Classical limit for the spatially homogeneous Boltzmann equation

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Contents

1. The (quantum) Boltzmann equations and the Fokker-Planck-Landau equation

Kinetic equation under consideraton without external force takes the form

$$\frac{\partial}{\partial t}f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = Q(f)$$

which describes time evolution of a dilute gas, where the solution $f = f(t, \mathbf{x}, \mathbf{v}) \geq 0$ is the number density of the gas, $Q(f)$ is an integral operator which gives the transport speed of particles per unite space volume. The structure of $Q(f)$ is determined by the physical feature of the gas. In this talk we only consider the spatially homogeneous equation, which means that the solution $f = f(t, \mathbf{v})$ does not depend on the position; the equation becomes

$$\frac{\partial}{\partial t}f(t, \mathbf{v}) = Q(f)(t, \mathbf{v})$$

with $Q(f)(t, \mathbf{v}) = Q(f(t, \cdot))(\mathbf{v})$ which is a nonlinear operator of functions of $\mathbf{v}$.

Specifically we consider the spatially homogeneous Fokker-Planck-Landau equation (**FPL**) associated to the Coulomb potential, the Maxwell-Boltzmann equation (**MB**), the Boltzmann equation for Fermi-Dirac particles (**FD**), and the Boltzmann equation for Bose-Einstein particles (**BE**), they are given as follows:

Eq.(FPL):

$$\frac{\partial}{\partial t} f(t, \mathbf{v}) = Q_L(f)(t, \mathbf{v}), \quad (t, \mathbf{v}) \in (0, \infty) \times \mathbb{R}^3 \tag{FPL}$$

$$Q_L(f)(\mathbf{v}) = \nabla_{\mathbf{v}} \cdot \int_{\mathbb{R}^3} A(\mathbf{v} - \mathbf{v}_*)\Big(f(\mathbf{v}_*)\nabla f(\mathbf{v}) - f(\mathbf{v})\nabla f(\mathbf{v}_*)\Big)\mathrm{d}\mathbf{v}_*,$$

$$A(\mathbf{u}) = \frac{1}{|\mathbf{u}|}\Big(\mathrm{I} - \frac{\mathbf{u}}{|\mathbf{u}|} \otimes \frac{\mathbf{u}}{|\mathbf{u}|}\Big) \in \mathbb{R}^{3\times 3}, \quad \mathbf{u} \in \mathbb{R}^3 \setminus \{\mathbf{0}\},$$

$$\mathbf{u} \otimes \mathbf{u} = (u_i u_j)_{3\times 3} \in \mathbb{R}^{3\times 3} \quad \text{for} \quad \mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3,$$

$$\nabla_{\mathbf{v}} = (\partial_{v_1}, \partial_{v_2}, \partial_{v_3}), \quad \nabla f(\mathbf{v}) = \nabla_{\mathbf{v}} f(\mathbf{v}), \quad \nabla f(\mathbf{v}_*) = \nabla_{\mathbf{v}_*} f(\mathbf{v}_*).$$

Eq.(MB), Eq.(FD), Eq.(BE):

$$\frac{\partial}{\partial t} f^{\varepsilon}(t, \mathbf{v}) = Q_0^{\varepsilon}(f^{\varepsilon})(t, \mathbf{v}), \quad (t, \mathbf{v}) \in (0, \infty) \times \mathbb{R}^3 \tag{MB}$$

$$\frac{\partial}{\partial t} f^{\varepsilon}(t, \mathbf{v}) = Q_{-1}^{\varepsilon}(f^{\varepsilon})(t, \mathbf{v}), \quad (t, \mathbf{v}) \in (0, \infty) \times \mathbb{R}^3 \tag{FD}$$

$$\frac{\partial}{\partial t} f^{\varepsilon}(t, \mathbf{v}) = Q_{+1}^{\varepsilon}(f^{\varepsilon})(t, \mathbf{v}), \quad (t, \mathbf{v}) \in (0, \infty) \times \mathbb{R}^3 \tag{BE}$$

with

$$Q_{\lambda}^{\varepsilon}(f)$$

$$= \int_{\mathbb{R}^3 \times \mathbb{S}^2} B_{\lambda}^{\varepsilon}(\mathbf{v} - \mathbf{v}_*, \omega)[f' f'_*(1 + \lambda\varepsilon^3 f)(1 + \lambda\varepsilon^3 f_*) - f f_*(1 + \lambda\varepsilon^3 f')(1 + \lambda\varepsilon^3 f'_*)] \mathrm{d}\omega \mathrm{d}\mathbf{v}_*$$

where $\varepsilon = 2\pi\hbar > 0$ is the Planck constant,

$$f_* = f(t, \mathbf{v}_*), \quad f' = f(t, \mathbf{v}'), \quad f'_* = f(t, \mathbf{v}'_*), \qquad \mathbf{v}, \mathbf{v}_* \in \mathbb{R}^3$$

$$\mathbf{v}' = \mathbf{v} - \langle \mathbf{v} - \mathbf{v}_*, \omega \rangle \omega, \quad \mathbf{v}'_* = \mathbf{v}_* + \langle \mathbf{v} - \mathbf{v}_*, \omega \rangle \omega, \quad \omega \in \mathbb{S}^2.$$

$$B^{\varepsilon}_{\lambda}(\mathbf{v}-\mathbf{v}_*,\omega) = \begin{cases} \dfrac{|\langle \mathbf{v}-\mathbf{v}_*,\omega\rangle|}{\varepsilon^4}\left(\left|\widehat{\phi}\left(\dfrac{|\mathbf{v}'-\mathbf{v}|}{\varepsilon}\right)\right|^2 + \left|\widehat{\phi}\left(\dfrac{|\mathbf{v}'_*-\mathbf{v}|}{\varepsilon}\right)\right|^2\right) & \text{for} \quad \lambda = 0 \\ \dfrac{|\langle \mathbf{v}-\mathbf{v}_*,\omega\rangle|}{\varepsilon^4}\left|\widehat{\phi}\left(\dfrac{|\mathbf{v}'-\mathbf{v}|}{\varepsilon}\right) + \lambda\widehat{\phi}\left(\dfrac{|\mathbf{v}'_*-\mathbf{v}|}{\varepsilon}\right)\right|^2 & \text{for} \quad \lambda = \pm 1 \end{cases}$$

$\lambda = 0:$ Maxwell – Boltzmann statistics

$\lambda = -1:$ Fermi – Dirac statistics

$\lambda = +1:$ Bose – Einstein statistics

The function $\widehat{\phi}$ is the Fourier transform

$$\widehat{\phi}(|\xi|) = 4\pi\int_0^\infty \rho^2\phi(\rho)\frac{\sin(\rho|\xi|)}{\rho|\xi|}\mathrm{d}\rho, \quad \xi\in\mathbb{R}^3$$

of the particle interaction potential ϕ which is real, radially symmetric and belonging to $L^1(\mathbb{R}^3)$. We also assume that

$$\sup_{r\geq 0} r|\widehat{\phi}(r)| < \infty, \quad \int_0^\infty r^3|\widehat{\phi}(r)|^2\mathrm{d}r = \frac{1}{2\pi}.$$

Existence Theorem for Eqs.(MB),(FD),(BE). Given any $\varepsilon = 2\pi\hbar > 0$.

(1) If $0 \le f_0 \in L^1_2\mathrm{Log}L(\mathbb{R}^3)$ (i.e. f_0 has finite mass, energy and entropy), then Eq.(MB) has a unique conservative mild solution f^ε with $f^\varepsilon|_{t=0} = f_0$, and f^ε satisfies the entropy identity.

(2) If $f_0 \in L^1_2(\mathbb{R}^3)$ satisfies $0 \le f_0 \le 1/\varepsilon^3$ on $\mathbb{R}^3$, then Eq.(FD) has the unique conservative mild solution f^ε with $f^\varepsilon|_{t=0} = f_0$ satisfying $0 \le f^\varepsilon \le 1/\varepsilon^3$ on $[0,\infty) \times \mathbb{R}^3$, and f^ε satisfies the entropy identity.

(3) If $F_0 \in \mathcal{B}^+_1(\mathbb{R}_{\ge 0})$ (i.e. if F_0 is a positive Borel measure on $\mathbb{R}_{\ge 0}$ satisfying $\int_{\mathbb{R}_{\ge 0}}(1+x)\mathrm{d}F_0(x) < \infty$), then Eq.(BE) has a conservative measure-valued isotropic weak solution $F^\varepsilon_t \in \mathcal{B}^+_1(\mathbb{R}_{\ge 0})$ with $F^\varepsilon_t|_{t=0} = F_0$.

$\Box$

This existence theorem is in fact a known result since the collision kernel $B^\varepsilon_\lambda(\mathbf{v} - \mathbf{v}_*, \omega)$ is a smooth cutoff of the hard sphere model.

The classical limit means that when the Planck constant $\varepsilon = 2\pi\hbar \to 0$, solution f^ε of one of the above quantum kinetic equations converges to a solution f of certain classical kinetic equation. In this talk we show that, at least in some cases, solutions of Eq.(MB), Eq.(FD), Eq.(BE) all converge weakly to a weak solution of Eq.(FPL).

Let us first look at the operator convergence: for any $\varphi \in C_c^2(\mathbb{R}^3)$ we have

$$\int_{\mathbb{R}^3} \varphi(\mathbf{v}) Q_\lambda^\varepsilon(f)(\mathbf{v})\mathrm{d}\mathbf{v} = \frac{1}{2}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\lambda^\varepsilon(\mathbf{v}-\mathbf{v}_*,\omega)(\varphi'+\varphi'_*-\varphi-\varphi_*)ff_*\mathrm{d}\omega\mathrm{d}\mathbf{v}_*\mathrm{d}\mathbf{v}$$

$$+\lambda\varepsilon^3\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B_\lambda^\varepsilon(\mathbf{v}-\mathbf{v}_*,\omega)(\varphi'+\varphi'_*-\varphi-\varphi_*)f'ff_*\mathrm{d}\omega\mathrm{d}\mathbf{v}_*\mathrm{d}\mathbf{v}.$$

Benedetto & Pulvirenti (2007) proved that $\forall\,\lambda \in \{0,-1,+1\}$,

$$\begin{aligned}\lim_{\varepsilon\to 0+}\int_{\mathbb{R}^3} \varphi(\mathbf{v}) Q_\lambda^\varepsilon(f)(\mathbf{v})\mathrm{d}\mathbf{v} &= \int_{\mathbb{R}^3} \varphi(\mathbf{v}) Q_L(f)(\mathbf{v})\mathrm{d}\mathbf{v} \\ &= \iint_{\mathbb{R}^3\times\mathbb{R}^3} L[\varphi](\mathbf{v},\mathbf{v}_*)f(\mathbf{v})f(\mathbf{v}_*)\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{v}_*\end{aligned}$$

for all $\varphi \in C_c^2(\mathbb{R}^3), f \in L^1 \cap L^p_\beta(\mathbb{R}^3)\ (p > 3/2, \beta > 3)$

where

$$\begin{aligned} L[\varphi](\mathbf{v},\mathbf{v}_*) &= \frac{-2}{|\mathbf{v}-\mathbf{v}_*|^3}\langle \nabla\varphi(\mathbf{v})-\nabla\varphi(\mathbf{v}_*),\mathbf{v}-\mathbf{v}_*\rangle \\ &+ \frac{1}{2}\mathrm{Trace}\Big(A(\mathbf{v}-\mathbf{v}_*)D^2\varphi(\mathbf{v})\Big)+\frac{1}{2}\mathrm{Trace}\Big(A(\mathbf{v}-\mathbf{v}_*)D^2\varphi(\mathbf{v}_*)\Big), \end{aligned}$$

$D^2\varphi(\mathbf{v}) = (\partial^2_{v_iv_j}\varphi(\mathbf{v}))_{3\times 3}$.

The proof uses the fact that for any $\delta > 0$

$$\lim_{\varepsilon\to 0+}\int_{\mathbb{S}^2,|\langle v-v_*,\omega\rangle|<\delta}\frac{|\langle v-v_*,\omega\rangle|^3}{\varepsilon^4}\left|\widehat{\phi}\Big(\frac{|\langle v-v_*,\omega\rangle|}{\varepsilon}\Big)\right|^2\mathrm{d}\omega = \frac{2}{|\mathbf{v}-\mathbf{v}_*|}$$

which implies that the grazing collision becomes dominant as $\varepsilon \to 0$, that is, the set

$$\{(\mathbf{v},\mathbf{v}_*,\omega)\in\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2 \mid \langle \mathbf{v}-\mathbf{v}_*,\omega\rangle\approx 0 \quad \text{i.e.} \quad (\mathbf{v}',\mathbf{v}_*')\approx(\mathbf{v},\mathbf{v}_*)\}$$

gives the main contribution to the total collision integration.

This effect of grazing collision and its asymptotics (by scaling, cutoff approximation, etc.) from the Boltzmann type kinetic equation to the Fokker-Planck-Landau equation have been studied by many authors:

A.A.Arsenev & O.E.Buryak (1991), On the connection between a solution of the Boltzmann equation and a solution of the Landau-Fokker-Planck equation, Math.USSR Sbornik.

P.Degond & B.Lucquin-Desreux (1992), The Fokker-Planck asymptotics of the Boltzmann collision operator in the Coulomb case, Math.Models Methods Appl.Sci.

L.Desvillettes (1992), On asymptotics of the Boltzmann equation when the collisions become grazing. Transp.Theory Stat.Phys.

T.Goudon (1997), On Boltzmann equations and Fokker-Planck asymptotics: influence of grazing collisions, J.Stat.Phys.

C.Villani (1998), On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations, Arch.Ration.Mech.Anal.

L.Desvillettes & V.Ricci (2001), A rigorous derivation of a linear kinetic equation of Fokker-Planck type in the limit of grazing collisions, J.Stat.Phys.

H.Guérinl & S.Méléard (2003), Convergence from Boltzmann to Landau processes with soft potential and particle approximations, J.Stat.Phys.

D.Benedetto & M.Pulvirenti (2007), The classical limit for the Uehling-Uhlenbeck operator, Bull.Inst.Math.Acad.Sin.N.S.

L.He (2014), Asymptotic analysis of the spatially homogeneous Boltzmann equation: grazing collisions limit, J.Stat.Phys.

Our main result of this work — the classical limit for the Boltzmann type equation — shows that this limit of grazing collision holds also for the quantum collision kernel and for the quantum Boltzmann equation.

2. Definition of weak solutions of Eq.(FPL) and the weak convergence Eq.(MB)→Eq.(FPL)

Logically, if the classical limit for the Boltzmann type equation holds true, it also gives a proof of the existence of certain weak solutions of Eq.(FPL). From the structure of the above " Landau kernel " $L[\varphi](\mathbf{v}, \mathbf{v}_*)$ we see that

$$|L[\varphi](\mathbf{v}, \mathbf{v}_*)| = O(1)\frac{1}{|\mathbf{v}-\mathbf{v}_*|} \qquad (|\mathbf{v}-\mathbf{v}_*| \to 0).$$

This leads to a problem of how to deal with the integrability :

$$\int_0^T \mathrm{d}t \int_{0<|\mathbf{v}-\mathbf{v}_*|<<1} |L[\varphi](\mathbf{v}, \mathbf{v}_*)| f(t,\mathbf{v}) f(t,\mathbf{v}_*) \mathrm{d}\mathbf{v}\mathrm{d}\mathbf{v}_*$$

$$= O(1)\int_0^T \mathrm{d}t \int_{0<|\mathbf{v}-\mathbf{v}_*|<<1} \frac{f(t,\mathbf{v}) f(t,\mathbf{v}_*)}{|\mathbf{v}-\mathbf{v}_*|} \mathrm{d}\mathbf{v}\mathrm{d}\mathbf{v}_* < \infty\,?$$

A recent paper

L.Desvillettes, Entropy dissipation estimates for the Landau equation in the Coulomb case and applications, J. Funct. Anal. (**2015**),

proves that

if $0 \le f_0 \in L^1_2(\mathbb{R}^3) \cap L\mathrm{Log}L(\mathbb{R}^3)$ (i.e. f_0 has finite mass, energy and entropy) and if $f(t, \mathbf{v})$ with $f(0, \cdot) = f_0$ is an H-solution of Eq.(FPL) (whose existence has been proven by **C. Villani (1998)**), then $f(t, \mathbf{v})$ has a weighted L^p integrability (or L^p production) for $p = 3$:

$$\int_0^T \left(\int_{\mathbb{R}^3} f(t, \mathbf{v})^3 (1 + |\mathbf{v}|^2)^{-9/2} \mathrm{d}\mathbf{v} \right)^{1/3} \mathrm{d}t < \infty \qquad (\forall\, 0 < T < \infty)$$

and thus $f(t, \mathbf{v})$ is also the usual weak solution of Eq.(FPL), that is, the integral operator in the weak formulation of Eq.(FPL) is well-defined in the sense of absolute convergence.

For our case, we don't know whether the classical limit of solutions of Eq.(MB) can be an H-solution of Eq.(FPL). We find however that if the the integral operation in the weak form of Eq.(FPL) is defined in the sense of **conditional convergence**, then the classical limit can be an weak solution of Eq.(FPL):

Definition. We say that $0 \le f \in L^{\infty}([0,\infty), L^1_2(\mathbb{R}^3))$ is a conservative weak solution of Eq.(FPL) with the initial datum $f(0,\cdot) = f_0 \in L^1_2(\mathbb{R}^3)$, if

(i) f conserves the mass, momentum and energy.

(ii) There exist a function $0 \le M \in L^1_{loc}([0,\infty))$, a constant $C > 0$ and a null set $Z_0 \subset [0,\infty)$ such that $\forall\, \varphi \in C^2_c(\mathbb{R}^3), \forall\, t \in [0,\infty) \setminus Z_0$, the limit

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} L[\varphi](\mathbf{v}, \mathbf{v}_*) f(t,\mathbf{v}) f(t,\mathbf{v}_*) \mathrm{d}\mathbf{v}\mathrm{d}\mathbf{v}_* := \lim_{\delta \to 0+} \iint_{|\mathbf{v}-\mathbf{v}_*| \ge \delta} L[\varphi](\mathbf{v}, \mathbf{v}_*) f(t,\mathbf{v}) f(t,\mathbf{v}_*$$

exists and

$$\left| \iint_{\mathbb{R}^3 \times \mathbb{R}^3} L[\varphi](\mathbf{v}, \mathbf{v}_*) f(t,\mathbf{v}) f(t,\mathbf{v}_*) \mathrm{d}\mathbf{v}\mathrm{d}\mathbf{v}_* \right| \le C \|D^2 \varphi\|_{\infty} M(t).$$

(iii) For any $\varphi \in C^2_c(\mathbb{R}^3)$, the function $t \mapsto \int_{\mathbb{R}^3} \varphi(\mathbf{v}) f(t,\mathbf{v}) \mathrm{d}\mathbf{v}$ is absolutely continuous on $[0,\infty)$ and for almost every $t \in [0,\infty)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \varphi(\mathbf{v}) f(t,\mathbf{v}) \mathrm{d}\mathbf{v} = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} L[\varphi](\mathbf{v}, \mathbf{v}_*) f(t,\mathbf{v}) f(t,\mathbf{v}_*) \mathrm{d}\mathbf{v}\mathrm{d}\mathbf{v}_*.$$

Remark. With the conditions (i),(ii), it is easily proved that the weak form in (iii) is equivalent to the general weak form:

$$\begin{aligned}\int_{\mathbb{R}^3}\varphi(t,\mathbf{v})f(t,\mathbf{v})\mathrm{d}\mathbf{v} &= \int_{\mathbb{R}^3}\varphi(0,\mathbf{v})f_0(\mathbf{v})\mathrm{d}\mathbf{v}+\int_0^t \mathrm{d}s\int_{\mathbb{R}^3} f(s,\mathbf{v})\frac{\partial}{\partial s}\varphi(s,\mathbf{v})\mathrm{d}\mathbf{v}\\ &+ \int_0^t \mathrm{d}s\iint_{\mathbb{R}^3\times\mathbb{R}^3} L[\varphi](s,\mathbf{v},\mathbf{v}_*)f(s,\mathbf{v})f(s,\mathbf{v}_*)\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{v}_*\end{aligned}$$

for all $\varphi\in C_c^2([0,\infty)\times\mathbb{R}^3)$ and all $t\in[0,\infty)$, where $L[\varphi](t,\mathbf{v},\mathbf{v}_*)=L[\varphi(t,\cdot)](\mathbf{v},\mathbf{v}_*)$.

Theorem 1. Let $0\le f_0\in L^1_2\cap L\mathrm{Log}L(\mathbb{R}^3)$ and let $\{f^\varepsilon\}_{\varepsilon>0}$ be a family of conservative mild solutions to the Eq.(MB) with $f^\varepsilon|_{t=0}=f_0$, and assume that f^ε satisfy the entropy inequality. Then for any sequence $\varepsilon_n\to 0^+$ $(n\to\infty)$, there exist a subsequence ε_{n_k}, and a conservative weak solution f of Eq.(FPL) with the same initial datum f_0, such that $f^{\varepsilon_{n_k}}(t,\cdot)\rightharpoonup f(t,\cdot)$ $(k\to\infty)$ weakly in $L^1(\mathbb{R}^3)$ for all $t\ge 0$.

Key step in the proof. The weak form of Eq.(MB) (i.e. $\lambda = 0$) is

$$\int_{\mathbb{R}^3} \varphi(\mathbf{v}) f^{\varepsilon_n}(t,\mathbf{v})\mathrm{d}\mathbf{v} = \int_{\mathbb{R}^3} \varphi(\mathbf{v}) f_0(\mathbf{v})\mathrm{d}\mathbf{v} + \int_0^t \mathcal{Q}^{\varepsilon_n}[\varphi](s)\mathrm{d}s \quad \forall\, t \geq 0,\, \forall\, \varphi \in C_c^2(\mathbb{R}^3)$$

where

$$\mathcal{Q}^{\varepsilon_n}[\varphi](t) = \iint_{\mathbb{R}^3\times\mathbb{R}^3} L_0^{\varepsilon_n}[\varphi](\mathbf{v},\mathbf{v}_*) f^{\varepsilon_n}(t,\mathbf{v}) f^{\varepsilon_n}(t,\mathbf{v}_*)\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{v}_*,$$

$$L_0^{\varepsilon_n}[\varphi](\mathbf{v},\mathbf{v}_*) = \frac{1}{2}\int_{\mathbb{S}^2} B_0^{\varepsilon_n}(\mathbf{v}-\mathbf{v}_*,\omega)\Big(\varphi(\mathbf{v}') + \varphi(\mathbf{v}_*' - \varphi(\mathbf{v}) - \varphi(\mathbf{v}_*)\Big)\mathrm{d}\omega.$$

The assumption implies that $\{f^{\varepsilon_n}\}_{n=1}^{\infty}$ is L^1 relatively weakly compact. Up to subsequence we may assume that

$$f^{\varepsilon_n}(t,\cdot) \rightharpoonup f(t,\cdot) \quad (n\to\infty) \quad L^1 \text{ weakly} \qquad \forall\, t \in [0,\infty).$$

Let

$$\mathcal{Q}_{\geq\delta}^{\varepsilon_n}[\varphi](t) = \iint_{|\mathbf{v}-\mathbf{v}_*|\geq\delta} L_0^{\varepsilon_n}[\varphi](\mathbf{v},\mathbf{v}) f^{\varepsilon_n}(t,\mathbf{v}) f^{\varepsilon_n}(t,\mathbf{v}_*)\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{v}_*,$$

$$\mathcal{Q}_{\geq\delta}[\varphi](t) = \iint_{|\mathbf{v}-\mathbf{v}_*|\geq\delta} L[\varphi](\mathbf{v},\mathbf{v}) f(t,\mathbf{v}) f(t,\mathbf{v}_*)\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{v}_*, \quad \delta > 0.$$

Then, by L^1-weak convergence,

$$\lim_{n\to\infty} \mathcal{Q}^{\varepsilon_n}_{\geq\delta}[\varphi](t) = \mathcal{Q}_{\geq\delta}[\varphi](t) \qquad \forall\,\delta > 0,\ \forall\, t \geq 0\,,\ \forall\,\varphi \in C^2_c(\mathbb{R}^3).$$

Using the Boltzmann H-theorem we have

$$|\mathcal{Q}_{\geq\delta}[\varphi](t)| = \liminf_{n\to\infty} |\mathcal{Q}^{\varepsilon_n}_{\geq\delta}[\varphi](t)| \leq C_{f_0}\|D^2\varphi\|_\infty \liminf_{n\to\infty} \sqrt{D^{\varepsilon_n}_0(f^{\varepsilon_n}(t))}$$

where $t \mapsto D^{\varepsilon_n}_0(f^{\varepsilon_n}(t))$ is the entropy dissipation and, as is well-known,

$$t \mapsto M(t) = \liminf_{n\to\infty} \sqrt{D^{\varepsilon_n}_0(f^{\varepsilon_n}(t))} \quad \text{belongs to} \quad L^1_{loc}([0,\infty))$$

Similarly we have, for all $0 < \delta < \eta$, for all $t \geq 0$,

$$\begin{aligned} |\mathcal{Q}_{\geq\delta}[\varphi](t) - \mathcal{Q}_{\geq\eta}[\varphi](t)| &= \liminf_{n\to\infty} |\mathcal{Q}^{\varepsilon_n}_{\geq\delta}[\varphi](t) - \mathcal{Q}^{\varepsilon_n}_{\geq\eta}[\varphi](t)| \\ &\leq C_{f_0}\|D^2\varphi\|_\infty \liminf_{n\to\infty} \sqrt{D^{\varepsilon_n}_0(f^{\varepsilon_n}(t))}\sqrt{\eta} = C_{f_0}\|D^2\varphi\|_\infty M(t)\sqrt{\eta}. \end{aligned}$$

So, for the null set $Z_0 = \{t \geq 0 \mid M(t) = \infty\}$,

$$\lim_{0<\delta<\eta\to 0+} |\mathcal{Q}_{\geq\delta}[\varphi](t) - \mathcal{Q}_{\geq\eta}[\varphi](t)| = 0 \qquad \forall\, t \in [0,\infty) \setminus Z_0.$$

So for any $t \in [0, \infty) \setminus Z_0$, the limit

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} L[\varphi](\mathbf{v}, \mathbf{v}) f(t, \mathbf{v}) f(t, \mathbf{v}_*) \mathrm{d}\mathbf{v} \mathrm{d}\mathbf{v}_* := \lim_{\delta \to 0+} \mathcal{Q}_{\geq \delta}[\varphi](t)$$

exists and

$$\left| \iint_{\mathbb{R}^3 \times \mathbb{R}^3} L[\varphi](\mathbf{v}, \mathbf{v}) f(t, \mathbf{v}) f(t, \mathbf{v}_*) \mathrm{d}\mathbf{v} \mathrm{d}\mathbf{v}_* \right| \leq C_{f_0} ||D^2 \varphi||_\infty M(t).$$

The rest of proof is easy hence f is a weak solution of Eq.(FPL) with $f|_{t=0} = f_0$. $\Box$

3. Weak convergence Eq.(FD)→Eq.(FPL) and Eq.(BE)→Eq.(FPL) for isotropic solutions

The usual weak formulation for Eq.(FD) and Eq.(BE) (i.e. $\lambda = \pm 1$) is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\varphi(\mathbf{v})f^{\varepsilon}(t,\mathbf{v})\mathrm{d}\mathbf{v}$$

$$=\frac{1}{2}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B^{\varepsilon}_{\lambda}(\mathbf{v}-\mathbf{v}_*,\omega)(\varphi'+\varphi'_*-\varphi-\varphi_*)f^{\varepsilon}f^{\varepsilon}_*\mathrm{d}\omega\mathrm{d}\mathbf{v}_*\mathrm{d}\mathbf{v}$$

$$+\lambda\varepsilon^3\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B^{\varepsilon}_{\lambda}(\mathbf{v}-\mathbf{v}_*,\omega)(\varphi'+\varphi'_*-\varphi-\varphi_*)f^{\varepsilon\prime}f^{\varepsilon}f^{\varepsilon}_*\mathrm{d}\omega\mathrm{d}\mathbf{v}_*\mathrm{d}\mathbf{v}.$$

For the classical limit, the main difficulty is how to prove the zero-limit of the cubic term, i.e.

$$\lim_{\varepsilon\to 0+}\varepsilon^3\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} B^{\varepsilon}_{\lambda}(\mathbf{v}-\mathbf{v}_*,\omega)(\varphi'+\varphi'_*-\varphi-\varphi_*)f^{\varepsilon\prime}f^{\varepsilon}f^{\varepsilon}_*\mathrm{d}\omega\mathrm{d}\mathbf{v}_*\mathrm{d}\mathbf{v}=0\,?$$

So far we can only prove that it holds for isotropic solutions.

Here the "isotropic" means that the solution $f(t,\mathbf{v})$ is radially symmetric with respect to the velocity variable $\mathbf{v}$. So we may write it as

$$f = f(t, |\mathbf{v}|^2/2) = f(t, x), \quad x = |\mathbf{v}|^2/2.$$

The isotropic weak form for Eq.(FD) and Eq.(BE) can be written

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}_{\geq 0}} \varphi(x) f^{\varepsilon}(t, x) \sqrt{x} \mathrm{d}x = \iint_{\mathbb{R}^2_{\geq 0}} \mathcal{J}^{\varepsilon}_{\lambda}[\varphi](y, z) f^{\varepsilon}(t, y) f^{\varepsilon}(t, z) \sqrt{yz} \mathrm{d}y \mathrm{d}z$$

$$+ \lambda \varepsilon^3 \iiint_{\mathbb{R}^3_{\geq 0}} \mathcal{K}^{\varepsilon}_{\lambda}[\varphi](x, y, z) f^{\varepsilon}(t, x) f^{\varepsilon}(t, y) f^{\varepsilon}(t, z) \sqrt{xyz} \mathrm{d}x \mathrm{d}y \mathrm{d}z$$

where

$$\mathcal{J}^{\varepsilon}_{\lambda}[\varphi](y, z) = \frac{1}{2} \int_0^{y+z} \mathcal{K}^{\varepsilon}_{\lambda}[\varphi](x, y, z) \sqrt{x} \mathrm{d}x,$$

$$\mathcal{K}^{\varepsilon}_{\lambda}[\varphi](x, y, z) = W^{\varepsilon}_{\lambda}(x, y, z) \Big(\varphi(x) + \varphi(x_*) - \varphi(y) - \varphi(z) \Big), \quad x_* = (y + z - x)_+$$

$W^{\varepsilon}_{\lambda}(x, y, z) \geq 0$ is an explicit function constructed by the collision kernel $B^{\varepsilon}_{\lambda}(\cdot, \cdot)$.

In order to cover the case of low temperature for Eq.(BE) (i.e. for the Bose-Einstein particles), the solution $f^\varepsilon(t,x)$ of Eq.(BE) should be generalized to a positive Borel measure F_t^ε, that is, the measure elements

$$f^\varepsilon(t,x)\sqrt{x}\mathrm{d}x, \quad f^\varepsilon(t,x)f^\varepsilon(t,y)\sqrt{xy}\,\mathrm{d}x\mathrm{d}y, \quad \text{etc.}$$

should be replaced by

$$\mathrm{d}F_t^\varepsilon(x), \quad \mathrm{d}F_t^\varepsilon(x)\mathrm{d}F_t^\varepsilon(y), \quad \text{etc.}$$

Then the isotropic weak form of Eq.(BE) for measure-valued solutions is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_{\geq 0}}\varphi(x)\mathrm{d}F_t^\varepsilon(x) = \iint_{\mathbb{R}^2_{\geq 0}}\mathcal{J}_\lambda^\varepsilon[\varphi](y,z)\mathrm{d}F_t^\varepsilon(y)\mathrm{d}F_t^\varepsilon(z)$$

$$+\varepsilon^3\iiint_{\mathbb{R}^3_{\geq 0}}\mathcal{K}_\lambda^\varepsilon[\varphi](x,y,z)\mathrm{d}F_t^\varepsilon(x)\mathrm{d}F_t^\varepsilon(y)\mathrm{d}F_t^\varepsilon(z).$$

The isotropic Eq.(FPL) can be written as

$$\frac{\partial}{\partial t} f(t,x)\sqrt{x} = Q_L(f)(t,x), \qquad x > 0, t > 0$$

with

$$Q_L(f)(t,x) = \frac{2}{3\pi}\frac{\partial}{\partial x}\int_{\mathbb{R}_{\geq 0}} \Big(f(t,y)\frac{\partial}{\partial x} f(t,x) - f(t,x)\frac{\partial}{\partial y} f(t,y)\Big)(x\wedge y)^{3/2} \mathrm{d}y.$$

Its weak form is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_{\geq 0}} \varphi(x) f(t,x)\sqrt{x}\,\mathrm{d}x = \iint_{\mathbb{R}^2_{\geq 0}} L[\varphi](x,y) f(t,x) f(t,y)\sqrt{x}\sqrt{y}\,\mathrm{d}x\mathrm{d}y$$

for all $t \geq 0$ and all $\varphi \in C_c^2(\mathbb{R}_{\geq 0})$, where

$$L[\varphi](x,y) = 4\pi\Big(\frac{2}{3}(\varphi''(x)+\varphi''(y))\frac{x\wedge y}{\sqrt{x\vee y}} - \frac{\varphi'(x)-\varphi'(y)}{\sqrt{x\vee y}}\mathrm{sign}(x-y)\Big),$$

$$a \wedge b = \min\{a,b\}, \quad a \vee b = \max\{a,b\}.$$

Measure-valued isotropic Eq.(FPL) can also be introduced as above:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_{\geq 0}}\varphi(x)F_t(x)=\iint_{\mathbb{R}^2_{\geq 0}}L[\varphi](x,y)\mathrm{d}F_t(x)\mathrm{d}F_t(y)$$

for all $t\geq 0$, all $\varphi\in C_c^2(\mathbb{R}_{\geq 0})$.

The following lemma together with the conservation of mass ensures that the cubic terms in the weak form of isotropic Eq.(FD) and Eq.(BE) converge to zero as $\varepsilon\to 0$.

Lemma. Let $\mathcal{B}^+(\mathbb{R}_{\geq 0})$ be the set of finite positive Borel measures F on $\mathbb{R}_{\geq 0}$, let $||F||=\int_{\mathbb{R}_{\geq 0}}\mathrm{d}F(x)$. Then for any $\varphi\in C_c^2(\mathbb{R}_{\geq 0}), \lambda\in\{-1,+1\}$,

$$\sup_{F\in\mathcal{B}^+(\mathbb{R}_{\geq 0}),\,||F||\leq 1}\left|\varepsilon^3\iiint_{\mathbb{R}^3_{\geq 0}}\mathcal{K}^{\varepsilon}_{\lambda}[\varphi](x,y,z)\mathrm{d}F(x)\mathrm{d}F(y)\mathrm{d}F(z)\right|\to 0\quad\text{as}\quad\varepsilon\to 0.$$

From this lemma, it is easy to prove the convergence Eq.(FD)→Eq.(FPL) and Eq.(BE)→Eq.(FPL):

Theorem 2.

Eq.(FD) $\to$ Eq.(FPL): Let $f^{\varepsilon}(t,x)$ be conservative isotropic solutions of Eq.(FD) with $f^{\varepsilon}|_{t=0} = f_0 \in L^1_1(\mathbb{R}_{\geq 0}, \sqrt{x}\mathrm{d}x) \cap L^{\infty}(\mathbb{R}_{\geq 0})$. Then for any sequence $\varepsilon_n \to 0^+$ $(n \to \infty)$ there exist a subsequence ε_{n_k} and a conservative isotropic weak solution $f(t,x)$ of Eq.(FPL) with the same initial datum f_0, such that $f^{\varepsilon_{n_k}}(t,\cdot) \rightharpoonup f(t,\cdot)$ $(k \to \infty)$ weakly in $L^1(\mathbb{R}_{\geq 0}, \sqrt{x}\mathrm{d}x)$ for all $t \geq 0$.

Eq.(BE) $\to$ Eq.(FPL): Let $F_0 \in B_1^+(\mathbb{R}_{\geq 0})$ and let $F_t^{\varepsilon} \in B_1^+(\mathbb{R}_{\geq 0})$ be conservative isotropic measure-valued weak solution of Eq.(BE) with $F_t^{\varepsilon}|_{t=0} = F_0$. Then for any sequence $\varepsilon_n \to 0^+$ $(n \to \infty)$ there exist a subsequence ε_{n_k} and a conservative isotropic measure-valued weak solution F_t of Eq.(FPL) with the same initial datum F_0, such that $F_t^{\varepsilon_{n_k}} \rightharpoonup F_t$ $(k \to \infty)$ weakly for all $t \geq 0$.

Thank You for Your Attention