Regularity for Diffuse Reflection Boundary Problem to the Stationary Linearized Boltzmann Equation in a Convex Domain

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Base on a joint work with Hsia and Kawagoe

Boltzmann Equation

$g(X, \xi, t)$ is the density of mass at position X and time t with velocity ξ. In other words, $g(X, \xi, t)\Delta X \Delta \xi$ is the mass inside the small volume $\Delta X \Delta \xi$ in the phase space.

$$\frac{\partial g}{\partial t} + \sum_{i=1}^{3} \xi_i \frac{\partial g}{\partial X_i} = J(g, g), \tag{1}$$

$$J(g, g) = \tag{2}$$

$$\int_{\mathbb{R}^3} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} (g(\xi')g(\xi'_*) - g(\xi)g(\xi_*))B(|\xi - \xi_*|\theta)d\theta d\phi d\xi_*.$$

$J(g, g)$ is called the collision operator.

Boundary Condition

Because of the hyperbolic nature of the equation, we can only give condition to the incoming part of the velocity distribution function. Let Ω be the domain. If $x \in \partial\Omega$, $n(x)$ denotes the outer normal

$$\Gamma_- = \{(X, \xi) | X \in \partial\Omega,\ \xi \cdot n(X) < 0\}. \quad (3)$$

Some commonly considered boundary conditions:

- In flow
- Spectular reflection
- Diffuse reflection

For $(X, \xi) \in \Gamma_-$,

- In flow

  $g(X, \xi, t) = h(X, \xi, t)$
- Spectular reflection

  $g(X, \xi, t) = g(X, \xi - 2(\xi \cdot n)n, t)$
- Diffuse reflection
  1. The distribution leaving the boundary is in thermal equilibrium with the boundary.
  2. No net flux on the boundary.

Maxwellian

$J(g, g) = 0$, if and only if g is a Gaussian with five parameters:

$$g = \frac{\rho}{(2\pi RT)^{3/2}} e^{-\frac{|\xi - v|^2}{2RT}} = M_{\rho, v, T}(\xi), \tag{4}$$

where R is the gas constant and $v = (v_1, v_2, v_3)$.

Linearized Boltzmann equation

We consider the standard Maxwellian

$$E(\zeta) = \pi^{-\frac{3}{2}} e^{-|\zeta|^2}. \tag{5}$$

We express g as a perturbation of the Maxwellian.

$$g = E + E^{\frac{1}{2}} f. \tag{6}$$

We have the equation for f.

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \zeta_i \frac{\partial f}{\partial x_i} = L(f) + \Gamma(f), \text{ where} \tag{7}$$

$$L(f) = E^{-\frac{1}{2}} \big(J(E^{\frac{1}{2}} f, E) + J(E, E^{\frac{1}{2}} f) \big), \tag{8}$$

$$\Gamma(f) = E^{-\frac{1}{2}} J(E^{\frac{1}{2}} f, E^{\frac{1}{2}} f). \tag{9}$$

We ignore Γ and get the linearized Boltzmann equation.

Cut-off hard potential and cut-off Maxwellian gas

We consider the cross-section:

$$B(|\zeta_* - \zeta|, \theta) = |\zeta_* - \zeta|^\gamma \beta(\theta), \tag{10}$$

where $0 \leq \gamma \leq 1$ and $0 \leq \beta(\theta) \leq C \cos\theta \sin\theta$.

Compare with Grad's angular cutoff:

$$0 \leq B \leq C|\zeta_* - \zeta|^\gamma \cos\theta \sin\theta.$$

Here, the cross section is for the binary collision operator

$$J(F, F) = \int_{\mathbb{R}^3} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} (F'F'_* - FF_*) B(|\zeta_* - \zeta|, \theta) d\theta d\epsilon d\xi_*.$$

Stationary linearized Boltzmann equation in a convex domain in $\mathbb{R}^3$

We consider

$$\begin{cases} \zeta \cdot \nabla f(x,\zeta) = L(f), \\ x \in \Omega, \\ \zeta \in \mathbb{R}^3, \end{cases} \tag{11}$$

where Ω is a C^2 strictly convex bounded domain in $\mathbb{R}^3$.

Existence of solutions:

- Convex domain: Guiraud (1970 J. de Mc.)
- General domain: Esposito, Guo, Kim, and Marra (2013 CMP)

Regularity:

- Continuous alway from the grazing set: Esposito, Guo, Kim, and Marra (2013 CMP)

- (C. 2016 (to appear SIMA)) Local Hölder continuity for incoming boundary condition.

  Key idea: Velocity averaging for stationary equation.

Regularity for the time evolutional problem (weakly nonlinear):

$$\frac{\partial}{\partial t} f(x, \zeta, t) + \zeta \cdot \nabla_x f(x, \zeta, t) = L(f) + \Gamma(f, f), \tag{12}$$

- Kim (2011 CMP): Discontinuity from boundary in a nonconvex domain.
- Guo, Kim, Tonon, and Trescases (2016 ARMA): BV estimate in a nonconvex domain.
- Guo, Kim, Tonon, and Trescases (2016 Inv. Math.) : Regularity in a convex domain.

All these results are NOT uniform in time.

In (2016 Inv. Math.), they establish weighted C^1 estimate, which grows severely with time. This motivates us to look at the regularity to the stationary solution directly.

Properties of the collision operator

$$L(f) = -\nu(|\zeta|)f + K(f),$$

$$K(f)(x,\zeta) = \int_{\mathbb{R}^3} k(\zeta,\zeta_*)f(x,\zeta_*)d\zeta_*,$$

$$\nu(|\zeta|) = \beta_0 \int_{\mathbb{R}^3} e^{-|\eta|^2}|\eta-\zeta|^\gamma d\eta$$

$$\nu_0(1+|\zeta|)^\gamma \leq \nu(|\zeta|) \leq \nu_1(1+|\zeta|)^\gamma.$$

Estimates for Kernel

Let $0 < \delta < 1$.

(Caflisch 1980):

$|k(\zeta, \zeta_*)| \leq$

$$C_1|\zeta - \zeta_*|^{-1}(1 + |\zeta| + |\zeta_*|)^{-(1-\gamma)}e^{-\frac{1}{4}(1-\delta)\left(|\zeta-\zeta_*|^2+(\frac{|\zeta|^2-|\zeta_*|^2}{|\zeta-\zeta_*|})^2\right)},$$

(C., Hsia 2015):

$|\bigtriangledown_\zeta k(\zeta, \zeta_*)| \leq$

$$C_2\frac{1 + |\zeta|}{|\zeta - \zeta_*|^2}(1 + |\zeta| + |\zeta_*|)^{-(1-\gamma)}e^{-\frac{1}{4}(1-\delta)\left(|\zeta-\zeta_*|^2+(\frac{|\zeta|^2-|\zeta_*|^2}{|\zeta-\zeta_*|})^2\right)}.$$

Definition of solution

We define

$$p(x,\zeta) : \text{backward trajectory} \cap \partial\Omega,$$
$$\tau_-(x,\zeta) : \text{traveling time.}$$

We write

$$\zeta \cdot \nabla f(x,\zeta) + \nu(|\zeta|)f(x,\zeta) = K(f). \tag{13}$$

Integral equation:

$$f(x,\zeta) = f(p(x,\zeta),\zeta)e^{-\nu\tau_-(x,\zeta)} + \int_0^{\tau_-(x,\zeta)} e^{-\nu s}K(f)(x-\zeta s,\zeta)ds. \tag{14}$$

We say $f(x,\zeta)$ is a solution to the stationary linearized Boltzmann equation if the integral equation is satisfied a. e.

Main Theorem

Theorem (C., Hsia, Kawagoe, 2017)

Let Ω be a C^2 strictly convex domain in $\mathbb{R}^3$ and $f \in L^\infty_{x,\zeta}$ be a stationary solution to the diffuse reflection boundary problem on Ω, (11)*, for hardsphere, cutoff hard potential, or Maxwellian molecular gases,* (10)*. Suppose the derivative of the boundary temperature is bounded. Then, for $\epsilon > 0$,*

$$\sum_{i=1}^{3} |\frac{\partial}{\partial_{x_i}} f(x,\zeta)| + \sum_{i=1}^{3} |\frac{\partial}{\partial_{\zeta_i}} f(x,\zeta)| \le C(1 + d_x^{-1})^{\frac{4}{3}+\epsilon}, \tag{15}$$

where d_x is the distance between x and $\partial\Omega$.

Sketch of proof

Diffuse reflection boundary condition for linearized Boltzmann equation:
Let $T(x)$ be the temperature on the boundary.
For $x \in \partial\Omega$ and $\zeta \cdot n(x) < 0$,

$$f(x,\zeta) = \sigma(x) M^{\frac{1}{2}} + T(x)(|\zeta|^2 - \frac{3}{2}) M^{\frac{1}{2}}, \tag{16}$$

where

$$M = M(\zeta) = \pi^{-\frac{3}{2}} e^{-|\zeta|^2}.$$

$$\sigma(x) = -\frac{1}{2} T(x) + 2\sqrt{\pi} \int_{\zeta \cdot n > 0} f(x,\zeta) |\zeta \cdot n| M^{\frac{1}{2}} d\zeta. \tag{17}$$

Let

$$\psi(x) := 2\sqrt{\pi}\int_{\zeta\cdot n>0} f(x,\zeta)|\zeta\cdot n|M^{\frac{1}{2}}d\zeta. \tag{18}$$

Substitute f above by the integral equation (14) and boundary condition (16) and (17).

$$\begin{aligned}\psi(x) =& 2\sqrt{\pi}\int_{\zeta\cdot n>0} T(p(x,\zeta))(|\zeta|^2-2)M(\zeta)e^{-\nu(|\zeta|)\tau_-(x,\zeta)}|\zeta\cdot n|d\zeta\\ &+2\sqrt{\pi}\int_{\zeta\cdot n>0}\psi(p(x,\zeta))M(\zeta)e^{-\nu(|\zeta|)\tau_-(x,\zeta)}|\zeta\cdot n|d\zeta\\ &+2\sqrt{\pi}\int_{\zeta\cdot n>0}\int_0^{\tau_-(x,\zeta)} e^{-\nu(|\zeta|)s}K(f)(x-s\zeta,\zeta)M^{\frac{1}{2}}(\zeta)|\zeta\cdot n|dsd\zeta\\ =:& B_T+B_\psi+D_f.\end{aligned} \tag{19}$$

Sketch of proof:

- ψ is bounded provided $f \in L^{\infty}_{x,\zeta}$
- First derivatives of B_T, B_ψ are bounded provided T, ψ are bounded.
- D_f is Hölder continuous provided $f \in L^{\infty}_{x,\zeta}$.

Now, we can conclude f is locally Hölder continuous by using analysis in (C. 2016). In this research, we further improve the regularity to differentiability.

- D_f is bounded differentiable provided f is locally Hölder up to boundary.
- We have the desired estimate for first derivatives of f provided derivatives of ψ, T are bounded and f is locally Hölder.

Recall

$$D_f(x) = 2\sqrt{\pi}\int_{\zeta\cdot n>0}\int_0^{\tau_-(x,\zeta)} e^{-\nu(|\zeta|)s} K(f)(x-s\zeta,\zeta) M^{\frac{1}{2}}(\zeta)|\zeta\cdot n| ds d\zeta. \tag{20}$$

Proposition (C. Hsia 2015)

Let $1 \leq p \leq \infty$.

$$\| \bigtriangledown_\zeta K(f)(x,\zeta)\|_{L^p_\zeta} \leq C\|f(x,\zeta)\|_{L^p_\zeta}. \tag{21}$$

Notice that $\| \bigtriangledown_\zeta k(\zeta,\zeta_*)\|_{L^\infty_\zeta L^1_{\zeta_*}}$ and $\| \bigtriangledown_\zeta k(\zeta,\zeta_*)\|_{L^\infty_{\zeta_*} L^1_\zeta}$ are bounded. By an argument similar to the proof of Young's inequality, we have the proposition.

Transfer regularity from velocity to space

Idea: Combination of averaging or collision and transport can transfer regularity in velocity to space. For time evaluational problem in whole space,

- Velocity averaging lemma (Golse, Perthame, Sentis 1985)
- Mixture lemma (Liu, Yu 2004)

In present research, we realize this effect for stationary problem in a convex domain by interplaying between velocity and space.

Let $-n(x),\ e_2,\ e_3$ be an orthonormal basis. Let

$$\begin{aligned}\zeta' &= -\rho\cos\theta n(x) + \rho\sin\theta\cos\phi e_2 + \rho\sin\theta\sin\phi e_3,\\ r &= \rho s,\\ \hat{\zeta}' &= \frac{\zeta'}{|\zeta'|}.\end{aligned} \tag{22}$$

Then,

$$\begin{aligned}D_f =& 2\pi^{-\frac{1}{4}} \int_0^\infty \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^{|x-p(x,\zeta)|}\\ & e^{-\frac{\nu(\rho)}{\rho}r} K(f)(x - r\hat{\zeta}, \zeta)|\hat{\zeta} \cdot n(x)| e^{-\frac{|\zeta|^2}{2}} \rho^2 \sin\theta dr d\phi d\theta d\rho.\end{aligned} \tag{23}$$

$$D_f = 2\pi^{-\frac{1}{4}} \underbrace{\int_0^\infty \int_0^{\frac{\pi}{2}} \int_0^{2\pi}}_{\zeta\cdot n(x)<0} \int_0^{|x-p(x,\zeta)|} e^{-\frac{\nu(\rho)}{\rho}r} K(f)(x - r\hat{\zeta}, \zeta)|\hat{\zeta} \cdot n(x)| e^{-\frac{|\zeta|^2}{2}} \rho^2 \sin\theta \, dr d\phi d\theta d\rho. \tag{24}$$

$$D_f = 2\pi^{-\frac{1}{4}} \int_0^\infty \underbrace{\int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^{|x-p(x,\zeta)|}}_{\Omega} e^{-\frac{\nu(\rho)}{\rho}r} K(f)(x - r\hat{\zeta}, \zeta)|\hat{\zeta} \cdot n(x)| e^{-\frac{|\zeta|^2}{2}} \rho^2 \sin\theta \, dr d\phi d\theta d\rho. \tag{25}$$

Let $y = x - r\hat{\zeta}$

$$D_f = 2\pi^{-\frac{1}{4}} \int_0^\infty \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^{|x-p(x,\zeta)|} e^{-\frac{\nu(\rho)}{\rho} r} K(f)(x - r\hat{\zeta}, \zeta)|\hat{\zeta} \cdot n(x)| e^{-\frac{\rho^2}{2}} \rho^2 \sin\theta dr d\phi d\theta d\rho$$
$$= 2\pi^{-\frac{1}{4}} \int_0^\infty \int_\Omega e^{-\frac{\nu(\rho)}{\rho}|x-y|} K(f)(y, \rho\frac{(x-y)}{|x-y|}) \frac{(x-y)\cdot n(x)}{|x-y|^3} e^{-\frac{\rho^2}{2}} \rho^2 dy d\rho$$
$$= 2\pi^{-\frac{1}{4}} \int_0^\infty \int_\Omega \int_{\mathbb{R}^3} e^{-\frac{\nu(\rho)}{\rho}|x-y|} k(\rho\frac{x-y}{|x-y|}, \zeta') f(y, \zeta') \frac{(x-y)\cdot n(x)}{|x-y|^3} e^{-\frac{\rho^2}{2}} \rho^2 d\zeta' dy d\rho. \quad (26)$$

Notice that the velocity is represented by space variable. However, if we differentiate directly, the integrant has a singularity of order

$$\frac{1}{|x-y|^3},$$

which is not integrable in $\Omega \subset \mathbb{R}^3$. This is the reason the result of (C. 2016) is restricted to Hölder continuity.

Suppose $g(t)$ is a smooth curve on $\partial\Omega$ passing x and

$$g(0) = x,$$
$$g'(0) = v,$$

where $v \in T_x\partial\Omega$. We define

$$\nabla_v^x F(x) = \frac{d}{dt} F(g(t))\Big|_{t=0}. \tag{27}$$

$$
\begin{aligned}
\nabla_v^x D_f(x) &= \int_0^\infty \int_\Omega \int_{\mathbb{R}^3} \nabla_v^x \left(e^{-\frac{\nu(\rho)}{\rho}|x-y|} k(\rho \frac{x-y}{|x-y|}, \zeta') \frac{(x-y)\cdot n(x)}{|x-y|^3} \right) \\
&\quad \cdot \left[f(y,\zeta') - f(x,\zeta') \right] e^{-\frac{\rho^2}{2}} \rho^2 d\zeta' dy d\rho \\
&- \int_0^\infty \int_{\mathbb{R}^3} \int_\Omega \mathrm{div}_y \\
&\quad \left(e^{-\frac{\nu(\rho)}{\rho}|x-y|} k(\rho \frac{x-y}{|x-y|}, \zeta') \frac{(x-y)\cdot n(x)}{|x-y|^3} e^{-\frac{\rho^2}{2}} \rho^2 f(x,\zeta') v \right) dy d\zeta' d\rho \\
&+ \int_0^\infty \int_{\mathbb{R}^3} \int_\Omega \\
&\quad e^{-\frac{\nu(\rho)}{\rho}|x-y|} k(\rho \frac{x-y}{|x-y|}, \zeta') \frac{(x-y)\cdot \nabla_v^x(n(x))}{|x-y|^3} e^{-\frac{\rho^2}{2}} \rho^2 f(x,\zeta') dy d\zeta' d\rho \\
&=: \nabla^x D_f^1 + \nabla^x D_f^2 + \nabla^x D_f^3.
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
\nabla^x D_f^{2,\epsilon} = &- \int_0^\infty \int_{\mathbb{R}^3} \int_{\partial\Omega \setminus B(x,\epsilon)} e^{-\frac{\nu(\rho)}{\rho}|x-y|} k(\rho \frac{x-y}{|x-y|}, \zeta') \frac{(x-y)\cdot n(x)}{|x-y|^3} \\
&\quad e^{-\frac{\rho^2}{2}} \rho^2 f(x,\zeta')[v \cdot n(y)] dA(y) d\zeta' d\rho \\
&- \int_0^\infty \int_{\mathbb{R}^3} \int_{\partial B(x,\epsilon)\cap\Omega} e^{-\frac{\nu(\rho)}{\rho}|x-y|} k(\rho \frac{x-y}{|x-y|}, \zeta') \frac{(x-y)}{|x-y|} \cdot n(x) \\
&\quad e^{-\frac{\rho^2}{2}} \rho^2 f(x,\zeta')[v \cdot \frac{x-y}{|x-y|}] \frac{1}{\epsilon^2} dA(y) d\zeta' d\rho \\
&=: S^\epsilon + B^\epsilon.
\end{aligned} \tag{29}
$$

Thank you!