The spatially homogeneous Boltzmann Equation for Debye-Yukawa potential

Yoshinori Morimoto

Kyoto University

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Collision cross section from Debye-Yukawa type potential

$$U(\rho) = \rho^{-1} e^{-\rho^s}, \quad \text{with } 0 < s < 2. \tag{1}$$

Here ρ is the distance between two interacting particles. Let $z = v - v_*$ be relative velocity and for $\sigma \in \mathbb{S}^2$ put $z \cdot \sigma / |z| = \cos\theta$. θ is the deviation angle after the collision. Denote $V = |z|$. Then the Boltzmann collision cross section $B(V, \cos\theta)$ is defined by

$$B(V, \cos\theta) = -\frac{V}{2\sin\theta} \frac{\partial p^2}{\partial\theta},$$

where $p = p(V, \theta, d_0)$ is the impact parameter determined by the conservation of energy and angular momentum respectively:

$$\begin{cases} \frac{\mu}{2}(\dot{\rho}^2 + \rho^2\dot{\varphi}^2) + U(\rho) = \frac{\mu}{2}V^2 + U(d_0), & (\rho \le d_0), \\ \rho^2\dot{\varphi} = pV^2, \end{cases}$$

where $\mu = mm_*/(m + m_*)$ is the reduced mass from masses of two particles. We choose $m = m_* = 2$, and so $\mu = 1$.

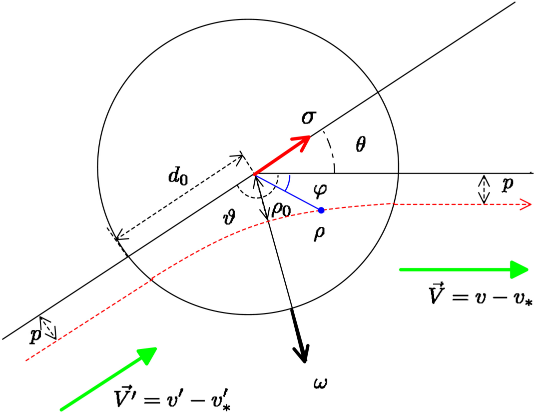


Figure: (ρ, φ), radial, angular coordinates in the plane of motion, d_0, the radius of the protection sphere, $\vartheta = (\pi - \theta)/2$

By using φ as the independent variable to eliminate the time derivative, after integration, we have

$$\vartheta = \frac{1}{\sqrt{2}} Vp \int_{\rho_0}^{d_0} \rho^{-2} \Big[\frac{V^2}{2}\Big(1 - \frac{p^2}{\rho^2}\Big) - U(\rho) + U(d_0) \Big]^{-1/2} d\rho + \sin^{-1}\left(\frac{p}{d_0}\right),$$

where ρ_0 is the smallest distance between two particles which satisfies

$$\frac{1}{2} V^2 \Big(1 - \frac{p^2}{\rho_0^2}\Big) = U(\rho_0) - U(d_0) > 0. \qquad (2)$$

Note that

$$p(V, \theta, d_0) < \rho_0(V, \theta, d_0) \le \rho \le d_0.$$

Lemma 1

For any $(V, \theta) \neq (0,0)$ *we have* $\limsup_{d_0 \to \infty} \rho_0(V, \theta, d_0) < \infty.$

Proof.

If $u_0 = (p/\rho_0)(V, \theta, d_0)$, then

$$\frac{\theta}{2} = \int_{p/d_0}^{1} \frac{dt}{\sqrt{1-t^2}} - \int_{\frac{p}{u_0 d_0}}^{1} \left[1 - t^2 + \frac{2}{V^2 u_0^2}\Big(U\big(\rho_0\big) - U\Big(\frac{\rho_0}{t}\Big)\Big)\right]^{-1/2} dt.$$

Suppose that $\limsup \rho_0 = \infty$. Then it follow from (2) that $\limsup u_0 = 1$, which leads us $\theta = 0$. This is a contradiction. □

Since the lemma shows $\lim_{d_0 \to \infty} \rho_0/d_0 = 0$, and the formula in the proof of Lemma 1 can be written as

$$\frac{\theta}{2} = \int_{p/d_0}^{\rho_0/d_0} \frac{dt}{\sqrt{1-t^2}} + \int_{\rho_0/d_0}^{1} \frac{1}{\sqrt{1-t^2}} \Big[1 - \Big(1 + \frac{2U(\rho_0) - 2U(\frac{\rho_0}{t})}{(1-t^2)\big(V^2 - 2U(\rho_0)\big)}\Big)^{-1/2}\Big] dt,$$

$\bar{\rho}_0 = \lim_{d_0 \to \infty} \rho_0(V, \theta, d_0)$ exists and satisfies

$$\frac{\theta}{2} = \int_0^1 \frac{1}{\sqrt{1-t^2}} \Big[1 - \Big(1 + \frac{2U(\bar{\rho}_0) - 2U(\frac{\bar{\rho}_0}{t})}{(1-t^2)\big(V^2 - 2U(\bar{\rho}_0)\big)}\Big)^{-1/2}\Big] dt.$$

We are interested in $\bar{\rho}_0$ and $\bar{p} = \lim_{d_0 \to \infty} p(V, \theta, d_0)$ for a small θ.

If θ is small, then $\frac{2U(\bar\rho_0)-2U(\frac{\bar\rho_0}{t})}{(1-t^2)\left(V^2-2U(\bar\rho_0)\right)}$ is small and $U(\bar\rho_0)$ so. Therefore by means of (2) we have $\bar\rho_0 \approx \bar p$, and

$$\frac{\theta}{2} \approx \int_0^1 \frac{1}{\sqrt{1-t^2}} \frac{U(\bar p) - U(\frac{\bar p}{t})}{(1-t^2)V^2} dt,$$

because $V^2 - 2U(\bar\rho_0) = V^2 \bar p^2/\bar\rho_0^2 \approx V^2$ and $(1+2\varepsilon)^{-1/2} \approx 1-\varepsilon$. Write p instead of $\bar p$. If $U(\rho) = \rho^{1-n}$ the inverse power law, then we have

$$\frac{V^2\theta}{2} \approx p^{1-n} \int_0^1 \frac{1-t^{n-1}}{(1-t^2)^{3/2}} dt.$$

Therefore

$$B = -\frac{V}{2\sin\theta} \frac{\partial p^2}{\partial\theta} \approx -\frac{V}{\theta}\frac{\partial}{\partial\theta}(V^2\theta)^{2/(1-n)} \approx \frac{V}{\theta^2}(V^2\theta)^{-2/(n-1)}.$$

If θ is small, then $\frac{2U(\bar\rho_0)-2U(\frac{\bar\rho_0}{t})}{(1-t^2)\left(V^2-2U(\bar\rho_0)\right)}$ is small and $U(\bar\rho_0)$ so. Therefore by means of (2) we have $\bar\rho_0 \approx \bar p$, and

$$\frac{\theta}{2} \approx \int_0^1 \frac{1}{\sqrt{1-t^2}} \frac{U(\bar p) - U(\frac{\bar p}{t})}{(1-t^2)V^2} dt,$$

because $V^2 - 2U(\bar\rho_0) = V^2\bar p^2/\bar\rho_0^2 \approx V^2$ and $(1+2\varepsilon)^{-1/2} \approx 1-\varepsilon$. Writing p instead of $\bar p$ and plugging $U(\rho) = \rho^{-1}e^{-\rho^s}$ into the above integral, we have

$$\frac{V^2\theta}{2} \approx \frac{1}{p}e^{-p^s}\int_0^1 (1-t^2)^{-3/2}\left(1 - te^{-p^s(t^{-s}-1)}\right)dt := \frac{1}{p}e^{-p^s}g(p^2).$$

It should be noted that this formula holds only for $p \gg 1$ (equivalently for $0 < V^2\theta \ll 1$).

If we put $f(p^2) = \frac{1}{p} e^{-p^s}$, then

$$\frac{V^2}{2} \approx \left.\frac{d(f(q)g(q))}{dq}\right|_{q=p^2} \frac{\partial p^2}{\partial \theta}$$

and

$$\frac{\partial p^2}{\partial \theta} \approx \frac{V^2 \theta}{2\theta (fg)'(p^2)} = \frac{1}{\theta} \left(\left.\frac{d \log(f(q)g(q))}{dq}\right|_{q=p^2} \right)^{-1}.$$

Notice that

$$\begin{aligned} 0 \leq \frac{dg(q)}{dq} &= \frac{d}{dq} \left(\int_0^1 (1-t^2)^{-3/2} \left(1 - te^{-q^{s/2}(t^{-s}-1)}\right) dt \right) \\ &= \int_0^1 (1-t^2)^{-3/2} t(t^{-s}-1) \frac{s}{2} q^{s/2-1} e^{-q^{s/2}(t^{-s}-1)} dt \\ &= \int_0^{(1+\delta)^{-1/s}} \cdots dt + \int_{(1+\delta)^{-1/s}}^1 \cdots dt = I_1(q) + I_2(q), \end{aligned}$$

for any $0 < \delta \ll 1$.

Since $g(q) \geq g(0)$, $I_1(q) \leq C_\delta q^{s/2-1} e^{-\delta q^{s/2}}$ and

$$0 < I_2(q) \lesssim \int_0^\delta \frac{q^{s/2-1} e^{-q^{s/2}\tau}}{\sqrt{\tau}} d\tau \sim q^{s/4-1},$$

we have

$$\frac{d \log(f(q)g(q))}{dq} \approx -q^{s/2-1} \text{ for } q \gg 1.$$

Therefore we see

$$B(V,\theta) = -\frac{V}{\sin\theta}\frac{\partial p^2}{\partial\theta} \approx \frac{V}{\theta^2} p^{2-s} \approx \frac{V}{\theta^2}\left(\log\frac{1}{V^2\theta}\right)^{\frac{2}{s}-1},$$

because we have $p^s \approx \log \frac{1}{V^2\theta}$, in view of $g(q) \lesssim q^{s/4}$.

Cauchy problem and Assumption of cross-section

Spatially homogeneous equation for the measure initial datum

$$\partial_t f = Q(f,f)\,, \;\; f(0,v) = F_0(v) \in P(\mathbb{R}^3), \tag{3}$$

where $P(\mathbb{R}^3)$ denotes the set of all probability measures. Furthermore, $P_\alpha(\mathbb{R}^3), \alpha \ge 0$, denotes the set of all probability measures $F \in P(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} |v|^\alpha dF(v) < \infty\,, \;\; \int_{\mathbb{R}^3} v_j dF(v) = 0, j = 1,2,3, \;\; \text{if } \alpha > 1.$$

Note $f(t,v)dv = dF_t(v)$ if F_t has a density $f(t,v)$.

Collision Integral Operator

$$Q(g,h)(v) = \int_{\mathbb{R}^3\times\mathbb{S}^2} B\left(v, \frac{(v-v_*)\cdot\sigma}{v}\right)\left\{g'_* h' - g_* h\right\} d\sigma dv_*$$

where $g'_* = g(v'_*), h' = h(v'), g_* = g(v_*), h = h(v)$ and , for $\sigma \in \mathbb{S}^2$,

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma.$$

$$\Downarrow \Uparrow$$

$$v' + v'_* = v + v_*, \quad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2.$$

Collision cross section from Debye-Yukawa type potential

$$B(V,\cos\theta) = \frac{V}{\theta^2}\left(\log\left(\frac{\pi}{V^2\theta} + e(V)\right)\right)^m, \quad m = \frac{2}{s} - 1 > 0, \quad (4)$$

where $C^\infty \ni e(V)$ satisfies $e(V) = e$ for $V \geq 1$ and $e(V) = 0$ for $V \leq 1/2$.

As usual, the range of θ can be restricted to $[0, \pi/2]$, by replacing B by its "symmetrized" version

$$\Big[B(V,\cos\theta) + B(V,\cos(\pi-\theta))\Big]\mathbf{1}_{0\leq\theta\leq\pi/2}.$$

Theorem 1 (Existence of measure valued solution, M-Wang-Yang)

For any $F_0 \in P_2(\mathbb{R}^3)$, *there exists a weak solution* $F_t \in C([0,\infty); P_2(\mathbb{R}^3))$ *to the Cauchy problem which satisfies:*

(1) *(conservation of energy)*

$$\int |v|^2 dF_t = \int |v|^2 dF_0, \text{ for all } t \geq 0;$$

(2) *(moment production)*

$$\text{for any } t_0 > 0 \text{ and } \ell > 0, \ \sup_{t \geq t_0} \int |v|^\ell dF_t(v) < \infty.$$

Here, for $\alpha > 0$, $F_t \in C([0,\infty); P_\alpha)$ means that the map $t \mapsto F_t \in P_\alpha$ is continuous at $\forall t_0 \in [0,\infty)$ in the weak topology, that is,

$$\begin{cases} \text{whenever } \psi \in C(\mathbb{R}^3) \text{ satisfies the growth condition} \\ \qquad \sup_{v\in\mathbb{R}^3} \dfrac{|\psi(v)|}{\langle v\rangle^\alpha} < \infty, \\ \lim_{t\to t_0} \int_{\mathbb{R}^3} \psi(v)dF_t(v) = \int_{\mathbb{R}^3} \psi(v)dF_{t_0}(v). \end{cases}$$

As for other equivalent conditions, see Villani ['03, Theorem 7.12] and Cho-M-Wang-Yang ['16, SIMA].
By the Fourier transform:

$$\varphi(\xi,t) = \int e^{-iv\cdot\xi} dF_t(v).$$

- Wasserstein distance $W_\alpha(F,G)$ in $P_\alpha(\mathbb{R}^d)$: For $F, G \in P_\alpha(\mathbb{R}^d)$,

$$W_\alpha(F,G) = \left(\inf_{L\in\Pi(F,G)} \int |v-w|^\alpha dL(v,w) \right)^{1/\alpha}, \ \text{if} \ \ \alpha \geq 1,$$
$$= \inf_{L\in\Pi(F,G)} \int |v-w|^\alpha dL(v,w), \ \text{if} \ \ 0<\alpha<1,$$

where $\Pi(F,G)$ denotes the set of all probability distributions L in $P_\alpha(\mathbb{R}^d \times \mathbb{R}^d)$ having F and G as marginal distributions, that is,

$$dF(v) = \int_{\mathbb{R}^d_w} dL(v,w), \ \ dG(w) = \int_{\mathbb{R}^d_v} dL(v,w).$$

- Toscani metric when $0 < \alpha \leq 2$: For $\varphi = \mathcal{F}(F), \tilde\varphi = \mathcal{F}(G)$,

$$||\varphi - \tilde\varphi||_\alpha = \sup_{\xi\in\mathbb{R}^d} \frac{|\varphi(\xi) - \tilde\varphi(\xi)|}{|\xi|^\alpha}.$$

Following Lu-Mouhot ['12 JDE], we introduce, for any $\psi \in C_b^2(\mathbb{R}^3)$,

$$L_B[\psi](v,v_*) = \int_{\mathbb{S}^2} B(V,\cos\theta)\Big(\psi(v') + \psi(v'_*) - \psi(v) - \psi(v_*)\Big)d\sigma.$$

Definition 1 (Measure valued weak solution)

For $F_0 \in P_2(\mathbb{R}^3)$, we say $F_t \in C([0,\infty); P_2(\mathbb{R}^3))$ is a measure valued solution to the Cauchy problem (3) *if it satisfies :*
For every $\psi(v) \in C_b^2(\mathbb{R}^3)$ and $t > 0$,

(1) $\displaystyle\int_0^t \iint_{\mathbb{R}^3\times\mathbb{R}^3} |L_B[\psi](v,v_*)| dF_\tau(v) dF_\tau(v_*) d\tau < \infty.$

(2) $$\int_{\mathbb{R}^3} \psi(v) dF_t = \int_{\mathbb{R}^3} \psi(v) dF_0 + \frac{1}{2}\int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} L_B[\psi](v,v_*) dF_\tau(v) dF_\tau(v_*) d\tau.$$

Existence for hard potential from inverse power potential

$$B(V,\cos\theta) = V^\gamma b(\cos\theta), \;\; b \approx K\theta^{-2-2s}$$

If $U(\rho) = \rho^{1-n}, n > 2$, then $\gamma = 1 - 4/(n-1)$, $s = 1/(n-1)$.

- Lu-Mouhot '12 JDE, $0 < \gamma \leq 2$.

- M-Wang-Yang '16 J.Stat.Phys. $-2 \leq \gamma \leq 2$.
  by means of the Fourier transform, Bobylev formula, Toscani metric

The uniqueness is unknown. cf., Desvillettes-Mouhot '09 ARMA, . Fournier-Mouhot '09 CMP, Fournier-Guérin '08 J.Stat.Phys.

Proposition 1 (moment production in general)

If $F_t \in C([0,\infty); P_2(\mathbb{R}^3))$ is an energy conservative measure valued solution to the Cauchy problem and if there exists a $\kappa > 0$ such that

$$\text{for any } T > 0, \quad \int_0^T \int_{\mathbb{R}^3} \langle v \rangle^{2+\kappa} dF_t(v) dt < \infty,$$

then F_t has the moment production :

$$\forall t_0 > 0, \ \forall \ell > 0, \ \sup_{t \geq t_0} \int |v|^\ell dF_t < \infty.$$

Mischler-Wennberg ‘99, Lu-Mouhot ‘12.

Povzner inequality

$$B(V,\cos\theta) = \frac{V}{\theta^2}\left(\log\left(\frac{\pi}{V^2\theta} + e(V)\right)\right)^m, \; m = \frac{2}{s} - 1.$$

Theorem 2 (smoothing effect, M-Wang-Yang)

Assume $\mathbb{N} \ni m \geq 2$. *Let* $F_t \in C([0,\infty); P_2(\mathbb{R}^3))$ *be an energy conservative measure valued solution satisfying the moment production property*

$$\forall t_0 > 0, \; \forall \ell > 0, \; \sup_{t \geq t_0} \int |v|^\ell dF_t < \infty.$$

If the initial datum $F_0 \in P_2(\mathbb{R}^3)$ *is not a single Dirac mass, then* $F_t \in C^\infty((0,\infty); \mathcal{S}(\mathbb{R}^3))$. *If* F_0 *has a density* $f_0(v)$ *belonging to* $L^1_2 \cap L \log L$, *then the same conclusion holds even for* $m = 1$.

Smoothing effect for inverse power models and etc

- Desvillettes-Wennberg '04-CPDE
$$B = \langle V\rangle^\gamma b(\cos\theta),\ \gamma > 0,\ b \approx \theta^{-2-2s},\ 0 < s < 1.$$
- Alexandre-ElSafadi '05, '09
- M-Ukai-Xu-Yang '09-DCDS,

Maxwellian molecule $B = b(\cos\theta) = \theta^{-2}\left(\log\frac{\pi}{\theta}\right)^m, m > 0$;

Gevrey smoothing effect for a linear Boltzmann model for
$$B = b(\cos\theta) \approx \theta^{-2-2s}.$$
Solved by Barbaroux-Dirk-Tobias-Semjon '17-ARMA, ('17-KRM).

- Huo-M-Ukai-Yang '08-KRM, $B = \langle V\rangle\theta^{-2}\left(\log\frac{\pi}{\theta}\right)^m$.
- Ultra-analytic, Gelfand-Silov regularity for Maxellian molecule case around the global equilibrium, $\mu + \sqrt{\mu}g$, by Lerner-M-PravdaStarov-Xu '14 JDE, Glangetas-Li-Xu '16 KRM

$$B = |V|^\gamma b(\cos\theta),\ b \approx \theta^{-2-2s},\ 0 < s < 1,\ \gamma > -2s.$$

- Chen-He '11-ARMA

- Alexandre-M-Ukai-Xu-Yang '12-Kyoto-J.

Smoothing effect for measure valued initial datum except for a single Dirac mass:

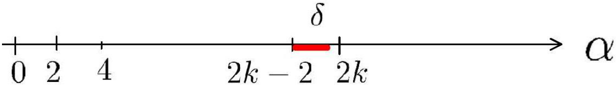
- M-Wang-Yang '16 J.Stat.Phys.

- M-Yang '15 AIHP for Maxwellian molecule case, including infinite energy solution, such as, Bobylev-Cercignani self-similar solution, M-Wang-Yang '15 JMPA, '17 AA, Cho-M-Wang-Yang '16 SIMA.

$$\mathcal{F}(P_\alpha(\mathbb{R}^d)),\ \ \alpha > 0.$$

Generalized Toscani metric

For $\alpha > 0$, we pose $\alpha = 2k - 2 + \delta$, $\delta \in [0, 2)$, $k = 1, 2, 3, \cdots$,



and set

$$\mathcal{M}_k^\delta = \{\varphi \in \mathcal{F}(P(\mathbb{R}^d))\,;\ \int \frac{\Delta^k \varphi(\xi)}{|\xi|^{d+2k-2+\delta}} d\xi < \infty\}, \tag{5}$$

where

$$\begin{aligned}\Delta^1 \varphi(\xi) &= \frac{2\varphi(0) - \varphi(\xi) - \varphi(-\xi)}{4} \\ &= \frac{1 - \mathrm{Re}\ \varphi(\xi)}{2} = \int \sin^2 \frac{v \cdot \xi}{2} dF(v),\end{aligned}$$

$$\Delta^2\varphi(\xi) = \frac{6\varphi(0) - 4\varphi(\xi) - 4\varphi(-\xi) + \varphi(2\xi) + \varphi(-2\xi)}{16}$$
$$= \frac{3 - 4\,\mathrm{Re}\ \varphi(\xi) + \mathrm{Re}\ \varphi(2\xi)}{8} = \int \sin^4 \frac{v \cdot \xi}{2} dF(v)$$

and generally for $k \in \mathbb{N}^+$,

$$\Delta^k\varphi(\xi) = \frac{1}{2}\sum_{j=0}^{k} c_{k,j}(\varphi(j\xi) + \varphi(-j\xi))$$
$$= \sum_{j=0}^{k} c_{k,j}\,\mathrm{Re}\ \varphi(j\xi) = \int \sin^{2k} \frac{v \cdot \xi}{2} dF(v).$$

Here, $c_{k,j}$ are the coefficients of the expansion

$$\sin^{2k}\frac{x}{2} = \sum_{j=0}^{k} c_{k,j}\cos(jx) \text{ for all } x \in \mathbb{R}, \tag{6}$$

and an inductive calculation gives

$$c_{k,0} = 2^{-2k}\binom{2k}{k}, \quad c_{k,j} = (-1)^j 2^{-2k+1}\binom{2k}{k+j}, \; j = 1, \cdots, k.$$

For $\varphi, \tilde{\varphi} \in \mathcal{M}_k^\delta$, put

$$||\varphi - \tilde{\varphi}||_{\mathcal{M}_k^\delta} = \int_{\mathbb{R}^d} \frac{|\Delta^k\varphi(\xi) - \Delta^k\tilde{\varphi}(\xi)|}{|\xi|^{d+2k-2+\delta}} d\xi,$$

and introduce the distance

$$dis_{k,\delta,\beta}(\varphi, \tilde{\varphi}) = ||\varphi - \tilde{\varphi}||_{\mathcal{M}_k^\delta} + ||\varphi - \tilde{\varphi}||_\beta, \tag{7}$$

where $\beta \in (0, \min\{\alpha, 2\}]$.

Proof of smoothing effect

Our proof is based on the time dependent mollifier in $[0, T]$,

$$M^\delta(t,\xi) = \frac{\langle \xi \rangle^{\lambda(t)}}{(1+\delta\langle\xi\rangle)^{N_0}}, \ \ \delta > 0,$$

where $\lambda(t) = Nt - (3/2+\varepsilon)$ and $N_0 > NT$.
If $F_t \in L^\infty([t_0, T]; P_\ell)$ then

$$M^\delta(t, D_v)\langle v\rangle^\ell F_t \in H^\varepsilon(\mathbb{R}^3), \ \ t \in [t_0, T].$$

Take a $\phi_c(V) \in C_0^\infty([0, 1/2])$ satisfying $\phi_c = 1$ near 0 and divide

$$Q(f,g) = Q_c(f,g) + Q_{\bar c}(f,g),$$

where Q_c is defined by B replaced by $B(V, \cos\theta)\phi_c(V)$.

Lemma 2 (Commutator for regular part, Huo-M-Ukai-Yang-'08)

If $M_\lambda^\delta(D_v) = \dfrac{\langle D_v \rangle^\lambda}{(1+\delta\langle D_v\rangle)^{N_0}}$ *then we have*

$$\left|\left(M_\lambda^\delta(D_v)Q_{\bar c}(f,g) - Q_{\bar c}(f, M_\lambda^\delta(D_v)g), h\right)\right| \lesssim \|f\|_{L^1_1}\|M_\lambda^\delta(D_v)g\|_{L^2_{1/2}}\|h\|_{L^2_{1/2}}.$$

Proof is done by using the Littlewood-Payley decomposition

$$\sum_{k=0}^\infty \phi_k(v) = 1, \quad \phi_k(v) = \phi(2^{-k}v) \text{ for } k \ge 1$$

$$\text{with } 0 \le \phi_0, \phi \in C_0^\infty(\mathbb{R}^3),$$

$$\text{supp } \phi_0 \subset \{|v| < 2\}, \quad \text{supp } \phi \subset \{1 < |v| < 3\}.$$

and the pseudo-differential calculus.

Lemma 3 (Commutator for singular part, AMUXY-Kyoto-J.'12)

Assume $N_0 - \lambda < 3$. *Then we have:*
1) If $\lambda < 3/2$, *then*

$$\left|\left(M_\lambda^\delta Q_c(f,g) - Q_c(f, M_\lambda^\delta g), h\right)\right| \lesssim ||f||_{L^1} ||M_\lambda^\delta g||_{L^2} \, ||h||_{L^2} \, .$$

2) If $\lambda \geq 3/2$, *then*

$$\left|\left(M_\lambda^\delta Q_c(f,g) - Q_c(f, M_\lambda^\delta g), h\right)\right|$$
$$\lesssim \left(||f||_{L^1} + ||f||_{H^{(\lambda-3)^+}}\right) ||M_\lambda^\delta g||_{L^2} \, ||h||_{L^2} \, .$$

Outline of the proof of Lemma 3

Note that

$$B(V,\cos\theta)\phi_c(V) = \frac{V}{\theta^2}\phi_c(V)\left(\log\frac{\pi}{V^2\theta}\right)^m, \quad m \in \mathbb{N},$$
$$= \sum_{j=0}^{m}\binom{m}{j}2^j V(-\log V)^j\phi_c(V)\theta^{-2}\left(\log\frac{\pi}{\theta}\right)^{m-j}$$

because $e(V) = 0$ on supp $\phi_c \subset [0,1/2]$. If we put $\Phi_{c,j}(v) = |v|(\log|v|)^j\phi_c(|v|)$ then we can write

$$B(V,\cos\theta)\phi_c(V) = \sum_{j=0}^{m}\Phi_{c,j}(v-v_*)b_j(\cos\theta).$$

It follows from the general Bobylev formula (see the Appendix of Alexandre-Desvillettes-Villani-Wennberg-'00) that

$$(2\pi)^3\Big(M_\lambda^\delta(D)\,Q_c(f,g) - Q_c(f, M_\lambda^\delta(D)\,g), h\Big)$$

$$= \sum_{j=0}^{m} \iiint b_j\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)[\hat{\Phi}_{c,j}(\xi_* - \xi^-) - \hat{\Phi}_{c,j}(\xi_*)]$$

$$\times\Big(M_\lambda^\delta(\xi) - M_\lambda^\delta(\xi-\xi_*)\Big)\hat{f}(\xi_*)\hat{g}(\xi-\xi_*)\overline{\hat{h}(\xi)}d\xi d\xi_* d\sigma,$$

where $\xi^- = \frac{1}{2}(\xi - |\xi|\sigma)$. Since for any small $\varepsilon > 0$ and any $\beta \in \mathbb{Z}^3_+$ there exists a $C_{\varepsilon,\beta} > 0$ such that

$$|\partial_\xi^\beta \hat{\Phi}_{c,j}(\xi)| \le C_{\varepsilon,\beta}\langle\xi\rangle^{-4-|\beta|+\varepsilon}, \quad \text{for all } \xi \in \mathbb{R}^3, \tag{8}$$

the proof of the lemma can be done by the same way as in Proposition 3.4 of AMUXY-Kyoto J. ’12.

For the proof of (8), we recall elementary formulas concerning the Fourier transform of $r^{-1}(\log r)^j$, $r = |x|, x \in \mathbb{R}^d, d \geq 2$. Note

$$\mathcal{F}_{x\to\xi}\left[\frac{1}{r^\alpha}\right](\xi) = \pi^{\alpha-\frac{d}{2}}\frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}\frac{1}{\rho^{d-\alpha}},\ \ 0<\alpha<d,\ \ \rho = |\xi|,$$

$$\mathcal{F}_{x\to\xi}\left[\frac{(-\log r)^j}{r}\right](\xi) = \mathcal{F}_{x\to\xi}\left[\frac{d^j}{d\alpha^j}\left(\frac{1}{r^\alpha}\right)\right](\xi)\Big|_{\alpha=1}$$

$$= \frac{d^j}{d\alpha^j}\left\{\pi^{\alpha-\frac{d}{2}}\frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}\frac{1}{\rho^{d-\alpha}}\right\}\Bigg|_{\alpha=1}$$

$$= \pi^{1-\frac{d}{2}}\frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\frac{(\log\rho)^j}{\rho^{d-1}} + \sum_{k=1}^{j} c_{d,k}\frac{(\log\rho)^{j-k}}{\rho^{d-1}},\ \ c_{d,k}\in\mathbb{R}.$$

Apply the last formula with $d = 3$ to $\Phi_j^0(v) = |v|^{-1}(\log|v|)^j$.

Put $\Phi^0_{c,j}(v) = |v|^{-1}(\log|v|)^j \phi_c(v)$. Then, for the proof of (8) it suffices to show

$$|\partial^\beta \hat{\Phi}^0_{c,j}(\xi)| \lesssim \langle \xi \rangle^{-2-|\beta|+\varepsilon},$$

because $\hat{\Phi}_{c,j}(\xi) = (-\Delta_\xi)\hat{\Phi}^0_{c,j}(\xi)$.
Let $\psi = \psi(\xi)$ be a smooth positive function supported on $|\xi| \le 1$ and equal to 1 for $|\xi| \le 1/2$. Then we have

$$\partial^\beta_\xi \hat{\Phi}^0_{c,j}(\xi) = \int_{\mathbb{R}^3_\eta} \hat{\Phi}^0_j(\xi - \eta)\partial^\beta_\eta \phi_c(\eta) d\eta = J_1 + J_2,$$

where

$$J_1 = \int_{|\xi-\eta|\le 1} \hat{\Phi}^0_j(\xi - \eta)\psi(\xi - \eta)\partial^\beta_\eta \phi_c(\eta) d\eta,$$

$$
\begin{aligned}
J_2 &= \int_{|\xi-\eta|\ge 1/2} \hat{\Phi}_j^0(\xi-\eta)(1-\psi(\xi-\eta))\partial_\eta^\beta \phi_c(\eta)d\eta \\
&= \int_{|\xi-\eta|\ge 1/2} \partial_\xi^\beta \hat{\Phi}_j^0(\xi-\eta)(1-\psi(\xi-\eta))\phi_c(\eta)d\eta \\
&- \sum_{\beta'+\beta''=\beta, \beta''\neq 0} \int_{1/2\le|\xi-\eta|\le 1} \left(\partial_\xi^{\beta'}\hat{\Phi}_j^0\right)(\xi-\eta)\left(\partial_\xi^{\beta''}\psi\right)(\xi-\eta)\phi_c(\eta)d\eta \\
&= J_{2,1} + J_{2,2}.
\end{aligned}
$$

Since $\phi_c \in \mathcal{S}$ and $\langle\xi\rangle \sim \langle\eta\rangle$ if $|\xi-\eta| \le 1$ for any $k > 0$ we have

$$|J_1| \lesssim \int_{|\xi-\eta|\le 1} \frac{1}{|\xi-\eta|^{2+\varepsilon}} \langle\eta\rangle^{-k} d\eta \lesssim \langle\xi\rangle^{-k}.$$

When $\beta \neq 0$, similarly we have

$$|J_{2,2}| \lesssim \int_{1/2\le|\xi-\eta|\le 1} \frac{1}{|\xi-\eta|^{2+|\beta'|+\varepsilon}} \langle\eta\rangle^{-k} d\eta \lesssim \langle\xi\rangle^{-k}.$$

On the other hand, choosing $k = 5 + |\beta|$ we have

$$|J_{2,1}| \lesssim \int_{|\xi-\eta|\ge 1/2} \frac{1}{|\xi-\eta|^{2+|\beta|-\varepsilon}} \langle\eta\rangle^{-k} d\eta$$

$$= \int_{|\xi-\eta|\ge 1/2} \left(\frac{1}{|\xi-\eta|\langle\eta\rangle}\right)^{2+|\beta|-\varepsilon} \frac{d\eta}{\langle\eta\rangle^{3-\varepsilon}} \lesssim \langle\xi\rangle^{-2-|\beta|+\varepsilon}.$$

Coercivity estimate

Proposition 2 (Fournier '15 A.A. Probab., Villani Chapter 2-6.2, '02)

Assume that the initial datum $F_0 \in P_2(\mathbb{R}^3)$ is not a single Dirac mass. Then, for any energy conservative weak solution $F_t \in C([0,\infty); P_2(\mathbb{R}^3))$, we have

$$\mathrm{supp}\, F_t = \mathbb{R}^3 \text{ for all } t > 0.$$

By this proposition and the method in [M-Yang '15], we see that for any $t_0 > 0$, $R > 0$ there exist $c_0 > 0$ and $0 < c_1 < c_2$ such that

$$\widehat{F_{t_0}\chi_{B(R)}}(0) - |\widehat{F_{t_0}\chi_{B(R)}}(\xi)| \geq c_0 \text{ if } c_1 \leq |\xi| \leq c_2, \tag{9}$$

where $\chi_{B(R)}(v)$ is a characteristic function on a ball $B(R)$ centered at the origin with a radius R.

Lemma 4 (Coercivity)

For any $f \in \mathcal{S}$ *we have*

$$-\Big(Q(F_{t_0}, f), f\Big) = \iiint B(f^2 - f'f)dvd\sigma dF_{t_0}(v_*)$$
$$\geq c_0' \|(\log\langle D_v\rangle)^{m/2} f\|^2_{L^2_{1/2}} - C_N \|(\log\langle v\rangle)^{m/2} f\|^2_{H^{-N}_{1/2+4N}} .$$

If $dF_0(v) = f_0(v)dv$ with $f_0 \in L^1_2 \cap L \log L$, then

$$-\Big(Q(f_0, f), f\Big)$$
$$\geq c_0' \|(\log\langle D_v\rangle)^{(1+m)/2} f\|^2_{L^2_{1/2}} - C_N \|(\log\langle v\rangle)^{(1+m)/2} f\|^2_{H^{-N}_{1/2+8N}} .$$

Indeed, it follows from Lemma 3 of ADVW '00 that, instead of (9), we have

$$\widehat{f_0\chi_{B(R)}}(0) - |\widehat{f_0\chi_{B(R)}}(\xi)| \geq c_0 \text{ if } c_1 \leq |\xi|. \tag{10}$$

Note $-2(Q(g,f),f) = \iiint Bg_*(f'-f)^2 + \iiint Bg_*(f^2 - f'^2)$.
The second term can be estimated by using the cancellation lemma of [ADVW]. Let $\phi_R(v)$ be a non-negative smooth function supported on $|v| \geq 2R$ and equal to 1 for $|v| \geq 4R$. We use the Littlewood-Payley decomposition $\sum_{2^k \gtrsim R} \phi_k(v)^2 = 1$ over supp $\varphi_R(v)$. Notice that

$$Bg_*(f'-f)^2 \geq B(g\chi_{B(R)})_* \sum_{2^k \gtrsim R} \phi_k(v)^2 \varphi_R^2(v)(f'-f)^2$$

$$\gtrsim \sum_{2^k \gtrsim R} \frac{2^k}{\theta^2}\left(\log\left(\frac{\pi}{2^{2k}\theta}\right)\right)^m (g\chi_{B(R)})_* \left[\frac{1}{2}\left((\phi_k f)' - \phi_k f\right)^2 - \left(\phi_k' - \phi_k\right)^2 f'^2\right]$$

Put $b_k(\cos\theta) = \dfrac{1}{\theta^2}\left(\log\left(\dfrac{\pi}{2^{2k}\theta}\right)\right)^m$. Then it follows from Corollary 3 of [ADVW] that

$$\iiint 2^k b_k(\cos\theta)(g\chi_{B(R)})_*\Big((\phi_k f)' - \phi_k f\Big)^2 dv dv_* d\sigma$$

$$\geq \frac{2^k}{2(2\pi)^2}\int_{|\xi|\geq c_2 2^{4k}/\pi}\Big(\widehat{g\chi_{B(R)}}(0) - |\widehat{g\chi_{B(R)}}(\xi^-)|\Big)|\widehat{\phi_k f}(\xi)|^2$$

$$\times\int_{2\sin^{-1}\frac{c_1}{|\xi|}}^{2\sin^{-1}\frac{c_2}{|\xi|}} \frac{\sin\theta}{\theta^2}\Big(\frac{1}{2}\log\frac{1}{\theta}\Big)^m d\theta d\xi$$

$$\geq c_0 c_m(c_1, c_2)\int_{|\xi|\geq c_2 2^{4k}/\pi} |(\log|\xi|)^{m/2} 2^{k/2}\widehat{\phi_k f}(\xi)|^2 d\xi,$$

where we used (9), that is,

$$0 < \exists c_1 < c_2, \exists c_0 > 0 \;\; \widehat{g\chi_{B(R)}}(0) - |\widehat{g\chi_{B(R)}}(\xi^-)| \geq c_0$$

$$\text{if } c_1 \leq |\xi^-| = |\xi|\sin\tfrac{\theta}{2} \leq c_2,$$

and the fact that

$$\pi/(2^{2k}\theta) \geq 1/\theta^{1/2} \text{ if } \theta \leq 2\sin^{-1}\frac{c_2}{|\xi|} \text{ and } |\xi| \geq 2^{4k}c_2/\pi.$$

Consequently we have

$$\iiint 2^k b(\cos\theta)(g\chi_{B(R)})_*\Big((\phi_k f)' - \phi_k f\Big)^2 dv dv_* d\sigma$$
$$\geq \frac{1}{2} c_0 c_m(c_1, c_2) ||(\log\langle D_v\rangle)^{m/2}\phi_k f||^2_{L^2_{1/2}}$$
$$- C||(\log\langle v\rangle)^{m/2}\langle v\rangle^{1/2+4N}\phi_k f||^2_{H^{-N}}.$$

As a conclusion we have the coercive estimate

$$-\Big(Q(g,f), f\Big) \geq \tilde{c}_0 ||(\log\langle D_v\rangle)^{m/2} f||^2_{L^2_{1/2}} - C_N ||(\log\langle v\rangle)^{m/2} f||^2_{H^{-N}_{1/2+4N}} .$$

Thank you for your attention !

ご静聴ありがとうございました。