Hypoelliptic Regularization for Kinetic Models with Rough Coefficients

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Fokker-Planck Equation

Unknown $f \equiv f(t,x,v)$; **data** $g \equiv g(t,x,v)$; time $t > -T$

$$(\partial_t + v \cdot \nabla_x) f(t,x,v) = \operatorname{div}_v (A(t,x,v) \nabla_v f(t,x,v)) + g(t,x,v)$$

Diffusion matrix $(t,x,v) \mapsto A(t,x,v) \in \mathbf{R}^{N \times N}$ measurable s.t.

$$\frac{1}{\Lambda} I \le A(t,x,v) = A(t,x,v)^T \le \Lambda I \,, \qquad \Lambda > 1$$

Energy inequality multiply by FP by $2f$ and integrate by parts

$$-T < t_0 < t \Rightarrow \|f(t)\|^2_{L^2_{x,v}} + \frac{2}{\Lambda} \int_{t_0}^{t} \|\nabla_v f(s)\|^2_{L^2_{x,v}} ds$$
$$\le \|f(t_0)\|^2_{L^2_{x,v}} + \int_{t_0}^{t} \underbrace{2\|g(s)\|_{L^2_{x,v}} \|f(s)\|_{L^2_{x,v}}}_{\le \|g(s)\|^2_{L^2_{x,v}} + \|f(s)\|^2_{L^2_{x,v}}} ds$$

Energy Inequality for Weak Solutions

By Gronwall: assuming $g \in L^2_{loc}(-T, \infty; L^2_{x,v})$

$$-T < t_0 < t \Rightarrow \|f(t)\|^2_{L^2_{x,v}} + \frac{2}{\Lambda}\int_{t_0}^{t} \|\nabla_v f(s)\|^2_{L^2_{x,v}} ds \leq (\|f(t_0)\|^2_{L^2_{x,v}} + \|g\|^2_{L^2(t_0,t;L^2_{x,v})})e^{T+t}$$

so that

$$f \in C(-T, \infty; L^2_{x,v}), \quad \nabla_v f \in L^2_{loc}(-T, \infty; L^2_{x,v})$$

Renormalized form of FP: for $\chi \equiv \chi(z) = O(z^2)$ of class C^2

$$(\partial_t + v \cdot \nabla_x)\chi(f) = \mathrm{div}_v(A\nabla_v \chi(f)) - \chi''(f)A : (\nabla_v f)^{\otimes 2} + \chi'(f)g$$

Notations

(1) For each $r > 0$ we set

$$Q[r] := (-r, 0] \times B(0, r)^2, \qquad \hat{Q} := (-\tfrac{3}{2}, -1] \times B(0, 1)^2$$

(2) for each $\Lambda > 1$ we set

$$S[\Lambda] := \{(t, x, v) \mapsto A = A^T(t, x, v) \in \mathbf{R}^{N \times N} \text{ s.t. } \Lambda^{-1} I \le A \le \Lambda I\}$$

(3) for each $A \in S[\Lambda]$ for some $\Lambda > 1$, each domain $\mathcal{O} \subset \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}^N$ and each $g \in L^1_{loc}(\mathcal{O})$, we set

$$FP[A, g, \mathcal{O}] := \{f \text{ s.t. } (\partial_t + v \cdot \nabla_x) f = \operatorname{div}_v(A \nabla_v f) + g \text{ on } \mathcal{O}\}$$

Hölder Regularity Theorem

Thm Let $\Lambda > 1$ and $A \in S[\Lambda]$, and let $g \in L^\infty(I \times \Omega)$ where Ω is a domain of $\mathbf{R}^N \times \mathbf{R}^N$ and I an open interval of $\mathbf{R}$.

Then, there exists $\sigma \in (0,1)$ such that

$$f \in C(I, L^2(\Omega)) \cap FP[A, g, I \times \Omega] \Rightarrow f \in C^{0,\sigma}(K)$$

for each compact $K \subset I \times \Omega$.

•Proof based on an adaptation of the DeGiorgi method to Fokker-Planck equation
•Alternative to Pascucci-Polidoro [Commun. Contemp. Math. 2004], Wang-Zhang [Science in China A 2009]

Remarks

•DeGiorgi's (1956-57) solution of the missing piece in the solution of Hilbert's 19th pbm — analyticity of solutions of Euler-Lagrange equations for functionals of the form

$$\int_\Omega F(\nabla w(x))dx \quad \text{with } F \text{ analytic+convex}$$

•Reduces to studying regularity of solutions of $\mathrm{div}_x(A(x)\nabla_x u) = 0$ where A is L^∞+elliptic, but not continuous

•For A in this regularity class, elliptic PDE does not "approach" Laplace's equation by zooming

•In the context of Hilbert's problem, $A(x) = F''(\nabla w(x))$ where w is the extremal; regularity achieved by bootstrap (using earlier results by Bernstein, Petrowsky, Schauder...)

PART I: FROM “ENERGY” TO L^∞

Thm 1 For each $\Lambda > 1$, $\gamma > 0$ and $q > 12N + 6$ there exists $\kappa[N, \Lambda, \gamma, q] \in (0, 1)$ satisfying the following property.

For each $A \in S[\Lambda]$, each $g \in L^q(Q[\frac{3}{2}])$ such that $\|g\|_{L^q} \leq \gamma$ and each $f \in C(-\frac{3}{2}, 0; L^2(B(0, \frac{3}{2})^2)) \cap FP[A, g, Q[\frac{3}{2}]]$

$$\int_{Q[\frac{3}{2}]} f_+^2 dtdxdv < \kappa \Rightarrow f \leq \tfrac{1}{2} \text{ a.e. on } Q[\tfrac{1}{2}]$$

Local Dissipation Rate

Weak formulation of renormalized equation with $\chi(f) := \frac{1}{2}(f-c)_+^2$; test function of the form

$$\psi(\tau, x, v) = \mathbf{1}_{s<\tau<t}\phi(x, v)\,, \quad \phi(x, v) = \eta(x)\eta(v)^2$$

Since $\chi''(f) = \mathbf{1}_{f>c}$, one has

$$\chi''(f)(\nabla_v f)^{\otimes 2} = (\nabla_v (f-c)_+)^{\otimes 2}$$

and then

$$\begin{aligned}&2\eta(v)A : \nabla_v \chi(f) \otimes \nabla\eta(v) + \eta(v)^2\chi''(f)A : (\nabla_v f)^{\otimes 2}\\ &= A : (\nabla_v(\eta(v)(f-c)_+))^{\otimes 2} - (f-c)_+^2 A : (\nabla_v \eta)^{\otimes 2}\end{aligned}$$

Local Energy Estimate

Identity above reminiscent of the (proof of the) Cacciopoli inequality for elliptic equation

Substitute rhs. of previous identity in the dissipation term of the "energy" equality, one finds that

$$\int \eta(x)\eta(v)^2\chi(f)(t)dxdv + \frac{1}{\Lambda}\int_s^t \int \eta(x)|\nabla_v(\eta(f-c)_+)|^2 dxdvd\tau$$

$$\leq \int \eta(x)\eta(v)^2\chi(f)(s)dxdv + \Lambda \int_s^t \int \eta(x)(f-c)_+^2|\nabla\eta|^2 dxdvd\tau$$

$$+ \int_s^t \int \eta(v)^2(\chi(f)v\cdot\nabla\eta(x) + g(f-c)_+\eta(x))dxdvd\tau$$

Dyadic Truncation

Set

$$R_k := \frac{1}{2}(1 + 2^{-k}) =: -T_k\,, \quad B_k := B(0, R_k)\,, \quad Q_k := Q[R_k]$$

Choose $\eta \in C^\infty(\mathbf{R}^N)$ so that

$$\eta_k \equiv 1 \text{ on } \overline{B_k}\,, \quad \eta_k \equiv 0 \text{ on } B_{k-1}^c\,, \quad \|\nabla \eta_k\|_{L^\infty} \leq 2^{k+2}$$

Finally, set

$$C_k = \frac{1}{2}(1 - 2^{-k})\,, \qquad f_k = (f - C_k)_+$$

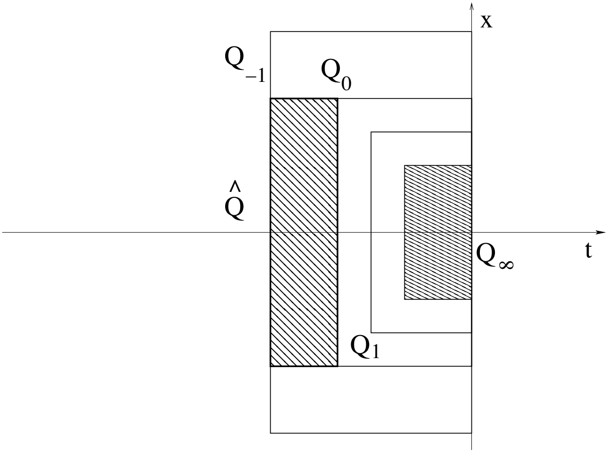

Figure: The nested system of space-time cylinders Q_k in the past centered at the origin, and the space-time cylinder $\hat{Q}$

Iterative Estimate

Set

$$U_k := \sup_{T_k \le t \le 0} \tfrac{1}{2} \int \eta(x)\eta(v)^2 f_k^2$$

$$+\frac{1}{\Lambda}\int_{T_k}^{0}\int \eta_k(x)|\nabla_v(\eta_k(v)f_k)|^2 dxdvd\tau$$

$$\le U_{k-1} \le \ldots \le U_1 \le U_0$$

(a) Local energy inequality for $\eta = \eta_k$, $c = C_k$ and $s \in (T_{k-1}, T_k)$

(b) **Average** both sides of resulting inequality over $s \in (T_{k-1}, T_k)$

$$U_k \le 2^{2k+3}(1+2\Lambda)\int_{Q_{k-1}} (f_k + |g|) f_k dxdvd\tau$$

Nonlinearization 1

Let $p > 2$ be s.t. $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$

$$\int_{Q_{k-1}} (|g| + f_k) f_k \, dx dv d\tau = \int_{Q_{k-1}} (|g| + f_k) f_k \mathbf{1}_{f_k > 0} \, dx dv d\tau$$

$$\leq \|f_k\|_{L^p(Q_{k-1})} \|g\|_{L^q(Q_{-1})} |\{f_k > 0\} \cap Q_{k-1}|^{1 - \frac{1}{p} - \frac{1}{q}}$$

$$+ \|f_k\|^2_{L^p(Q_{k-1})} |\{f_k > 0\} \cap Q_{k-1}|^{1 - \frac{2}{p}}$$

By the Bienaymé-Chebyshev inequality

$$|\{f_k > 0\}| = |\{f > C_k\}| = |\{f_{k-1} > C_k - C_{k-1} = 2^{-k-1}\}|$$

$$\leq 2^{2k+2} \|f_{k-1}\|^2_{L^2(Q_{k-1})} \leq 2^{2k+2} |T_{k-1}| U_{k-1} \leq 3 \cdot 2^{2k+1} U_{k-1}$$

Nonlinearization 2

Hence

$$U_k \leq 3(1+2\Lambda)\cdot 2^{4k+4} U_{k-1}^{1-\frac{2}{p}} \|f_k\|^2_{L^p(Q_{k-1})}$$
$$+3\gamma \cdot 2^{2k+1} U_{k-1}^{1-\frac{1}{p}-\frac{1}{q}} \|f_k\|_{L^p(Q_{k-1})}$$

Now, if one can prove that

$$(*) \qquad \|f_k\|^2_{L^p(Q_{k-1})} \leq C U_{k-2}^{1-\frac{2}{q}}$$

then, for some $\rho > 1$, one has

$$U_k \leq \rho^k U_{k-2}^{\alpha}\,, \quad \alpha = \min(2-\tfrac{2}{p}-\tfrac{2}{q}, 1+\tfrac{1}{2}-\tfrac{1}{p}-\tfrac{2}{q}) > 1$$

Nonlinearization 3

Hence

$$U_{2k} \leq \left(\rho^{\frac{\alpha}{\alpha-1}} U_0\right)^{\alpha^k} \to 0 \quad \text{if} \quad U_0 < \rho^{-\frac{\alpha}{\alpha-1}}$$

Since

$$U_0 \leq 8(1+2\Lambda)\|f_+\|_{L^2(Q_{-1})}(\|f_+\|_{L^2(Q_{-1})} + \gamma|Q_{-1}|^{\frac{1}{2}-\frac{1}{q}})$$

we pick $\kappa > 0$ small enough so that

$$8(1+2\Lambda)\kappa\left(\kappa + \gamma|Q_{-1}|^{\frac{1}{2}-\frac{1}{q}}\right) < \rho^{-\frac{\alpha}{\alpha-1}}$$

and conclude that

$$\|f_+\|_{L^2(Q_{-1})} < \kappa \Rightarrow f \leq \tfrac{1}{2} \text{ a.e. on } Q_\infty = (-\tfrac{1}{2}, 0) \times B(0, \tfrac{1}{2})^2$$

Barrier Function

Set $F_k(t,x,v) := f_k(t,x,v)\eta_k(x)\eta_k(v)^2$ and

$$S_k = S_{k,1} + \operatorname{div}_v S_{k,2} := g\mathbf{1}_{f>C_k}\eta_k(x)\eta_k(v)^2 + f_k\eta_k(v)^2 v\cdot\nabla\eta_k(x)$$
$$-\eta_k(x)A:\nabla_v f_k\otimes\nabla\eta_k(v)^2 - \operatorname{div}_v(\eta_k(x)f_kA\nabla\eta_k(v)^2)$$

and let G_k be the solution of

$$\begin{cases}(\partial_t + v\cdot\nabla_x)G_k - \operatorname{div}_v(A\nabla_v G_k) = S_k \quad \text{on } Q_{k-1}\\ G_k\big|_{\Gamma_-} = 0 \qquad\qquad G_k\big|_{t=T_{k-1}} = 0\end{cases}$$

where

$$\Gamma_- := (T_{k-1},0)\times\{|v| = R_{k-1}\,,\ \text{or } |x| = R_{k-1} \text{ and } v\cdot x < 0\}$$

Then

$$0 \le F_k \le G_k \quad \text{a.e. on } Q_{k-1}$$

Energy inequality tells us that

$$\|G_k\|^2_{L^2(Q_{k-1})} \le 4T^2_{k-1}\|S_{k,1}\|^2_{L^2(Q_{k-1})} + 2\Lambda|T_{k-1}|\|S_{k,2}\|^2_{L^2(Q_{k-1})}$$
$$\|\nabla_v G_k\|^2_{L^2(Q_{k-1})} \le \Lambda|T_{k-1}|\|S_{k,1}\|^2_{L^2(Q_{k-1})} + \Lambda^2\|S_{k,2}\|^2_{L^2(Q_{k-1})}$$

while

$$(\partial_t + v\cdot\nabla_x)G_k = S_{k,1} + \operatorname{div}_v(S_{k,2} + A\nabla_v G_k)$$

Apply Velocity Averaging [Bouchut, JMPA2002] to find that

$$\|D^{1/3}_{t,x,v}G_k\|^2_{L^2} \le b[N,\Lambda,\gamma]^2(1+U_0^{\frac{2}{q}})2^{2k}U_{k-1}^{1-\frac{2}{q}}$$

so that, by Sobolev embedding with $\frac{1}{p} = \frac{1}{2} - \frac{1}{6N+3}$

$$\|f_{k+1}\|^2_{L^p(Q_k)} \le \|F_k\|^2_{L^p} \le \|G_k\|^2_{L^p} \le K_S^2 b^2 2^{2k}(1+U_0^{\frac{2}{q}})U_{k-1}^{1-\frac{2}{q}}$$

PART II: FROM L^∞ TO $C^{0,\sigma}$

Zooming and Fokker-Planck

Zooming transformation

$$\mathcal{T}_\epsilon[t_0, x_0, v_0]F(s, y, \xi) := F(t_0 + \epsilon^2 s, x_0 + \epsilon^3 y + \epsilon^2 s v_0, v_0 + \epsilon\xi)$$

Action of zooming transformation on Fokker-Planck

$$\mathcal{T}_{\omega/6}[t_0, x_0, v_0]FP[A, G, (-\tfrac{\omega^2}{24}, 0) \times B(0, \tfrac{\omega^3}{144}) \times B(0, \tfrac{\omega}{4})] = FP[a, g, Q[\tfrac{3}{2}]]$$

where

$$a := \mathcal{T}_{\omega/6}[t_0, x_0, v_0]A\,, \quad g := \left(\tfrac{\omega}{6}\right)^2 \mathcal{T}_{\omega/6}[t_0, x_0, v_0]G$$

Example Assume $|G| \leq 1$ on $Q[\frac{\omega}{4}]$, then

$$\int_{Q[\frac{\omega}{4}]} F_+^2 dtdxdv \leq \left(\tfrac{\omega}{6}\right)^{4N+2} \kappa[N, \Lambda, \tfrac{\omega^2}{36}, \infty] \Rightarrow F \leq \tfrac{1}{2} \text{ on } Q[\tfrac{\omega^3}{432}]$$

Lemma 2 Let $\Lambda > 1$, $\eta > 0$ and $\omega \in (0, 1 - 2^{-N})$. Then there exist $\theta \in (0, \frac{1}{2})$ and $\alpha > 0$ satisfying the following property.

For all $A \in S[\Lambda]$ and all f, g s.t. $f \in FP[A, g, \hat{Q} \cup Q[1]]$ and

$$f, |g| \leq 1 \text{ on } \hat{Q} \cup Q[1] \quad \text{ and } |\{f \leq 0\} \cap \hat{Q}| \geq \tfrac{1}{2}|\hat{Q}|$$

the following conclusion holds

(a) $|\{f \geq 1 - \theta\} \cap Q[\frac{\omega}{4}]| < \eta$

or

(b) $|\{0 < f < 1 - \theta\} \cap (\hat{Q} \cup Q[1])| \geq \alpha$

Sketch of the proof If wrong, there exists $A_n \in S[\Lambda]$ and f_n, g_n s.t. $f_n \in FP[A_n, g_n, \hat{Q} \cup Q[1]]$ and

$$\begin{cases} f_n \le 1\,, \quad |g_n| \le 1\,, \quad |\{f_n \le 0\} \cap \hat{Q}| > \frac{1}{2}|\hat{Q}| \\ |\{0 < f_n < 1 - 2^{-n}\} \cap (\hat{Q} \cup Q[1])| < 2^{-n} \\ |\{f_n \ge 1 - 2^{-n}\} \cap Q[\frac{\omega}{4}]| > \eta \end{cases}$$

Implies that $f_n^+ \to \mathbf{1}_P(t,x)$ in $L^p(\hat{Q} \cup Q[1])$ for $1 \le p < \infty$, and that $A_n \nabla_v f_n^+ \rightharpoonup h$ in $L^2_{loc}((\hat{Q} \cup Q[1])$, with

$$(\partial_t + v \cdot \nabla_x)\mathbf{1}_P \le \operatorname{div}_v h + 1 \Rightarrow (\partial_t + v \cdot \nabla_x)\mathbf{1}_P \le 0$$

Condition $|\{f_n \le 0\} \cap \hat{Q}| > \frac{1}{2}|\hat{Q}|$ implies $|(P \times B(0,1)) \cap Q[\frac{\omega}{4}]| = 0$: contradiction

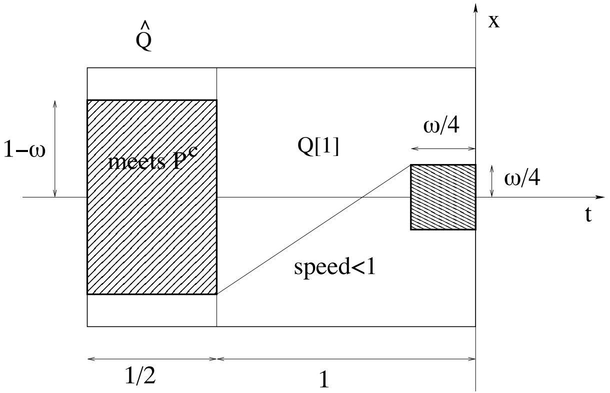

Figure: Any point in the left shaded region traveling with speed < 1 can reach the right shaded region. Since P^c meets the left shaded region and $\mathbf{1}_P$ is nonincreasing along characteristics, the right shaded region meets P in a set of measure 0

Oscillation for $f : X \to \mathbf{R}$, we denote

$$\operatorname{osc}_X f := \sup_{x \in X} f(x) - \inf_{x \in X} f(x)$$

Lemma 3 For all $\omega \in (0, 1-2^{-N})$, there exist $\beta, \mu \in (0,1)$ satisfying the following property.

For all f, g s.t. $f \in FP[A, g, \hat{Q} \cup Q[1]]$ with

$$|f| \leq 1 \quad \text{and} \quad |g| \leq \beta \qquad \text{on } \hat{Q} \cup Q[1]$$

one has

$$\operatorname{osc}_{Q[\frac{\omega^3}{432}]} f \leq \mu \operatorname{osc}_{Q[\omega/2]} f$$

Sketch of the proof Given ω, set $\eta := \left(\frac{\omega}{6}\right)^{4N+2}\kappa[N,\Lambda,\frac{\omega^2}{36},\infty]$
Lemma 2 gives $\theta \in (0,\frac{1}{2})$ and $\alpha > 0$; choose $0 < \beta \ll 1$ so that

$$\ln\frac{1}{\beta} \geq \left(\frac{\frac{1}{2}|\hat{Q}|+|Q[1]|}{\alpha}+2\right)\ln\frac{1}{\theta}$$

Assume that $|\{f \geq 0\} \cap \hat{Q}| \geq \frac{1}{2}|\hat{Q}|$, and define

$$f_k := \tfrac{1}{\theta}(f_{k-1}-1)+1 \leq f_{k-1} \leq \ldots \leq f_0 = f \leq 1$$
$$f_k \in FP[A,\theta^{-k}g,\hat{Q}\cup Q[1]]$$

There exists $1 \leq \hat{k} \leq [\ln\beta/\ln\theta]$ s.t. Lemma 2 (a) holds

$$\int_{Q[\frac{\omega}{4}]}(f_{\hat{k}+1})_+^2\,dtdxdv \leq \int_{Q[\frac{\omega}{4}]}\mathbf{1}_{f_{\hat{k}}\geq 1-\theta}\,dtdxdv < \eta$$
$$\Rightarrow f_{\hat{k}+1} \leq \tfrac{1}{2} < 1-\theta \Rightarrow f < 1-\theta^{\hat{k}+2} \text{ on } Q[\tfrac{\omega^3}{432}]$$

Conclusion

•Theorem 1+Lemma 3+zooming transformation implies the Hölder Regularity Theorem

•**Extensions/Applications**

(a) Regularity issues on the Landau equation (there are many...)

(b) Nonlocal (in v) variants of the FP result — btw. DeGiorgi can be adapted to nonlocal pbms (eg. Caffarelli-Vasseur paper on geostrophic eqn. Ann. Math. 2010)

(c) Application to the Boltzmann equation without angular cutoff in the collision kernel [Imbert-Silvestre arXiv:1608.07571]