Nonexistence of subsonic irrotational flow around bodies with several protruding corners

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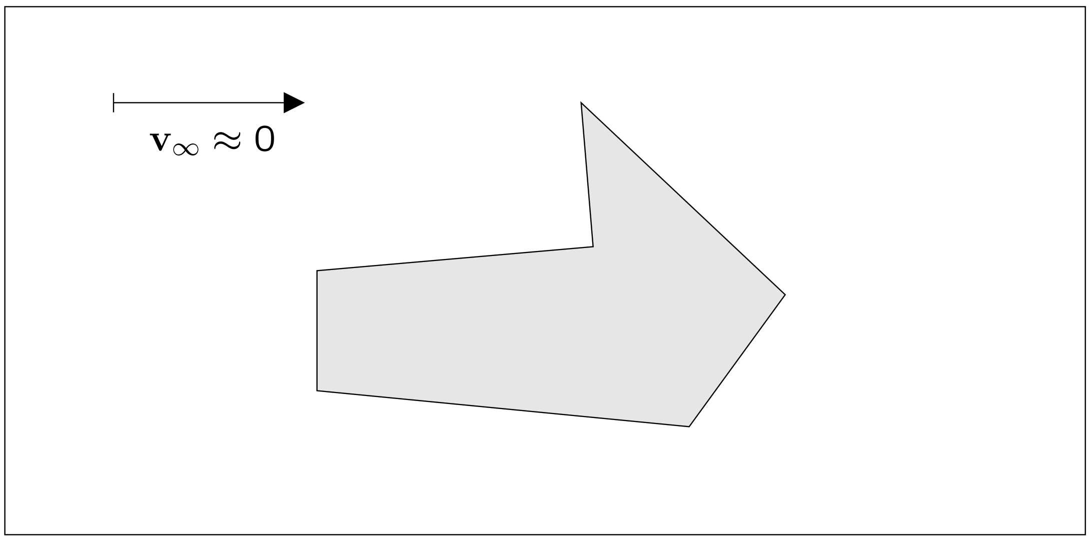


1. **Speed of sound**

Compressible (isentropic) Euler equations:

$$0 = \overbrace{(\partial_t + \mathbf{v}\cdot\nabla)}^{=D_t}\,\varrho + \varrho\nabla\cdot\mathbf{v} \qquad [\varrho \text{ mass density, } \mathbf{v} \text{ velocity}]$$

$$0 = (\partial_t + \mathbf{v}\cdot\nabla)\,\mathbf{v} + \nabla p \qquad [p = \widehat{p}(\varrho),\ \widehat{p}_\varrho = \varrho^{-1}\widehat{P}_\varrho,\ P = \widehat{P}(\varrho) \text{ pressure}]$$

Linearize around $\varrho = \overline{\varrho} = \text{const} > 0$, $\mathbf{v} = \overline{\mathbf{v}} = 0$:

$$\partial_t| \quad 0 = \partial_t\varrho + 0\cdot\nabla\varrho + \mathbf{v}\cdot\overbrace{\nabla\overline{\varrho}}^{=0} + \varrho\nabla\cdot 0 + \overline{\varrho}\nabla\cdot\mathbf{v}$$

$$-\nabla\cdot| \quad 0 = \partial_t\mathbf{v} + 0\cdot\nabla\mathbf{v} + \mathbf{v}\cdot\nabla 0 + \widehat{p}_\varrho(\overline{\varrho})\nabla\varrho$$

$$\Rightarrow \quad 0 = \partial_t^2\varrho - \overline{c}^2\Delta\varrho, \quad \overline{c} = \sqrt{\overline{\varrho}\widehat{p}_\varrho(\overline{\varrho})} \text{ sound speed}$$

2. **Vorticity**

$$
\begin{aligned}
0 &= v^x_t + v^x v^x_x + v^y v^x_y + p_x \\
&= v^x_t + v^x v^x_x + v^y v^y_x + p_x \qquad - v^y (v^y_x - v^x_y) \\
&= v^x_t + \underbrace{\Big(\frac{(v^x)^2 + (v^y)^2}{2} + p \Big)}_{=B=\text{Bernoulli const.}}{}_x \qquad - v^y \underbrace{\nabla \times \mathbf{v}}_{=\omega=\text{vorticity}} \\
&= v^x_t + B_x - v^y \omega \quad \Rightarrow \qquad 0 = v^x_{yt} + B_{xy} - v^y \omega_y - v^y_y \omega \\
0 &= v^y_t + B_y + v^x \omega \quad \Rightarrow \qquad 0 = v^y_{xt} + B_{yx} + v^x \omega_x + v^x_x \omega
\end{aligned}
$$

$$
\begin{aligned}
0 &= \omega_t + \mathbf{v} \cdot \nabla \omega + \omega \nabla \cdot \mathbf{v} \\
0 &= \varrho_t + \mathbf{v} \cdot \nabla \varrho + \varrho \nabla \cdot \mathbf{v} \\
0 &= (\omega/\varrho)_t + \mathbf{v} \cdot \nabla (\omega/\varrho)
\end{aligned}
$$

Smooth flow: ω/ϱ conserved along streamlines.

$\omega = 0$ at $t = 0 \quad \Rightarrow \quad 0 = \omega$ for *all* t

3. **Bernoulli relation**

2d steady flow ($\partial_t = 0$): $\omega = 0$ yields

$$0 = v_t^x + B_x - v^y \omega$$
$$0 = v_t^y + B_y + v^x \omega$$
$$B = \text{Bernoulli constant} = \frac{1}{2}|\mathbf{v}|^2 + \widehat{p}(\varrho)$$
$$\varrho = \widehat{p}^{-1}\Big(\text{const} - \frac{1}{2}|\mathbf{v}|^2\Big)$$

Polytropic equation of state:

$$\widehat{p}(\varrho) = \varrho^{\gamma-1} \quad \Rightarrow \quad \widehat{p}^{-1}(x) = x^{\frac{1}{\gamma-1}}$$

$\gamma > 1$ (helium $\gamma = 5/3$, $\gamma = 7/5$ air, ...)

$$\varrho = \Big(\text{const} - \frac{1}{2}|\mathbf{v}|^2\Big)^{5/2} \quad \rightsquigarrow \quad \text{undefined for } |\mathbf{v}| > \sqrt{2\text{const}}$$

$$\mathbf{v} = \sqrt{2\text{const}} \quad \Rightarrow \quad \varrho = 0 = c \qquad \text{(vacuum, cavitation)}$$
$$\mathbf{v} = 0 \quad \Rightarrow \quad \varrho = \varrho_{\text{max}}, \ c = c_{\text{max}}$$

Compressible flow does not allow unbounded velocity fields

4. **Stream function formulation**

$0 = \nabla \times \mathbf{v} \quad \Rightarrow \quad \mathbf{v} = \nabla\phi \quad$ [ϕ scalar (multi-valued) *potential*]

$0 = \nabla \cdot (\varrho\mathbf{v}) \quad \Rightarrow \quad \varrho\mathbf{v} = -\nabla^{\perp}\psi \quad$ [ψ scalar *stream function*]

Bernoulli const $= \hat{p}(\varrho) + \frac{1}{2}|\mathbf{v}|^2 = \hat{p}(\varrho) + \frac{1}{2}\varrho^{-2}|\nabla\psi|^2 =: f(\varrho, |\nabla\psi|)$

$\frac{\partial f}{\partial \varrho} = \hat{p}_\varrho(\varrho) - \varrho^{-3}|\nabla\psi|^2 = \varrho^{-1}(c^2 - |\mathbf{v}|^2) \quad > 0 \quad$ if $\quad |\mathbf{v}| < c \quad$ [subsonic]

Implicit function thm.: $\quad \varrho = \hat{\varrho}(|\nabla\psi|) \quad$ for some function $\hat{\varrho}$

$$0 = \omega = \nabla \times \mathbf{v} = -\nabla \times (\frac{1}{\varrho}\nabla^{\perp}\psi) = -\nabla \cdot \Big(\frac{1}{\hat{\varrho}(|\nabla\psi|)}\nabla\psi\Big)$$

$$0 = \big(c^2 - (v^x)^2\big)\psi_{xx} - 2v^x v^y \psi_{xy} + \big(c^2 - (v^y)^2\big)\psi_{yy}$$

$$\boxed{0 = (I - c^{-2}\mathbf{v}^2) : \nabla^2\psi} \qquad [A : B = \sum_{i,j} A_{ij}B_{ij}, \mathbf{a}^2 = \mathbf{a}\mathbf{a}^T]$$

Eigenvalues 1 and $1 - \frac{|\mathbf{v}|}{c}$: elliptic if and only if $|\mathbf{v}| < c$ (subsonic).

5. **Incompressible limit**

Limit of *Mach number* $M := |\mathbf{v}|/c$ vanishing:

sequence (ψ^n) of potential flows for *same* $\hat{p}$ and Bernoulli constant,

$$\sup_x |M^n| \to 0 \quad \Rightarrow \quad \mathbf{v}^n \to 0 \quad , \quad c^n \to c_{\text{max}} \quad , \quad \varrho^n \to \varrho_{\text{max}}$$

Assume $\nabla\psi^n \to \nabla\psi^\infty$ in $\mathcal{C}^0$ (after scaling; proof later).

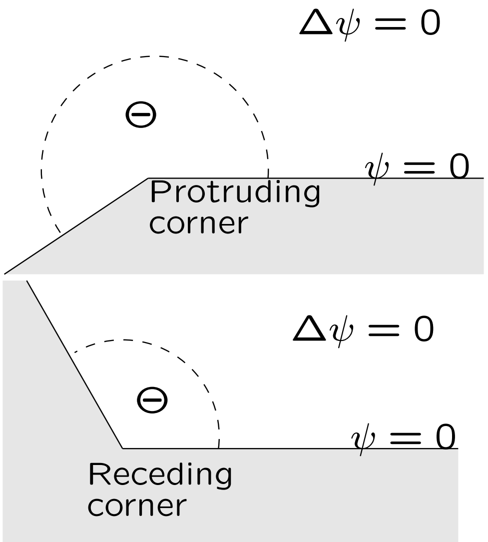
$$0 = \nabla \cdot \Big(\frac{1}{\varrho^n}\nabla\psi^n\Big) \to \nabla \cdot \Big(\frac{1}{\varrho_{\text{max}}}\nabla\psi^\infty\Big) \quad \Rightarrow \quad 0 = \Delta\psi^\infty$$

Harmonic ψ represent *incompressible potential flows*:

$$\nabla \times \mathbf{v} = 0 = \nabla \cdot \mathbf{v} \quad , \quad \varrho = \text{const} \quad , \quad 0 = \mathbf{v}\cdot\nabla\mathbf{v} + \nabla p$$

(p found by solving $0 = \nabla \cdot [(\mathbf{v}\cdot\nabla\mathbf{v}) + \nabla p] = |\nabla\mathbf{v}|^2 + \Delta p$.)

6. **Slip condition at solid boundaries**



Slip condition at solid boundaries:

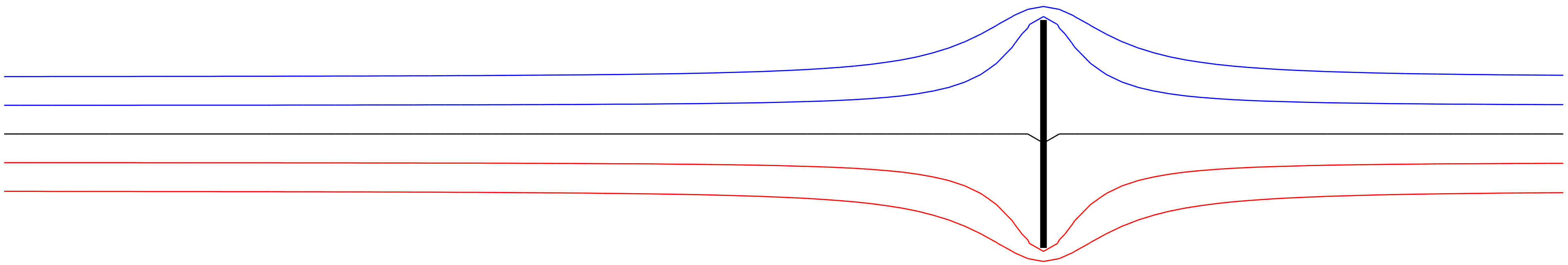
($\mathbf{n}$ normal, $\mathbf{s}$ tangent):

$0 = \mathbf{n} \cdot \mathbf{v} = \mathbf{n} \cdot \nabla^{\perp}\psi = \mathbf{s} \cdot \nabla\psi$;

integrate: $\boxed{\psi = \text{const} = 0}$

7. **Incompressible flow around vertical plate**

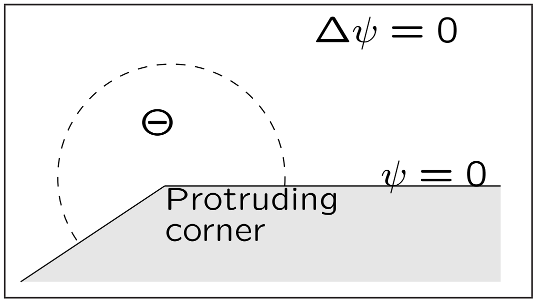
Level sets of ψ (positive, negative, zero)



8. **Elliptic corner regularity**

Polar coordinates (r, θ):

$$0 = \Delta\psi = r^{-1}(r\psi_r)_r + r^{-2}\psi_{\theta\theta} \overset{\ell=\log r}{=} r^{-2}(\psi_{\ell\ell} + \psi_{\theta\theta}) \quad \text{in interior}$$



Homogeneous solutions:

$$\psi = e^{\alpha(\ell+i\theta)} = r^{\alpha}e^{i\alpha\theta} \qquad \leftarrow \text{not } H^1 \text{ if } \operatorname{Re}\alpha < 0$$

Dirichlet boundaries $\rightsquigarrow$ asymptotic expansion

$$\psi = a_1 r^{\pi/\Theta} \sin\frac{\pi\theta}{\Theta} + a_2 r^{2\pi/\Theta} \sin\frac{2\pi\theta}{\Theta} + a_3 r^{3\pi/\Theta} \sin\frac{3\pi\theta}{\Theta} + \ldots$$

$$\rho\mathbf{v} = \nabla^{\perp}\psi \sim a_1 \underbrace{r^{\pi/\Theta-1}\ldots}_{\text{blowup!}} + a_2 \underbrace{r^{2\pi/\Theta-1}\ldots}_{\text{slow decay}} + a_3 \underbrace{r^{3\pi/\Theta-1}\ldots}_{\text{faster decay}} + \ldots$$

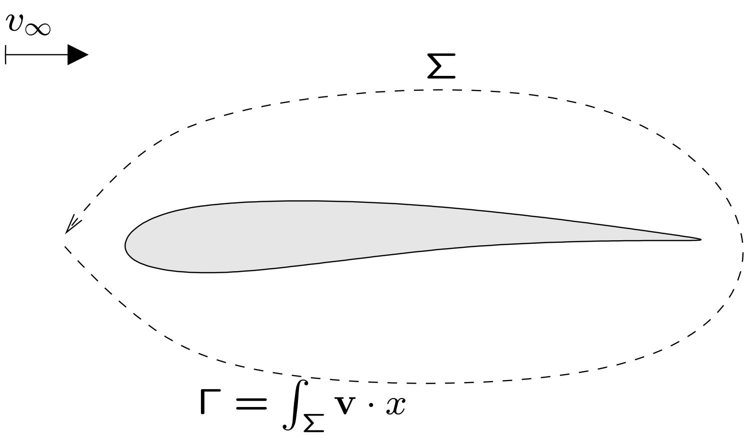
Protruding: $\pi < \Theta < 2\pi$, so $\pi/\Theta - 1 < 0$ — unbounded velocity!

a_1 must happen to be 0. If so: $\psi = 0$ or

$$\psi = \overbrace{\underbrace{a_k}_{\neq 0} r^{k\pi/\Theta}}^{\text{slowest decay}} \underbrace{\sin\frac{k\pi\theta}{\Theta}}_{\text{sign change!}} + a_{k+1} r^{(k+1)\pi/\Theta} \sin\frac{(k+1)\pi\theta}{\Theta} + \ldots$$

Hence path from corner into $\{\psi > 0\}$ and same for $\{\psi < 0\}$.

9. **Kutta-Joukowsky condition**



$$\nabla \times \mathbf{v} = 0 \quad \Rightarrow \quad \text{circulation } \Gamma = \int_\Sigma \mathbf{v} \cdot dx$$

constant as Σ shifted

Can prove:

1. $\forall\ \Gamma\ \exists_1$ incompressible potential flow
2. Exactly one Γ yields bounded $\mathbf{v}$

Kutta-Joukowsky theory: use flow for that Γ as physical model

$\rightsquigarrow$ useful formula for lift

Compressible case:

Frankl/Keldysh '34, Shiffman '52, Bers '54, Finn/Gilbarg '57, ...

10. **New theorems**

2 protruding corners: for certain profiles (incl. flat plates),
no $\mathbf{v} \neq 0$ compressible flows in "most" cases,
existence in "special" cases [E., CPAA, to appear]

≥ 3 protruding corners: non nonzero uniformly subsonic flows if:

- $\hat{P}(\varrho)$ analytic [E., submitted], e.g. $\hat{P}(\varrho) = \varrho^{\gamma}$, or
- $\sup_x M \ll 1$ around nondegenerate polygon [E., JDE 2017]

Theorem: given profile with at least one protruding corner,
if no incompressible flows with *bounded* $\mathbf{v} \neq 0$,
then no nontrivial low-Mach number compressible flows

Theorem: no nontrivial uniformly subsonic flows around infinite protruding rounded corners [E., submitted]

$\rightsquigarrow$ $\omega = 0$ flow rather limited beyond one-corner case

11. **Expansion at infinity**

Incompressible: $\nabla \times \mathbf{v} = 0 = \nabla \cdot \mathbf{v} \quad \Leftrightarrow$
complex velocity $w = v^x - iv^y$ holomorphic in $z = x + iy$.
Laurent expansion: w_∞ velocity at ∞ (real positive), Γ *circulation*

$$w(z) = w_\infty + \frac{\Gamma}{2\pi i} z^{-1} + c_2\, z^{-2} + ... = \frac{d\Phi}{dz}$$

where Φ *complex velocity potential*:

$$\Phi = \phi + i\psi = w_\infty z + \frac{\Gamma}{2\pi i} \log z + ...$$

Compressible [Finn/Gilbarg 1957 and earlier]:

$$\psi = \varrho_\infty \left(v^x_\infty y - \frac{\Gamma}{2\pi} \beta \log \sqrt{x^2 + \beta^2 y^2} + ...\right)$$

$\beta = \sqrt{1 - M_\infty^2}$ *Prandtl-Glauert factor.*

12. **Uniqueness** (incompressible, polygon)

$$\psi = \operatorname{Im}\left(w_\infty z + \frac{\Gamma}{2\pi i} \log z + c + o(1) \right) \quad \text{as } |z| \to \infty$$

Theorem (classical): given w_∞ there is at most one bounded-$\mathbf{v}$ flow ψ around a nondegenerate polygon.

Proof: assume there are two, ψ_0, ψ_1, difference $d = \psi_0 - \psi_1$.

- Nondegenerate polygon: at least one protruding corner, ∇d bounded
  $\Rightarrow d$ attains both signs nearby
- $d = 0$ at body (slip condition)
- Exterior of the body: $\Delta d = 0 \Rightarrow d$ no extrema unless constant
  $\Rightarrow d$ attains nonzero extrema only at infinity.
- At infinity:

$$d = \psi_0 - \psi_1 = \underbrace{(w_\infty - w_\infty)}_{0} y - \frac{\Gamma_0 - \Gamma_1}{2\pi} \log r + c_0 - c_1 + o(1) \quad \text{as } |x + iy| \to \infty$$

1. $\Gamma_0 - \Gamma_1 \neq 0 \Rightarrow \log r$, d single-signed near $\infty \Rightarrow$ everywhere ↯
2. $c_0 - c_1 \neq 0 \Rightarrow d$ single-signed near $\infty \Rightarrow$ everywhere ↯
3. Rest is $o(1)$, so $d = 0$, trivial ↯

Compressible (non-polygon) case: Finn/Gilbarg 1957 & earlier

13. **Incompressible limit (rigorous)**

Assume: $\exists$ potential flows (ψ_n) for *same* $\widehat{p}$ and Bernoulli constant,

$$M^n \neq 0 \text{ at } |x| \to \infty, \text{ but } \sup_x M^n \to 0, \quad \text{so} \quad \mathbf{v}^n \to 0,\ c^n \to c_{\max},\ \varrho^n \to \varrho_{\max}$$

Strategy: $\mathbf{v}^n \to$ incompressible *nonzero* bounded $\mathbf{v} \rightsquigarrow$ contradiction

$$\text{Pick } \epsilon^n \searrow 0 \text{ so that } \operatorname{Var} \nabla\psi^n/\epsilon^n := \operatorname{diam}\{\nabla\psi^n/\epsilon^n\} = 1$$

$\nabla\psi^n = 0$ at every profile *corner*, so $\nabla\psi^n/\epsilon^n$ bounded, n-uniformly

$$0 = \overbrace{\left(I - (\frac{\mathbf{v}^n}{c^n})^2\right)}^{n,x\text{-unif. elliptic}} : \nabla^2 \frac{\psi^n}{\epsilon^n} \underset{(1932)}{\overset{\text{Morrey}}{\Rightarrow}} \nabla \frac{\psi^n}{\epsilon^n} \in \mathcal{C}^{0,\alpha}(\overline{\Omega} \cup \{\infty\}),\ n\text{-uniformly}$$

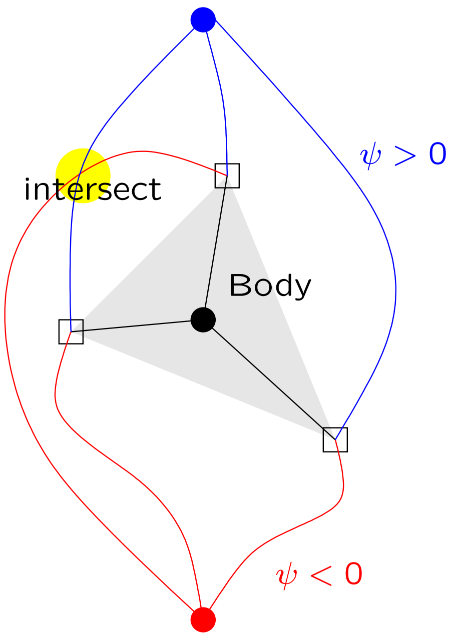
Subsequence: $\nabla\psi^n/\epsilon^n \to \nabla\overline{\psi}$ in $\mathcal{C}^0(\overline{\Omega} \cup \{\infty\})$

$$1 = \operatorname{Var} \nabla\psi^n/\epsilon^n \to \operatorname{Var} \nabla\overline{\psi} \quad \Rightarrow \quad \text{nontrivial limit}$$

$$0 = \nabla \cdot \left(\frac{1}{\varrho^n} \nabla\psi^n/\epsilon^n\right) = \nabla \cdot \left(\frac{1}{\varrho_{\max}} \nabla\overline{\psi}\right) = \frac{1}{\varrho_{\max}} \Delta\overline{\psi}$$

An incompressible flow with bounded nontrivial $\mathbf{v}$ exists!

14. **Incompressible flows with ≥ 3 protruding corners**



Slip condition: $\overline{\psi} = 0$ at body

Strong maximum principle for $0 = \Delta\overline{\psi}$

$\Rightarrow \{\overline{\psi} > 0\}, \{\overline{\psi} < 0\}$ no bounded components

Near ∞: $\overline{\psi}$ smooth, $-\varrho_\infty \mathbf{v}_\infty^\perp = \nabla\overline{\psi}(\infty) \neq 0$

$\Rightarrow \{\overline{\psi} > 0\}, \{\overline{\psi} < 0\} \approx$ halfspaces, connected

If $\mathbf{v}$ bounded at a corner:

local path through $\{\overline{\psi} > 0\}, \{\overline{\psi} < 0\}$ to corner

$\rightsquigarrow$ 3 local paths —; connect,

make disjoint except for 1 exterior vertex •

same for $\{\overline{\psi} < 0\}$ — •

same for body — •

All paths disjoint $\Rightarrow$ contradiction to "utility graph theorem":

Round dots = 3 utilities (gas, water, electricity)

Empty squares = 3 houses

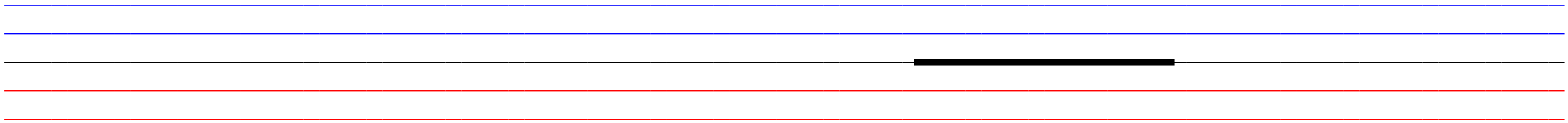
No planar graph with 1 edge between each house and each utility

Conclusion: any nontrivial incompressible flow has unbounded $\mathbf{v}$

Conclusion 2: no nontrivial low-Mach compressible flows

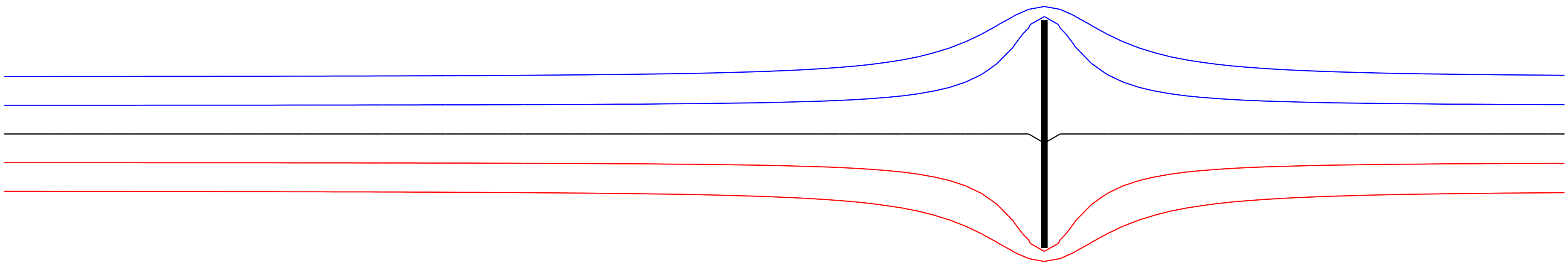
15. **Trivial flow around horizontal plate**

Level sets of ψ (positive, negative, zero)



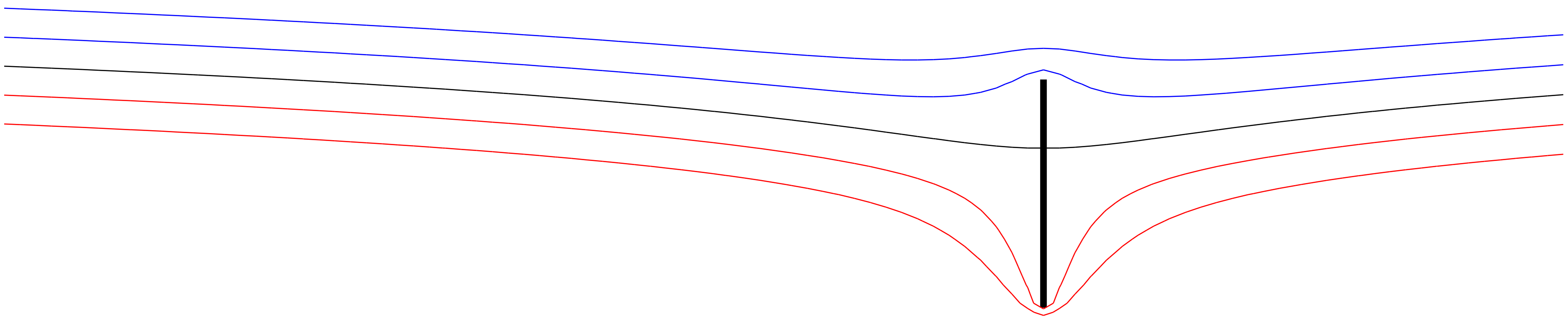
16. **Incompressible flow around vertical plate**

Level sets of ψ (positive, negative, zero)



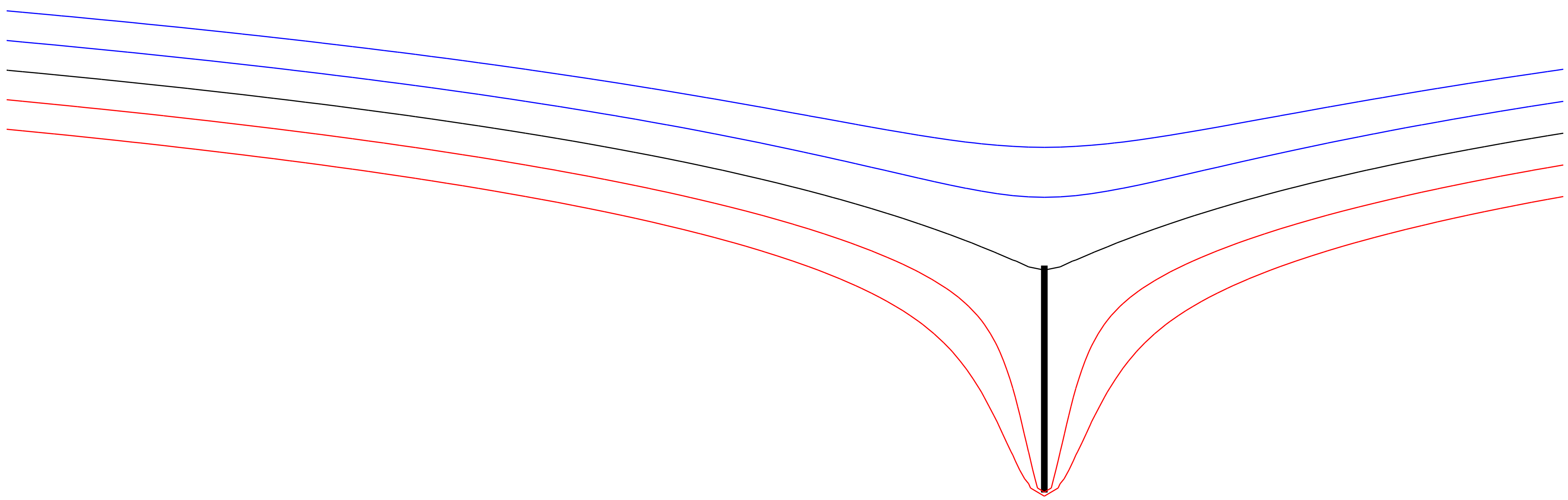
17. **Vertical plate with circulation**

Level sets of ψ (positive, negative, zero)



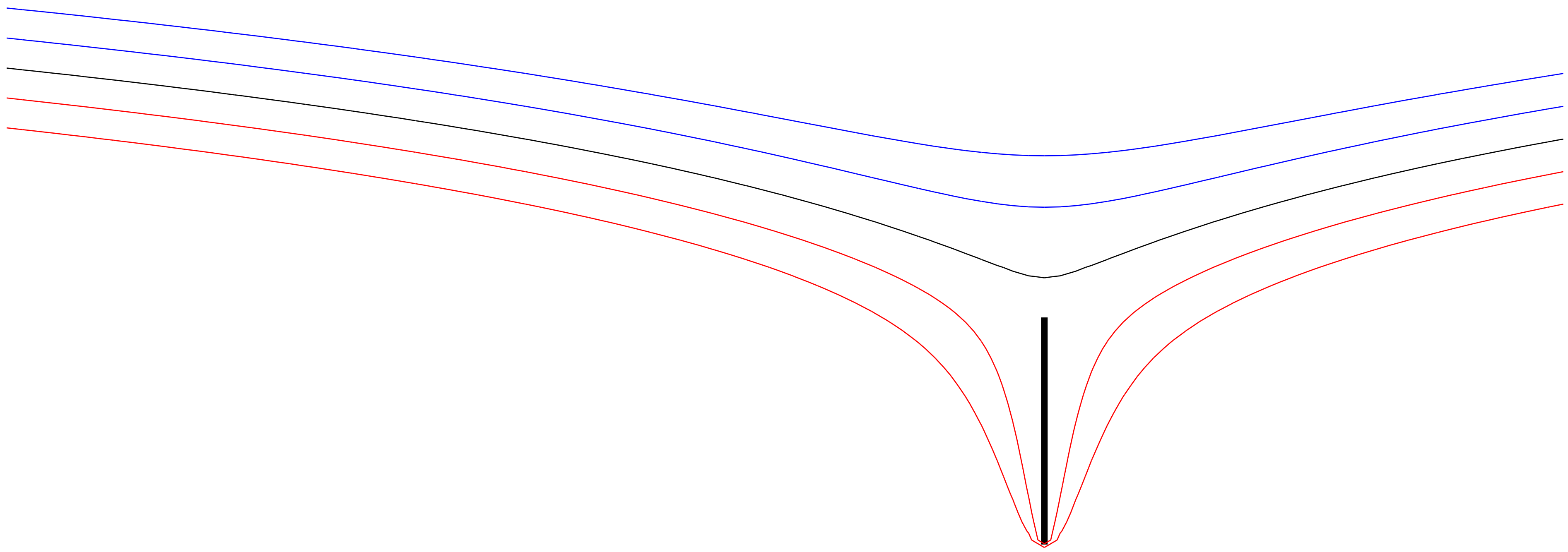
18. **Vertical plate with lots of circulation**

Level sets of ψ (positive, negative, zero)



19. **Vertical plate with even more circulation**

Level sets of ψ (positive, negative, zero)



20. **No nontrivial flows around vertical plate**

Vertical symmetry: if $\psi(x,y)$ solution for $\mathbf{v}_\infty$, then also

$$\tilde{\psi}(x,-y) := -\psi(x,y)$$

Uniqueness: $\psi(x,y) = \tilde{\psi}(x,y) = \psi(x,-y)$.
Hence $\varrho(x,y) = \varrho(x,-y)$, $v^x(x,y) = v^x(x,-y)$, $v^y(x,y) = -v^y(x,-y)$,

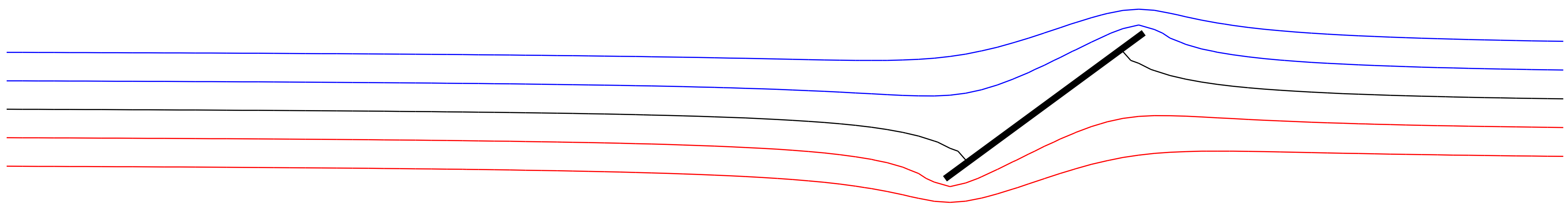
$$\psi(x,0) = 0$$

ψ attains both signs near each corner $\Rightarrow \{\psi > 0\}$ disconnected.
Near infinity: $\psi = \varrho_\infty v^x_\infty y + ...$, positive in upper halfplane (roughly)
$\Rightarrow$ only one unbounded connected components of $\{\psi > 0\}$
$\Rightarrow$ disconnected, so at least one bounded component
$\Rightarrow$ must attain $\psi > 0$ extremum in interior of bounded component
$\lightning$ violated strong maximum principle!

Proof applies to other *vertically symmetric* bodies with two off-axis protruding corners

21. **Incompressible flow around non-horizontal plates**

Level sets of ψ (positive, negative, zero)



22. **No nontrivial low-Mach flows around non-horizontal plates**

1. Simple argument:
(nonzero) incompressible flow has unbounded $\nabla\psi$ near each corner
$\Rightarrow$ no (nonzero) low-Mach flows exist
2. More elaborate argument: trivial constant-velocity flow at infinity:

$$\psi_\infty = \varrho_\infty v^x_\infty y \quad , \quad \varrho_\infty, v^x_\infty > 0$$

Difference to actual flow ("reflection"):

$$\tilde\psi := \psi - \psi_\infty = c_1 \log(x^2 + \beta^2 y^2) + c_2 + o(1) \quad \text{near } \infty$$

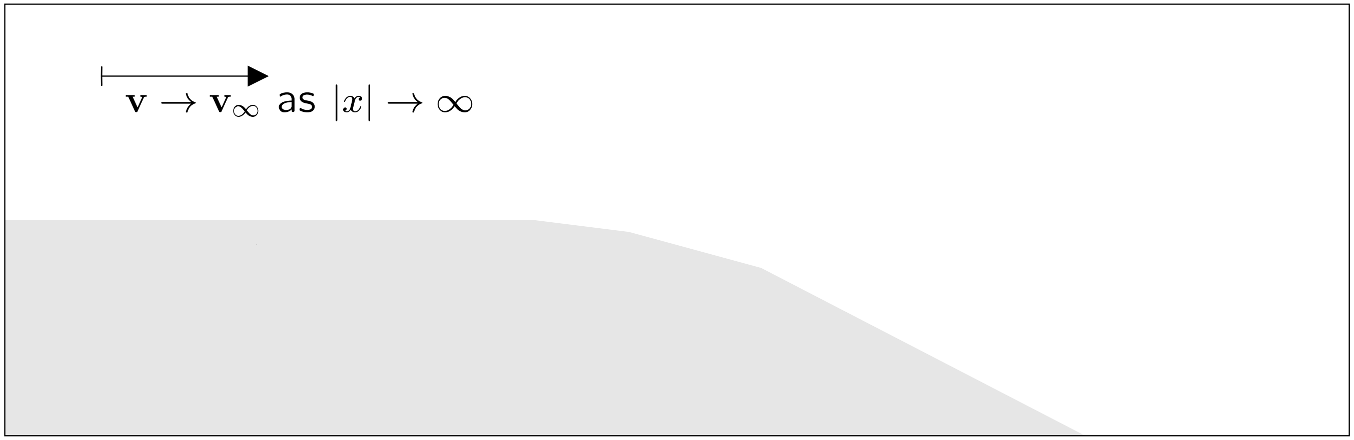
$\psi = 0$ on body, ψ_∞ attains max in upper, min in lower corner
$\Rightarrow$ $\tilde\psi$ attains *min* in upper, *max* in lower
$\tilde\psi$ must attain both signs near each corner
Assume $c_1 > 0$; $\log(x^2 + \beta^2 y^2) > 0$ near ∞, so $\tilde\psi > 0$ there
$\tilde\psi$ attains global min at upper corner, $\tilde\psi - \min\tilde\psi \geq 0$ everywhere, $= 0$ at corner, not mixed-sign near corner ↯ strong maximum principle of

$$A : \nabla^2\tilde\psi = A : \nabla^2\psi - A : \underbrace{\nabla^2\psi_\infty}_{=0}$$

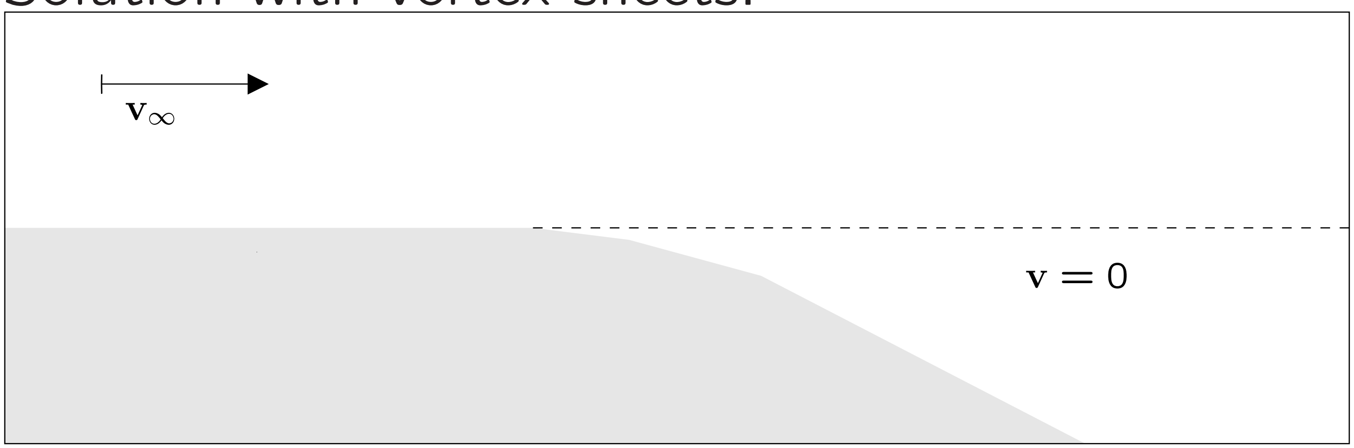
$c_1 < 0$ analogous, so $= 0$. Analogously $c_2 = 0$. $\tilde\psi = o(1)$ analogous (global extrema at both corners)
$\Rightarrow$ no (nonzero) uniformly subsonic flows exist

23. **Discussion**

Theorem (E., submitted): NO nontrivial compressible $\omega = 0$ solutions:



Solution with vortex sheets:



Protruding corners require

1. *Vortex sheets* downstream from corners (E., DCDS-A 2016)
2. *Super*sonic bubbles (at smooth convex boundaries: observed)