odd cycle has a $k$-colouring.
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**Brooks’ Theorem:**
Any connected graph with maximum degree $k$ which is not a complete graph nor an odd cycle has a $k$-colouring.
Let $P(G, \lambda)$ be the function of $\lambda$ such that $P(G, k)$ counts the number of $k$-colourings of $G$ whenever $k$ is a positive integer.
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$P(G, \lambda)$ is called the *chromatic polynomial* of $G$, and it is indeed a polynomial in $\lambda$. 
The chromatic polynomial was introduced by Birkhoff in 1912.
The chromatic polynomial was introduced by Birkhoff in 1912 with the hope of proving 4CC by applying the computation of chromatic polynomials.
Example

For any null graph $N_n$, $P(N_n, \lambda) = \lambda^n$.

For any complete graph $K_n$, $P(K_n, \lambda) = \lambda(\lambda - 1)\cdots(\lambda - n + 1)$.

For any tree $T$ of order $n$, $P(T, \lambda) = \lambda(\lambda - 1)^{n-1}$.

For any cycle $C_n$ of order $n$, $P(C_n, \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1)^{n-1}$.
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- For any cycle $C_n$ of order $n$,
  \[ P(C_n, \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1). \]
Basic properties

$P(G, \lambda)$ is a monic polynomial in $\lambda$ of degree $n$, the order of $G$. $P(G, \lambda)$ can be expressed as:

$$P(G, \lambda) = \lambda^n - a_{n-1}\lambda^{n-1} + \cdots + (-1)^n c_0,$$

where $c$ is the number of components of $G$ and $a_i$ is a positive integer for all $c \leq i \leq n$.

$G$ is connected if and only if $\lambda^2$ is not a factor of $P(G, \lambda)$. 

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Basic properties

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- $P(G, \lambda)$ can be expressed as
  $$P(G, \lambda) = \lambda^n - a_{n-1} \lambda^{n-1} + \cdots + (-1)^{n-c} a_c \lambda^c,$$
  where $c$ is the number of components of $G$ and $a_i$ is a positive integer for all $c \leq i \leq n$.
- $G$ is connected if and only if $\lambda^2$ is not a factor of $P(G, \lambda)$.
Important Results by Sokal

(Sokal 2001) For any simple graph $G$ with maximum degree $D$, the zeros of $P(G, z)$ are within the disc $|z| < 7.963907D$.

(Sokal 2004) Chromatic polynomials of graphs have dense complex zeros in the whole plane.
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Important Results by Thomassen

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Important Results by Thomassen

(Thomassen 1997) chromatic polynomials of graphs have dense real zeros in $[32/27, \infty)$. 

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Important Results by Jackson

Chromatic polynomials of graphs have no real zeros in the following intervals:

$(-\infty, 0)$,

$(0, 1)$,

$(1, 32/27)$.

$(1, 32/27)$ was determined by Jackson in 1993.
Important Results by Jackson

- chromatic polynomials of graphs have no real zeros in the following intervals:
  \(( -\infty, 0 ), (0, 1), (1, 32/27) \).

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Results associated with 4CT

For any real $\epsilon > 0$, $P(G, \lambda) = 0$ holds for some plane graph $G$ and some $\lambda \in (4 - \epsilon, 4)$. (Gordon Royle 2008)
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- $4\text{CT} \iff P(G, 4) > 0$ for any loopless plane graph $G$. 

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Results associated with 4CT

\[ P(G, 2 + \tau) > 0 \text{ for all loopless plane graphs } G, \]

where \( \tau \approx 1.618033 \cdots \) is the golden ratio, i.e., the real root > 1 of \[ x(x - 1) = 1. \]
Results associated with 4CT

(Tutte 1970)

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the real root > 1 of \( x(x - 1) = 1 \).
Results associated with 4CT

The roots of the chromatic polynomials of planar graphs are dense in the interval $(32/27, 4)$, except possibly in a small interval $(t_1, t_2)$ with $t_2 - t_1 < 0.000324$ and $t_1 < \tau + 2 < t_2$. 

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(Perrett and Thomassen 2016)
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Birkhoff and Lewis’s Conjecture

Theorem (Birkhoff and Lewis 1946)

\[ P(G, \lambda) > 0 \] for any loopless plane graph \( G \) and any real \( \lambda \geq 5 \).

Conjecture (Birkhoff and Lewis 1946)

\[ P(G, \lambda) > 0 \] for any loopless plane graph \( G \) and any real \( \lambda \in (4, 5) \).

So far there is no any progress. It is even unknown if there is a plane graph \( G \) and a real \( \lambda \in (4, 5) \) such that \( P(G, \lambda) \leq 0 \).
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Question:
What is a hypergraph?
**Question:** What is a hypergraph?

**Figure:** Hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$

$\mathcal{V} = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$

$\mathcal{E} = \{e_1, e_2, e_3, e_4\}$

$e_1 = \{v_1, v_2, v_3\}$; $e_2 = \{v_3, v_4, v_5, v_6\}$;

$e_3 = \{v_5, v_7, v_8\}$; $e_4 = \{v_8, v_9\}$
Hypergraphs

A hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is an extension of a simple graph in which edges can contain more than two vertices:

$$\mathcal{E} = \{ e \subseteq \mathcal{V} : |e| \geq 2 \}.$$
A hypergraph \( \mathcal{H} = (\mathcal{V}, \mathcal{E}) \) is an extension of a simple graph in which edges can contain more than two vertices:

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\mathcal{E} = \{ e \subseteq \mathcal{V} : |e| \geq 2 \}.
\]

A hypergraph is also called a set system or a family of subsets of a universal set \( \mathcal{V} \).
Special hypergraphs

H is called r-uniform if $|e| = r$ ($r \geq 2$) each edge $e \in E(H)$.

2-uniform hypergraphs are normal graphs.

H is called linear if no two edges intersect in more than one vertex.

A hypergraph is called Sperner if no edge in the hypergraph is a subset of another edge.
\( \mathcal{H} \) is called \textit{r-uniform} if \( |e| = r \) \((r \geq 2)\) each edge \( e \in \mathcal{E}(\mathcal{H}) \).
Special hypergraphs

- $\mathcal{H}$ is called \textit{\textbf{r-uniform}} if $|e| = r \ (r \geq 2)$ each edge $e \in \mathcal{E}(\mathcal{H})$.

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- 2-uniform hypergraphs are normal graphs.

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Special hypergraphs

- $\mathcal{H}$ is called **$r$-uniform** if $|e| = r$ ($r \geq 2$) each edge $e \in \mathcal{E}(\mathcal{H})$.

- 2-uniform hypergraphs are normal graphs.

- $\mathcal{H}$ is called **linear** if no two edges intersect in more than one vertex.

- A hypergraph is called **Sperner** if no edge in the hypergraph is a subset of another edge.
Example

Figure: 3-uniform linear Sperner hypergraph $\mathcal{H}$
For a hypergraph $H = (V, E)$ and a positive integer $\lambda$, a (weak) $\lambda$-colouring of $H$ is a mapping $f : V \rightarrow \{1, \ldots, \lambda\}$ such that $|\{f(u) : u \in e\}| \geq 2$ for each $e \in E$, i.e., for each edge $e$ of $H$, there exist at least two vertices $u, v \in e$ for which $f(u) \neq f(v)$. 
For a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ and a positive integer $\lambda$, 
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For a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ and a positive integer $\lambda$, a \textit{(weak) $\lambda$-colouring} of $\mathcal{H}$ is a mapping $f : \mathcal{V} \rightarrow \{1, \cdots, \lambda\}$ such that $|\{f(u) : u \in e\}| \geq 2$ for each $e \in \mathcal{E}$, i.e., for each edge $e$ of $\mathcal{H}$, there exist at least two vertices $u, v \in e$ for which $f(u) \neq f(v)$. 
(Weak) $\lambda$-colouring

The (weak) colouring of hypergraphs was introduced by Erdös and Hajnal in 1966.
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Example

Figure: This hypergraph can be 2-coloured in 6 different ways
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For any positive integer $\lambda$, this hypergraph has $\lambda^3 - \lambda$ different $\lambda$-colourings.
Example

Figure: two different 2-colourings of $\mathcal{H}$
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Question?
How many different 2-colourings on $\mathcal{H}$?
Example

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Question?

How many different 2-colourings on $\mathcal{H}$? Answer: 54
The chromatic polynomial of a hypergraph $\mathcal{H}$, denoted by $P(\mathcal{H}, \lambda)$, counts the number of $\lambda$-colourings of $\mathcal{H}$ whenever $\lambda$ is a positive integer.
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This concept appeared in Helgason’s work in 1972 and it may have appeared in V. Chvátal’s Ph.D. thesis.
Examples

If $H$ has $p$ vertices and one edge only, which contains all vertices, then $P(H, \lambda) = \lambda^{p-1}$.

If $H$ is the following hypergraph, then $P(H, \lambda) = (\lambda^3 - \lambda) / \lambda$. 

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Examples

- If $\mathcal{H}$ has $p$ vertices and one edge only, which contains all vertices, then

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- If \( \mathcal{H} \) has \( p \) vertices and one edge only, which contains all vertices, then

\[
P(\mathcal{H}, \lambda) = \lambda^p - \lambda.
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- If \( \mathcal{H} \) is the following hypergraph, then

\[
P(\mathcal{H}, \lambda) = (\lambda^3 - \lambda)^2 / \lambda.
\]
Basic properties

**Basic properties**

- $P(H, \lambda)$ is a monic polynomial in $\lambda$ of degree $p$, the number of vertices in $H$.
- The constant term in the polynomial $P(H, \lambda)$ is 0, i.e., $P(H, 0) = 0$.
- If $H$ has components $H_1, \ldots, H_r$, then $P(H, \lambda) = \prod_{1 \leq i \leq r} P(H_i, \lambda)$.
- If $H$ is the union of $H_1$ and $H_2$ with $|V(H_1) \cap V(H_2)| = 1$, then $P(H, \lambda) = P(H_1, \lambda)P(H_2, \lambda) / \lambda$.
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- If $\mathcal{H}$ is the union of $\mathcal{H}_1$ and $\mathcal{H}_2$ with $|V(\mathcal{H}_1) \cap V(\mathcal{H}_2)| = 1$, then
  \[ P(\mathcal{H}, \lambda) = P(\mathcal{H}_1, \lambda)P(\mathcal{H}_2, \lambda) / \lambda. \]
Computing of $P(\mathcal{H}, \lambda)$

$P(\mathcal{H}, \lambda) = \lambda p$ if $\mathcal{H}$ has $p$ vertices and no edges. (R.P. Jones, 1976)

Deletion-contraction formula

For any edge $e$ in $\mathcal{H}$, then

$P(\mathcal{H}, \lambda) = P(\mathcal{H} - e, \lambda) - P(\mathcal{H} \cdot e, \lambda),$

where $\mathcal{H} \cdot e$ is obtained from $\mathcal{H}$ by identifying all vertices in $e$.

Repeatedly apply the Deletion-contraction formula until each hypergraph obtained has no edges.
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Example

Figure: $\mathcal{T}_2^3$: 3-uniform hypertree with 2 edges
Example

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\[
P(\mathcal{H}, \lambda) = P(\mathcal{H} - e_2, \lambda) - P(\mathcal{H} \cdot e_2, \lambda) \\
= \lambda^2 \cdot (\lambda^3 - \lambda) - (\lambda^3 - \lambda) \\
= \lambda(\lambda^2 - 1)^2.
\]
Example

Figure: $C_3^3$: 3-uniform elementary hypercycle

$$P(C_3^3, \lambda) = (\lambda^2 - 1)^3 - (\lambda - 1)$$
(Tomescu 1998) Let $\mathcal{H}$ be a hypergraph with $n$ vertices. Then

$$P(\mathcal{H}, \lambda) = \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda,$$

where

$$a_i = \sum_{j \geq 0} (-1)^j N(i, j)$$

and $N(i, j)$ denotes the number of spanning sub-hypergraphs of $\mathcal{H}$ with $i$ components and $j$ edges.
Interpretation of coefficients

(Tomescu 1998) Let $\mathcal{H}$ be a hypergraph with $n$ vertices. Then

$$P(\mathcal{H}, \lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda,$$

where $a_i = \sum_{j \geq 0} (-1)^j N(i, j)$ and $N(i, j)$ denotes the number of spanning sub-hypergraphs of $\mathcal{H}$ with $i$ components and $j$ edges.
Problem

Do chromatic polynomials of hypergraphs have properties not held for chromatic polynomials of graphs?
Results for chromatic polynomials of graphs

$P(G, -1)$ always counts the number of acyclic orientations of a graph $G$;
chromatic polynomials of graphs are zero-free on the intervals $(-\infty, 0)$, $(0, 1)$, and $(1, 32/27]$.

$G$ is connected $\iff \lambda^2 \neq |P(G, \lambda)|$;
for a connected graph $G$, $G$ is non-separable $\iff (\lambda - 1)^2 \neq |P(G, \lambda)|$. 
Results for chromatic polynomials of graphs

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$G$ is connected $\iff \lambda^2 \not| P(G, \lambda)$;

For a connected graph $G$, $G$ is non-separable $\iff (\lambda - 1)^2 \not| P(G, \lambda)$.
The Tutte polynomial $T_G(x, y)$ is defined as:

$$T_G(x, y) = \sum_{A \subseteq E} (x - 1)^{r(E) - r(A)} (y - 1)^{|A| - r(A)},$$

where $r(A)$ is the rank of $A$, i.e., $r(A) = |V| - c(A)$, and $c(A)$ is the number of components of the spanning subgraph of $G$ with edge set $A$. 

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Tutte polynomial

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where $r(A)$ is the rank of $A$, i.e., $r(A) = |V| - c(A)$ and $c(A)$ is the number of components of the spanning subgraph of $G$ with edge set $A$. 
Stanley’s result on $P(G, -1)$

The chromatic polynomial:

$$P(G, x) = x^c(G) (-1)^r(G) T_G(1 - x, 0).$$

In particular, $$(-1)^n P(G, -1) = T_G(2, 0).$$

(Stanley 1970) For any simple graph $G$ of order $n$, $$(-1)^n P(G, -1)$$ is equal to the number of acyclic orientations of $G$. 

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Stanley’s result on $P(G, -1)$

- The chromatic polynomial:
  \[
P(G, x) = x^{c(G)}(-1)^{r(G)}T_G(1 - x, 0).
  \]
  In particular, \((-1)^nP(G, -1) = T_G(2, 0)\).
Stanley’s result on $P(G, -1)$

- The chromatic polynomial:

$$P(G, x) = x^{c(G)}(-1)^{r(G)}T_G(1-x, 0).$$

In particular, $(-1)^nP(G, -1) = T_G(2, 0)$.

- (Stanley 1970) For any simple graph $G$ of order $n$, $(-1)^nP(G, -1)$ is equal to the number of acyclic orientations of $G$. 

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The number $T_G(0,2)$

It is known that $T_G(0,2)$ is equal to the number of totally cyclic orientations of $G$.

Does $T_G(0,2)$ have a relation with chromatic polynomial?
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- Does $T_G(0, 2)$ have a relation with chromatic polynomial?
Hypergraph $\mathcal{H}_G$

For a given multigraph $G = (V, E)$, let $\mathcal{H}_G = (\mathcal{V}, \mathcal{E})$ be the hypergraph with $\mathcal{V} = V \cup \{w_e : e \in E\}$ and $\mathcal{E} = \{\{u_e, v_e, w_e\} : e \in E\}$, where $u_e$ and $v_e$ are the two ends of $e$. 
Hypergraph $\mathcal{H}_G$

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![Diagram of $G$ and $\mathcal{H}_G$]
\( P(\mathcal{H}_G, -1) \)

(Zhang and Dong, 2017)

For any \((n, m)\)-graph \(G\) with \(c\) components,

\[
P(\mathcal{H}_G, -1) = \lambda^{m-n+2c} \cdot (-1)^{n+c} T_G(1 - \lambda^2, (\lambda-1)/\lambda^2),
\]

\(T_G\) is equal to the number of totally cyclic orientations of \(G\).
For any \((n, m)\)-graph \(G\) with \(c\) components,

\[
P(\mathcal{H}_G, -1) = \frac{(-1)^m + c}{m} \cdot T_G\left(1 - \lambda^2, \left(\frac{\lambda - 1}{\lambda}\right)\right).\]

(Zhang and Dong, 2017)

For any \((n, m)\)-graph \(G\) with \(c\) components,

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P(\mathcal{H}_G, \lambda) = \lambda^{m-n+2c} \cdot (-1)^{n+c} \cdot T_G\left(1 - \lambda^2, (\lambda - 1)/\lambda\right),
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In particular, \(P(\mathcal{H}_G, -1) = (-1)^{m+c}T_G(0, 2)\).
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\((-1)^{m+c}P(\mathcal{H}_G, -1)\) is equal to the number of totally cyclic orientations of \(G\).
Zero-free intervals for chromatic polynomials

Chromatic polynomials of graphs have no zeros in the following intervals:

\[ (-\infty, 0), (0, 1), (1, 32/27) \]

Do chromatic polynomials of hypergraphs have zero-free intervals?

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May 2017
Zero-free intervals for chromatic polynomials

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Do chromatic polynomials of hypergraphs have zero-free intervals?
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For any simple graph \( G = (V, E) \), let \( \mathcal{H}_G \) be the hypergraph with vertex set \( V = V \cup \{w\} \) and edge set \( E = \{\{u, v, w\} : uv \in E\} \), i.e., \( \mathcal{H}_G \) is obtained from \( G \) by adding a new vertex \( w \) and changing each edge \( \{u, v\} \) in \( G \) to an edge \( \{u, v, w\} \) in \( \mathcal{H}_G \).
Hypergraph $\mathcal{H}_G$

For any simple graph $G = (V, E)$, let $\mathcal{H}_G$ be the hypergraph with vertex set $V = V \cup \{w\}$ and edge set $E = \{\{u, v, w\} : uv \in E\}$, i.e., $\mathcal{H}_G$ is obtained from $G$ by adding a new vertex $w$ and changing each edge $\{u, v\}$ in $G$ to an edge $\{u, v, w\}$ in $\mathcal{H}_G$

For example, if $G$ is $K_3$ with vertex set $\{u_1, u_2, u_3\}$,
For any simple graph $G = (V, E)$, let $\mathcal{H}_G$ be the hypergraph with vertex set $V = V \cup \{w\}$ and edge set $\mathcal{E} = \{\{u, v, w\} : uv \in E\}$, i.e., $\mathcal{H}_G$ is obtained from $G$ by adding a new vertex $w$ and changing each edge $\{u, v\}$ in $G$ to an edge $\{u, v, w\}$ in $\mathcal{H}_G$.

For example, if $G$ is $K_3$ with vertex set $\{u_1, u_2, u_3\}$, then $\mathcal{H}_G$ is the hypergraph with vertex set $\{w, u_1, u_2, u_3\}$ and three edges $e_1, e_2, e_3$:

$$e_1 = \{w, u_1, u_2\}, e_2 = \{w, u_1, u_3\}, e_3 = \{w, u_2, u_3\}.$$
For any simple graph $G = (V, E)$ of order $n$,

$$P(H \circ_G \lambda) = \lambda(\lambda - 1)^n I(G, 1/(\lambda - 1)),$$

(Zhang and Dong, 2017)
For any simple graph $G = (V, E)$ of order $n$,

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where $I(G, x) = \sum_A x^{|A|}$ is the independence polynomial of $G$. 

(Zhang and Dong, 2017)
For any simple graph $G = (V, E)$ of order $n$,

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where $I(G, x) = \sum_{A \text{ ind.}} x^{|A|}$ is the independence polynomial of $G$.

Brown, Hickman and Nowakowski in 2004 showed that real roots of independence polynomials are dense in $(-\infty, 0]$.
Thus the real roots of $P(\mathcal{H}_G, \lambda)$’s for all graphs $G$ are dense in $(-\infty, 1]$. 
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It can be shown further that the real zeros of chromatic polynomial of hypergraphs are dense in $(-\infty, \infty)$. 
Thus the real roots of $P(\mathcal{H}_G, \lambda)$'s for all graphs $G$ are dense in $(-\infty, 1]$.

It can be shown further that the real zeros of chromatic polynomial of hypergraphs are dense in $(-\infty, \infty)$.

Hence chromatic polynomial of hypergraphs have no zero-free intervals.
$P(G, \lambda)$ and connectivity of graphs

Known: $G$ is connected $\iff \lambda^2 \not| P(G, \lambda)$. 
$P(G, \lambda)$ and connectivity of graphs

Known: $G$ is connected $\iff \lambda^2 \nmid P(G, \lambda)$.

Is this property true for chromatic polynomials of hypergraphs?
Assume that $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is connected.
Assume that $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is connected.

An edge $e$ in $\mathcal{H}$ is called as a *bridge* of $\mathcal{H}$ if $\mathcal{H} - e$ (i.e., the hypergraph obtained from $\mathcal{H}$ by removing $e$) is disconnected.
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Let $B(\mathcal{H})$ be the set of bridges of $\mathcal{H}$. 
Bridges in a hypergraph

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Let $B(\mathcal{H})$ be the set of bridges of $\mathcal{H}$.

(Zhang and Dong, 2017)

If $B(\mathcal{H})$ is a proper subset of $\mathcal{E}$ and the sub-hypergraph $(\mathcal{V}, B(\mathcal{H}))$ is connected, then $\lambda^2$ is a factor of $P(\mathcal{H}, \lambda)$. 
Example for Bridges

B \subseteq E and (V, B(H)) is connected.

So \lambda_2 | P(H, \lambda).
$B(\mathcal{H})$ is a proper subset of $\mathcal{E}$ and $(\mathcal{V}, B(\mathcal{H}))$ is connected.
Example for Bridges

$B(\mathcal{H})$ is a proper subset of $\mathcal{E}$ and $(\mathcal{V}, B(\mathcal{H}))$ is connected.

So $\lambda^2 | P(\mathcal{H}, \lambda)$. 
Known: If $G$ is connected, then $G$ is separable $\iff (\lambda - 1)^2 | P(G, \lambda)$.
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Is this property true for chromatic polynomials of hypergraphs?
Example

a separable hypergraph
Example

a separable hypergraph

But its chromatic polynomial is $\lambda(\lambda - 1)(\lambda^2 + \lambda - 1)$, no factor $(\lambda - 1)^2$. 
Assume that no vertex is contained in all edges of $\mathcal{H}$.
Separable at $w$

Assume that no vertex is contained in all edges of $\mathcal{H}$. Assume that $\mathcal{V}_1$ and $\mathcal{V}_2$ are two proper subsets of vertices of $\mathcal{H}$ with sizes at least 2 such that

(a) $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$;
(b) $\mathcal{V}_1 \cap \mathcal{V}_2 = \{w\}$;
(c) for each $e \in E$, either $w \in e$ or $e \subseteq \mathcal{V}_i$ for some $i$.

Thus $\mathcal{H} - w$ is disconnected.

(Zhang and Dong, 2017) $(\lambda - 1)^2 \not\mid P(\mathcal{H}, \lambda)$ if (i) $e \not\subseteq \mathcal{V}_1$ for all $e \in E$ and (ii) $(\lambda - 1)^2 \not\mid P(\mathcal{H} \cdot \mathcal{V}_1, \lambda)$. 

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Separable at $w$

Assume that no vertex is contained in all edges of $H$.

Assume that $V_1$ and $V_2$ are two proper subsets of vertices of $H$ with sizes at least 2 such that

(a) $V_1 \cup V_2 = V$;
Separable at \( w \)

Assume that no vertex is contained in all edges of \( \mathcal{H} \).
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(a) \( \mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V} \);

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$\lambda - 1 \neq \left| P(\mathcal{H}, \lambda) \right|$ if

(i) $e \not\subseteq \mathcal{V}_1$ for all $e \in \mathcal{E}$ and

(ii) $\lambda - 1 \neq \left| P(\mathcal{H} - \mathcal{V}_1, \lambda) \right|$. 

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Separable at $w$

Assume that no vertex is contained in all edges of $\mathcal{H}$.

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Thus $\mathcal{H} - w$ is disconnected.
Assume that no vertex is contained in all edges of $H$.

Assume that $V_1$ and $V_2$ are two proper subsets of vertices of $H$ with sizes at least 2 such that

(a) $V_1 \cup V_2 = V$;
(b) $V_1 \cap V_2 = \{w\}$;
(c) for each $e \in E$, either $w \in e$ or $e \subseteq V_i$ for some $i$.

Thus $H - w$ is disconnected.

(Zhang and Dong, 2017) $(\lambda - 1)^2 \triangledown P(H, \lambda)$ if
(i) $e \nsubseteq V_1$ for all $e \in E$ and (ii) $(\lambda - 1)^2 \triangledown P(H \cdot V_1, \lambda)$. 
Open problems

Result (Dong 2000): For every graph $G$ of order $n$, where $n \geq 1$, when real $\lambda \geq n$, we have
$$(\lambda - 1)n \prod_{\lambda}(G, \lambda) - \lambda n \prod_{\lambda - 1}(G, \lambda - 1) \geq 0.$$ This result proved "the shameful conjecture" proposed by Bartels and Welsh in 1995.

Conjecture: For any hypergraph $H = (V, E)$ with $|V| = n$, $$(\lambda - 1)n \prod_{\lambda}(H, \lambda) - \lambda n \prod_{\lambda - 1}(H, \lambda - 1) \geq 0$$ holds for all real $\lambda \geq n$. 

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Open problems

- **Result** (Dong 2000): For every graph $G$ of order $n$, where $n \geq 1$, when real $\lambda \geq n$, we have

$$ (\lambda - 1)^nP(G, \lambda) - \lambda^n P(G, \lambda - 1) \geq 0. $$
Open problems

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\end{eqnarray*}$$

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- **Conjecture**: For any hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ with $|\mathcal{V}| = n$,

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\end{eqnarray*}$$

holds for all real $\lambda \geq n$. 
Open problems

Conjecture (Welsh, 1970 and Brenti, 1992)

For any graph $G$ and any integer $k > 0$, \[
(P(G, k))^2 \geq P(G, k+1)P(G, k-1).
\]

Counterexamples were found by Seymour in 1998 for $k = 6$. 
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Open problems

Conjecture (Modified)
Let $G$ be a graph of order $n$. For all real $\lambda \geq n - 1$,
\[
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Conjecture (Zhang and Dong, 2016)
The above conjecture also holds for hypergraphs.
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The above conjecture also holds for hypergraphs.
THANK YOU!