Boltzmann Equation and Hydrodynamics beyond Navier-Stokes

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Chapman-Enskog method and Burnett equations

Notation:

$f(x, v, t)$ - distribution function ($x \in \mathbb{R}^3,\ v \in \mathbb{R}^3,\ t \geq 0$)

Boltzmann equation:

$$\mathcal{D}f = \frac{1}{\varepsilon} Q(f, f),\ \mathcal{D} = \partial_t + v \cdot \partial_x,$$

$$\varepsilon \text{ - Knudsen number},\ \varepsilon \to 0^+$$

Macroscopic variables: (in the notation $\langle f, g \rangle = \int_{\mathbb{R}^3} dv f(v) g(v)$)

density $\rho = \langle f, 1 \rangle \in \mathbb{R}_+$

bulk velocity $u = \frac{1}{\rho} \langle f, v \rangle \in \mathbb{R}^3$

temperature $T = \frac{1}{3\rho} \langle f, |c|^2 \rangle \in \mathbb{R}_+,\ c = v - u$ (thermal velocity)

Equations of hydrodynamics

(based on: $\langle \Psi, Q(f,f) \rangle = 0$ if $\Psi = 1, v, |v|^2$)

$$\begin{cases} \rho_t + div\rho u = 0 \\ \rho \mathcal{D}_0 u_\alpha + \dfrac{\partial p}{\partial x_\alpha} + \varepsilon \dfrac{\partial \pi_{\alpha\beta}}{\partial x_\beta} = 0 \\ \dfrac{3}{2}\rho \mathcal{D}_0 T + pdivu + \varepsilon \left(\pi_{\alpha\beta} \dfrac{\partial u_\alpha}{\partial x_\beta} + divq \right) = 0 \end{cases}$$

where

$$\mathcal{D}_0 = \partial_t + u \cdot \partial_x, \; p = \rho T \text{ (pressure)}$$

Fluxes: $$\varepsilon \pi_{\alpha\beta} = \langle f, c_\alpha c_\beta - \frac{|c|^2}{3}\delta_{\alpha\beta} \rangle, \; c = v - u,$$
$$\varepsilon q_\alpha = \frac{1}{2}\langle f, c_\alpha |c|^2 \rangle, \; \alpha, \beta = 1, 2, 3$$

Main problem:

How to express fluxes $\pi(f)$ and $q(f)$ in terms of macroscopic variables (ρ, u, T)?

Chapman-Enskog Expansion

We use the famous Chapman-Enskog expansion

(Hilbert 1912, Enskog 1917, Chapman 1917,
Burnett 1935, Chapman & Cowling 1939)

and obtain "Equations of hydrodynamics" in symbolic form

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ u\\ T\end{pmatrix}=\left(A_0^{E}+\varepsilon A_1^{N-S}+\varepsilon^2 A_2^{B}+\dots\right)\begin{pmatrix}\rho\\ u\\ T\end{pmatrix},$$

where all $A_k^{\cdots}$, $k=0,1,\dots$ are nonlinear operators

Chapman-Enskog Expansion

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho \\ u \\ T\end{pmatrix} = \left(A_0^E + \varepsilon A_1^{N-S} + \varepsilon^2 A_2^B + \ldots\right)\begin{pmatrix}\rho \\ u \\ T\end{pmatrix},$$

Truncation problem

The standard truncation rule

"Neglect all terms of order $O\left(\varepsilon^{n+1}\right)$"

does not work for *n=2:* *Burnett equations are ill-posed.*

What shall we do?

Is the way of truncation unique?

General Approach (Poincaré,...)

Take an abstract evolution equation (or simply a vector ODE)

$$\frac{dy}{dt} = T(y;\varepsilon) = A(y) + \varepsilon B(y) + \varepsilon^2 C(y) + ...,$$

$y \in Y;\ A(y), B(y), ...$ are differentiable non-linear operators:

$A(y+h) - A(y) = A'(y)h + O(||h||^2)$

1. Let the "natural" truncation

$$Y_t = A(y) + \varepsilon B(y) + \varepsilon^2 C(y) + O(\varepsilon^3)$$

   does not work. What then?

2. Use the "change of variables" (all operators are time-independent):

$$z = y + \varepsilon^2 R(y)$$

   Then $y \simeq z - \varepsilon^2 R(z)$ and simple calculation yields

$$Z_t = A(z) + \varepsilon B(z) + \varepsilon^2 \left\{ C(z) + [R,A](z) \right\} + O(\epsilon^3),$$

   where $[R,A](z) = R'(z)A(z) - A'(z)R(z)$.

Proposition *Any truncated equation*

$$y_t = A(y) + \varepsilon B(y) + \varepsilon^2 C(y) + ... \tag{2}$$

is formally equaivalent to a family of equations

$$z_t = A(z) + \varepsilon B(z) + \varepsilon^2 \tilde{C}(z) + ..., \tag{3}$$

where

$$z = y + \varepsilon^2 R(y) \Longleftrightarrow y = z - \varepsilon^2 R(z) + ...,$$

and

$$\tilde{C}(z) = C(z) + R'(z)A(z) - A'(z)R(z)$$

with any differentiable operator R.

Remark: Note that even classical Navier-Stokes Eqs. are not defined uniquely (only Euler Eqs. are unique).

Remaining problem: Find the operator $R(z)$ that makes Eqs. (3) "good".

Hierarchy of problems (with increasing difficulty)

Linearized BE in $\mathbb{R}^3$ or $\boldsymbol{Tor}^3$

Nonlinear BE in $\mathbb{R}^3$ or $\boldsymbol{Tor}^3$

Linearized BE in other domains + boundary conditions

Nonlinear BE in other domains + boundary conditions

Separate question: stationary BE

Main goal is to understand how to improve the Navier-Stokes results

EXAMPLE **Linearized Burnett equations for 1d solutions**

$$x \in \mathbb{R}, \quad \rho = \rho(x,t), \quad T = T(x,t), \quad u = (u(x,t), 0, 0)$$

Hydrodinamic vector: $$z = \begin{pmatrix} \rho \\ u \\ T \end{pmatrix} \in \mathbb{R}^3$$

Linearized Burnett equations:

$$\underbrace{z_t + M_0 z_x}_{Euler} = \underbrace{\varepsilon M_1 z_{xx}}_{Navier-Stokes} + \underbrace{\varepsilon^2 M_2 z_{xxx}}_{Burnett}$$

$$M_0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2/3 & 0 \end{pmatrix}, \qquad M_{1,2} = \dots$$

Comment Equations are ill-posed, but we do not need to solve them. Instead we transform the variables.

Proposition A transformation

$$y = R_0\, z + \varepsilon\, R_1\, z_x + \varepsilon^2 R_2\, z_{xx}$$

leads (formally) to "diagonal" Burnett equations

$$y_t + c_0 \Lambda_0\, (y_x + \alpha \varepsilon^2 y_{xxx}) = \beta\, \varepsilon\, \Lambda_1 y_{xx} + O(\varepsilon^3),$$

$$c_0 = \sqrt{5/3}, \quad \Lambda_0 = diag(1, -1, 0), \quad \Lambda_1 = diag(1, 1, \gamma),$$

with some positive numbers α, β, γ

Comment LBEs for general 3d flows can be also reduced to diagonal form by using the Fourier transform

Sketch of proof (in $\mathbb{R}$)

- Step1 **Fourier transform**

$$\hat{\rho}(k,t) = \mathfrak{F}(\rho) = \int_{-\infty}^{\infty} dx\, e^{-ikx} \rho(x,t),$$

$$\hat{u} = \mathfrak{F}(u), \quad \hat{T} = \mathfrak{F}(T).$$

Then $\hat{z} = \begin{pmatrix} \hat{\rho} \\ \hat{u} \\ \hat{T} \end{pmatrix}$ and we obtain ODEs

$$\hat{z}_t + ikM_0\hat{z} = -\varepsilon k^2 M_1 \hat{z} + \varepsilon^2 (ik)^3 M_2 \hat{z}$$

Note that M_0 has three distinct eigenvalues

$$\lambda_0 = 0, \quad \lambda_\pm = \pm\sqrt{5/3} = \pm c_0.$$

- Step 2 **Diagonalization of** $\boldsymbol{M}_0$

New variables $s = \hat{\rho} - \frac{3}{2}\hat{T}, \quad w_{\pm} = \hat{\rho} + \hat{T} \pm c_0 \hat{u}$

Then $w = \begin{pmatrix} w_+ \\ w_- \\ s \end{pmatrix}$ and we obtain

$$w_t + ikc_0\Lambda_0 w = -\varepsilon k^2 \bar{M}_1 w + \varepsilon^3 (ik)^3 \bar{M}_2 w \,,$$

where $\Lambda_0 = \mathrm{diag}(1, -1, 0)$

• <u>Step 3</u> **Follows from**

Lemma

If $w_t + \Lambda w = \varepsilon A w,\ w \in \mathbb{R}^n,\ \Lambda = \mathrm{diag}(\lambda_1,\ldots,\lambda_n),\ \lambda_i \neq \lambda_j$ for $i \neq j$,

then there is a unique matrix *B* such that

(a) $\mathrm{diag}\, B = 0$ and (b) the substitution $w = y + \varepsilon B y$

leads to equation $$y_t + \Lambda y = \varepsilon\left(\mathrm{diag}\, A\right) y + O\left(\varepsilon^2\right).$$

Proof

$y = \left(1 + \varepsilon B\right)^{-1} w$ and therefore $y_t + \Lambda y = \varepsilon\left(A + \left[B, \Lambda\right]\right) y + O\left(\varepsilon^2\right).$

Let $A = \left\{a_{ij}\right\},\ B = \left\{b_{ij}\right\}$, then $\left[B, \Lambda\right]_{ij} = \left(B\Lambda - \Lambda B\right)_{ij} = b_{ij}\left(\lambda_j - \lambda_i\right).$

Hence $B = \left\{ b_{ij} = \dfrac{a_{ij}}{\lambda_i - \lambda_j},\ i \neq j;\, b_{ii} = 0 \right\}$ completes the proof.

Structure of Chapman-Enskog expansion for linearized Boltzmann equation

Linearization of BE : $f = m + g\sqrt{m}, \quad m = (2\pi)^{-3/2} e^{-|v|^2/2}$

$g(x,v,t):$

$$g_t + v \cdot g_x = \frac{1}{\varepsilon} Lg, \quad g\big|_{t=0} = g_0; \qquad x \in \mathbb{R}^3, v \in \mathbb{R}^3;$$

$$\langle g_1, g_2 \rangle = \int_{\mathbb{R}^3} dv\, g_1^*(v) g_2(v), \quad \|g\|^2 = \langle g, g \rangle, \quad \int_{\mathbb{R}^3} dx \|g_0\|^2 < \infty.$$

Fourier transform:

$$\hat{g}(k,v,t) = \int_{\mathbb{R}^3} dx\, e^{-ik\cdot x} g(\ldots), \qquad \hat{g}_t + ik \cdot v\hat{g} = \frac{1}{\varepsilon} L\hat{g}, \quad \hat{g}\big|_{t=0} = g_{0,} \quad k \in \mathbb{R}^3.$$

Null-space of *L*:

$$N(L)=\left\{g\in L_2\left(\mathbb{R}^3\right),\ Lg=0\right\}$$

$$N=Span\left\{\sqrt{m},\quad v\sqrt{m},\quad |v|^2\sqrt{m}\right\},\qquad dim\,N=5.$$

Basis in *N(L)*:

$$\left\{e_i\left(v,\,{k}\big/{|k|}\right),\quad i=1,\ldots,5\right\}\ \text{such that}\quad \left\langle e_i,e_j\right\rangle=\delta_{ij},$$

$$\left\langle (k\cdot v)\,e_i,e_j\right\rangle=\sqrt{\frac{5}{3}}\,|k|\,\gamma_i\,\delta_{ij},\qquad \gamma_1=-\gamma_2=1,\ \gamma_3=\gamma_4=\gamma_5=0.$$

Hydrodynamic vector:

$$w=\begin{pmatrix}w_1\\ \vdots\\ w_5\end{pmatrix},\qquad w_i=\left\langle e_i,\,g(k,v,t)\right\rangle,$$

$$g(k,v,t)=exp\left[\frac{t}{\varepsilon}A\left(i\varepsilon|k|,\frac{k}{|k|}\right)\right]g_0(k,v),\quad \text{where}$$

$$A(\mu,\omega)=L-\mu(\omega\cdot v),\quad \mu\in\mathbb{C},\quad \omega\in S^2.$$

Lemma (Arsen'ev, 1965).

For hard spheres with diameter $d = \pi^{-1/2}$ there is a number $r_0 > 0$

such that the eigenvalue problem $\lambda g = A(\mu,\omega)\, g, \quad |\mu| \le r_0,$

has exactly **5** linearly independent solutions $\{\lambda_j, g_j;\ \ j = 1,\dots,5\}$

having the following form:

$$\lambda_j = \lambda_j(\mu) = \sum_{n=1}^{\infty} \mu^n \, \lambda_n^{(j)},$$

$$g_j(v;\mu,\omega) = e_j(v;\omega) + \sum_{n=1}^{\infty} \mu^n \, g_n^{(j)}(v;\omega), \quad \langle e_j, g_n^{(j)} \rangle = 0.$$

Connection with Chapman-Enskog expansion

Chapman-Enskog method leads to equation

$$w(k,t): \quad w_t = Uw = \left(\sum_{n=0}^{\infty} \varepsilon^n U_n \right) w,$$

where

$$U_0 = \sqrt{\frac{5}{3}} |k| \operatorname{diag}(1,-1,0,0,0), \quad U_1 = -|k|^2 U^{N-S}, \quad U_2 = (i|k|)^3 U^B, \ldots$$

What can be said about the structure of U ?

Connection with Chapman-Enskog expansion

Proposition

$$U = B\Lambda B^{-1}, \quad \text{where} \quad \Lambda = \frac{1}{\varepsilon}\mathrm{diag}\left\{\lambda_1\left(i\varepsilon|k|\right),\dots,\lambda_5\left(i\varepsilon|k|\right)\right\},$$

$$B = \left\{b_{ij};\, i,j = 1,\dots,5\right\}, \qquad b_{ij} = \left\langle e_i, g_j\left(i\varepsilon|k|, \frac{k}{|k|}\right)\right\rangle.$$

Obviously the series

$$\lambda_j = \sum_{j=1}^{\infty} \lambda_n^{(j)}\left(i\varepsilon|k|\right)^n, \qquad b_{ij} = \delta_{ij} + \sum_{n=1}^{\infty} b_n^{(ij)}\left(i\varepsilon|k|\right)^n; \quad i,j = 1,\dots,5;$$

converge for such $k \in \mathbb{R}^3$ that $|k| \leq r_0/\varepsilon$.

Asymptotic expansion of solutions

Chapman - Enskog equations of hydrodynamics :

$$w_t = B\Lambda B^{-1} w, \qquad w\big|_{t=0} = w_0^{(as)},$$

where $B(\varepsilon), \Lambda(\varepsilon), w_0^{(as)}$ are analytic at $\varepsilon = 0.$

"Diagonal" equations:

$$w = By \quad \Rightarrow \quad y_t = \Lambda y, \quad y\big|_{t=0} = y_0 = B^{-1} w_0^{(as)}$$

(the connection with "changes of variables" is obvious).

Then $w(t) = B\, e^{\Lambda t} y_0$ and we can use **two** independent truncations:

$$w_{nl}(t) = B^{(l)}\, e^{\Lambda^{(n)} t}\, y_0^{(l)}, \qquad n \geq 0,\ l \geq 0, \qquad \text{where}$$

$$\mathrm{B} = \mathrm{B}^{(l)} + O\left(\varepsilon^{l+1}\right), \quad y_0 = y_0^{(l)} + O\left(\varepsilon^{l+1}\right), \quad \Lambda = \Lambda^{(n)} + O\left(\varepsilon^{n+1}\right).$$

Final estimates $\left(\text{for } n = 0,1,2\right)$:

$$\mathfrak{F}_{k \to x}^{-1}\left[w(k,t) - w_{ne}(k,t)\right] = O\left(\varepsilon^{n}\right) + O\left(\varepsilon^{l+1}\right)$$

uniformly in time.

Note that

n=1 for Navier-Stokes and ***n=2*** for Burnett equations.

Conclusions

1

We began with the problem regularization of classical Burnett equations and show that this can be done *by transformation to new hydrodynamic variables.*

It was also shown that the way of truncation of the Chapman-Enskog expansion *is not unique* (even at the Navier-Stokes level).

2

What is the meaning of these transformations?
How to find the optimal one?   Does it exist?

These questions were considered in detail for the linearized Boltzmann equation.

It was shown that the Chapman-Enskog expansion has a special structure which allows to pass to "*diagonal*" equations of hydrodynamics (5 independent equations).

This is the meaning of the optimal transformation in the linear case.

The "diagonal" Navier-Stokes equations are:
three heat equations for two viscous and
one thermal modes respectively.

Plus two linearized Burgers equations for two sound modes.

The corresponding solutions are valid with error
$O(\varepsilon)$ uniformly in time.

4

The "diagonal" Burnett equations are:

the same three heat equations plus two linearized Burgers-KdV equations for sound modes.

The uniform in time error is quadratic in ε in this case provided that the diagonalizing transformation and the asymptotic initial data are also computed with quadratic error.

THANK YOU!......