Global Well-Posedness of the Non-Cutoff Boltzmann Equation with Polynomial Decay Perturbations

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International Conference on Nonlinear Analysis: Kinetic Theory, Gas Dynamics, and Related Fields
Institute of Mathematics, Academia Sinica, Taipei, Taiwan
October 30, 2017

Outline

- Basic Setting;
- Main result and sketch of proof:
    - Framework: three main steps;
    - Outline of each step.
- Future Questions.

Setting: Boltzmann Equation

- Consider the Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F = Q(F,F)\,,$$
$$F(0,x,v) = F^{in}(x,v) \geq 0\,,$$

where

$$Q(F,F) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)|v-v_*|^\gamma \left(F_*'F' - F_*F\right) \,\mathrm{d}\sigma\,\mathrm{d}v_*\,.$$

- $x \in \mathbb{T}^3$, $v \in \mathbb{R}^3$, $\cos\theta = \dfrac{v-v_*}{|v-v_*|}\cdot\sigma$.
- (v, v_*), (v', v_*'): pre- and post-collision pairs or vice versa.
- Hard potential: $\gamma > 0$.

Setting: Collision Kernel and Initial Condition

- Collision kernel $b(\cos\theta)$ satisfies

$$b(\cos\theta) \sim \frac{C}{\theta^{2+2s}} \quad \text{near } \theta = 0\,,$$

with $0 < s < 1/2$.

Setting: Collision Kernel and Initial Condition

- Collision kernel $b(\cos\theta)$ satisfies

$$b(\cos\theta) \sim \frac{C}{\theta^{2+2s}} \quad \text{near } \theta = 0\,,$$

with $0 < s < 1/2$.

- Initial data $F^{in}(x, v)$ satisfies

$$F^{in} = M(v) + f^{in}\,,$$

where M is the absolute Maxwellian and f^{in} satisfies

$$\langle v \rangle^K f^{in} \in H^2_x L^2_v$$

for some $K \in \mathbb{N}$ large enough and $\langle v \rangle = \sqrt{1 + |v|^2}$.

Literature Reivew: Wellposedness Theory

- Non-cutoff Boltzmann
  - Renormalized solutions: Alexandre-Villani (2001, CPAM)
  - Global well-posedness with Gaussian tails:
    - Gressman-Strain (JAMS 2011),
    - Alexandre-Morimoto-Ukai-Xu-Yang (AMUXY) (CMP 2011, JFA 2012)
  - Polynomial decay perturbation
    - Local existence for the non-cutoff case with soft potentials:

      Morimoto-Yang (Analy. Appl. 2014)
    - Global well-posedness for the cut-off case with hard potentials: Gualdani-Mischler-Mouhot (to appear as Mémoire de la Société Mathématique de France).
    - Global well-posedness for the non-cutoff with hard potentials:

      Hérau,Tonon, and Tristani (arXiv1710.01098, 2017)

What is different between the polynomial decay and Gaussian tail?

Difference: Polynomial and Exponential

- Exponential: $F = M + \sqrt{M}\, f$
  - Linearized operator is self-adjoint:

$$
\begin{aligned}
L_M f &= \frac{1}{\sqrt{M}} \Big(Q(M, \sqrt{M} f) + Q(\sqrt{M} f, M) \Big) \\
&= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b(\cos\theta) |v - v_*|^\gamma \left(\sqrt{M'_*} f' + \sqrt{M'} f'_* - \sqrt{M} f_* - \sqrt{M_*} f \right) \mathrm{d}\sigma \, \mathrm{d}v_*
\end{aligned}
$$

  - Null space of L_M:

$$
\mathrm{Null} L_M = \mathrm{Span} \left\{ \sqrt{M}, \ \ \sqrt{M}\, v, \ \ \sqrt{M}\, |v|^2 \right\} .
$$

[1] AMUXY, JFA, 2012. See also Gressman-Strain, JAMS, 2011.

Difference: Polynomial and Exponential

- Exponential: $F = M + \sqrt{M}\, f$
  - Linearized operator is self-adjoint:

$$L_M f = \frac{1}{\sqrt{M}}\Big(Q(M, \sqrt{M} f) + Q(\sqrt{M} f, M)\Big)$$
$$= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)|v - v_*|^\gamma \left(\sqrt{M'_*} f' + \sqrt{M'} f'_* - \sqrt{M} f_* - \sqrt{M_*} f\right) \mathrm{d}\sigma\, \mathrm{d}v_*$$

  - Null space of L_M:

$$\mathrm{Null} L_M = \mathrm{Span}\left\{\sqrt{M},\ \sqrt{M}\, v,\ \sqrt{M}\, |v|^2\right\}.$$

  - Coercivity estimate [1]: $f \in (\mathrm{Null} L)^\perp$

$$\langle f,\ L_M f\rangle_{L^2_v} \le -c_0 \left(\|f\|^2_{H^s_{\gamma/2}} + \|f\|^2_{L^2_{s+\frac{\gamma}{2}}} \right).$$

[1] AMUXY, JFA, 2012. See also Gressman-Strain, JAMS, 2011.

- Polynomial: $F = M + f$
  - Linearized operator is non-self-adjoint:

$$Lf = Q(M, f) + Q(f, M) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} (M'_* f' + M' f'_* - M f_* - M_* f)\, b(\cos\theta)|v - v_*|^\gamma \, \mathrm{d}\sigma \, \mathrm{d}v_* \, .$$

  - Coercivity estimate [2]:

$$\langle Q(M, f), \; f\rangle \le -c_0 \, \|f\|^2_{H^s_{\gamma/2}} + C \, \|f\|^2_{L^2_{\gamma/2}} \quad .$$

[2] AMUXY, Kyoto J. Math. 2012.

One Main Tool: Enlargement Theorem

- Enlargement Theorem (A special case)

Theorem (Gualdani-Mischler-Mouhot, to appear)

Let $E \subset \mathcal{E}$ and $L = A + B$. If $L : E \to E$ satisfies

- *Essential spectrum of L is contained in $\{z \in \mathcal{C} \mid Re z \leq -a_0 < 0\}$,*
- *(spectrum of L) $\cap \{z \in \mathcal{C} \mid Re z > -a_0\} = \{\lambda_1, \lambda_2, \cdots, \lambda_N\}$,*

and

- *$A : \mathcal{E} \to \mathcal{E}$ is closed and strictly dissipative,*
- *$B : \mathcal{E} \to E$ is bounded,*
- *$\left\|(B\mathcal{S}_A)^{*n}(t)\right\|_{\mathcal{E}\to E} \leq Ce^{-a_0 t}$ for some n.*

then $L : \mathcal{E} \to \mathcal{E}$ satisfies that

$$(\text{spectrum of } L) \cap \{z \in \mathcal{C} \mid Re z > -a_0\} = \{\lambda_1, \lambda_2, \cdots, \lambda_N\}.$$

Main A Priori Estimate

- Equation in terms of the perturbation f:

$$\partial_t f + v \cdot \nabla_x f = Lf + Q(f,f)\,, \qquad (0.1)$$
$$f(0,x,v) = f^{in}(x,v)\,,$$

where $\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} f^{in}\phi(v)\,\mathrm{d}x\,\mathrm{d}v = 0$ for any $\phi \in \mathrm{Null}L$.

Theorem (Alonso-Morimoto-S.-Yang, 2017)

Suppose f is a classical solution to (0.1). Suppose $0 < s < 1/2$ and $\gamma > 0$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}|||f|||^2 \leq (-c_0 + C|||f|||)\left|\left|\left|\langle v\rangle^{\gamma/2} f\right|\right|\right|^2.$$

Function Space

- Define the weight function [3]:

$$W(v) = \begin{cases} \langle v \rangle\,, & \text{if } 0 < s \le 1/4\,, \\ \langle v \rangle^{\frac{2s}{1-2s}}\,, & \text{if } 1/4 < s < 1/2\,. \end{cases}$$

- Define

$$Y_K = \{h \in L^2_{x,v} \big| \, W^{K-|\alpha|}\partial^\alpha f \in L^2_{x,v}, \quad |\alpha| \le 2, \,, \; K \ge 2 + k_0\}\,,$$

with k_0 large enough. Define

$$\|f\|_{Y_K} = \sum_{|\alpha|\le 2} \left\| W^{K-|\alpha|}\partial_x^\alpha f \right\|_{L^2_{x,v}}\,.$$

[3] AMUXY, AA, 2014.

Wellposedness Theorem

- Main Theorem

Theorem (Alonso-Morimoto-S.-Yang, 2017)

Suppose $0 < s < 1/2$ *and* $\gamma > 0$. *Then there exist* K *large enough and* $\epsilon_0 > 0$ *small enough such that if the initial data satisfies* $F^{in} = M + f^{in} \geq 0$ *with*

$$\left\| f^{in} \right\|_{Y_K} < \epsilon_0 \,, \qquad \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f^{in} \phi(v) \, \mathrm{d}v \, \mathrm{d}x = 0$$

for any $\phi \in NullL$, *then the non-cutoff Boltzmann equation has a unique nonnegative solution in* $C([0, \infty), Y_K)$.

- Iteration scheme: [4]

$$\partial_t F_{n+1} + v \cdot \nabla_x F_{n+1} = Q(F_n, F_{n+1}) \,, \qquad F_{n+1} = F^{in} \geq 0 \,.$$

[4] AMUXY, ARMA, 2011.

Closed Energy Estimate

Three components

- Three main components:
  - Propagation of moments;
  - Spectral gap of $\mathcal{L} - v \cdot \nabla_x$;
  - Regularization of the semigroup generated by $\mathcal{L} - v \cdot \nabla_x$.

Propagation of Moments

First Component: Propagation of Moments

- First key component:
  - Recall

$$Y_K = \left\{ h \in L^2_{x,v} \middle| \, W^{K-|\alpha|}\partial_x^\alpha f \in L^2_{x,v}, \quad |\alpha| \leq 2, \, , \; K \geq 2 + k_0 \right\} ,$$

and

$$\|f\|_{Y_K} = \sum_{|\alpha|\leq 2} \left\| W^{K-|\alpha|}\partial_x^\alpha f \right\|_{L^2_{x,v}} .$$

  - Main estimate: Propagation of moments

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{Y_K} \leq -c_0 \left(\frac{1}{4} - C_K \|f\|_{Y_K} \right) \left\| \langle v \rangle^{\gamma/2} f \right\|_{Y_K}$$
$$- c_1 \sum_{|\alpha|\leq 2} \left\| \langle v \rangle^{(K-|\alpha|)} \partial_x^\alpha f \right\|^2_{H^s_{\gamma/2}} + C_K \|f\|^2_{Y_K} .$$

Propagation of Moments

- Equation

$$\partial_t f + v \cdot \nabla_x f = Q(F, f) + Q(f, M),$$

  with $F = M + f \geq 0$.

- Denote $\|f\|_{L^2_k} = \left\|W^k f\right\|_{L^2}$.
- Direct bilinear estimate [5] does not work for $Q(f, f)$ or $Q(f, M)$:

$$\left\|Q(f, g)\right\|_{H^{-s}_k} \leq C \|f\|_{L^1_{k+\gamma+2s}} \|g\|_{H^s_{k+\gamma+2s}} .$$

[5] AMUXY, ARMA 2010.

Propagation of Moments

- Bound for $Q(F, f)$:

$$\int_{\mathbb{T}^3} \int_{\mathbb{R}^3} Q(F, f)\, f\, W^{2k} \leq -c_0 \left(1 - C_k \sup_{\mathbb{T}^3} \int_{\mathbb{R}^3} |f|^{1+\gamma}\, W^{\gamma_0}\, \mathrm{d}v\right) \|f\|^2_{L^2_{k+\frac{\gamma}{2}}} + C_k \left(1 + \sup_{\mathbb{T}^3} \int_{\mathbb{R}^3} |f|\, W^{\gamma_1}\, \mathrm{d}v\right) \|f\|^2_{L^2_k} .$$

[6] Desvillettes-Mouhot, Ann. I. H. Poincaré, 2005.

Propagation of Moments

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- Structure: [6]

$$\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} Q(F,f)\, f\,\langle v\rangle^{2k}\,\mathrm{d}v\,\mathrm{d}x$$
$$= \int b(\cos\theta)|v-v_*|^\gamma F_* \left(|f|\,|f'|\langle v'\rangle^k\langle v\rangle^k\cos^k\left(\tfrac{\theta}{2}\right) - |f|^2\langle v\rangle^{2k}\right)\mathrm{d}\mu\,\mathrm{d}x$$
$$+ \int b(\cos\theta)|v-v_*|^\gamma F_*\,|f|\,|f'|\langle v'\rangle^k\left(\langle v'\rangle^k - \langle v\rangle^k\cos^k\left(\tfrac{\theta}{2}\right)\right)\mathrm{d}\mu\,\mathrm{d}x\,,$$

where $\mathrm{d}\mu = \mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v$.

[6] Desvillettes-Mouhot, Ann. I. H. Poincaré, 2005.

Propagation of Moments

- Bound for $Q(f, M)$: for k large enough,

$$\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} Q(f,M) f \langle v\rangle^{2k} \le C_{\delta,k} \|f\|^2_{L^2_k} + \delta \left\|\langle v\rangle^k f\right\|^2_{L^2_{k+\frac{\gamma}{2}}} ,$$

where $\delta > 0$ can be chosen as small as needed.

[7] Alexandre-Desvillettes-Villani-Wennberg, ARMA, 2000.

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  where $\delta > 0$ can be chosen as small as needed.

- Structure:

$$\begin{aligned}
&\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} Q(f,M)f\langle v\rangle^{2k}\,\mathrm{d}v\,\mathrm{d}x \\
&= \int b(\cos\theta)|v-v_*|^\gamma f_*\left(M'f'\langle v'\rangle^{2k} - Mf\langle v\rangle^{2k}\right)\mathrm{d}\mu\,\mathrm{d}x \\
&\quad + \int b(\cos\theta)|v-v_*|^\gamma f_* f'\langle v'\rangle^{k}\left(M\langle v\rangle^{k} - M'\langle v'\rangle^{k}\right)\mathrm{d}\mu\,\mathrm{d}x \\
&\quad + \int b(\cos\theta)|v-v_*|^\gamma f_* f'\langle v'\rangle^{k} M\left(\langle v'\rangle^{k} - \langle v\rangle^{k}\right)\mathrm{d}\mu\,\mathrm{d}x\,.
\end{aligned}$$

- Cancellation Lemma [7] + large k.

[7] Alexandre-Desvillettes-Villani-Wennberg, ARMA, 2000.

Regularization of $\mathcal{S}_{\mathcal{L}}$

Second Component: Regularization

- Second key component:
  - Define the pseudo-differential operator $\mathcal{M}$ with the symbol

$$\mathcal{M}(t,\ell,\xi) = \left(1 + \delta \int_0^{T-t} \langle \xi - 2\pi\tau\ell \rangle^{2s} \, \mathrm{d}\tau\right)^{-1/2-\epsilon}, \quad 0 < t < T$$

    for $0 < \epsilon < \frac{2s}{1-s}$ and $\delta > 0$ small enough. Here ξ, ℓ are the Fourier variables of v, x respectively.

  - Roughly,

$$\mathcal{M} \sim \left(1 + \delta\left(\langle \xi \rangle^{2s} + \langle \eta \rangle^{2s}\right)(T-t)\right)^{-1/2-\epsilon}.$$

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  - Roughly,

$$\mathcal{M} \sim \left(1 + \delta \left(\langle \xi \rangle^{2s} + \langle \eta \rangle^{2s}\right)(T - t)\right)^{-1/2-\epsilon}.$$

  - Regularization: for any $T_0 > 0$ small enough and $k \geq 0$,

$$\int_0^{T_0} \left\| \langle v \rangle^k h(t,\cdot,\cdot) \right\|_{L^2_{x,v}}^2 \mathrm{d}t \leq \frac{1}{c_0 \epsilon \delta} \left\| \langle v \rangle^k h^{in} \right\|_{H^{-s}_{x,v}}^2$$

    and

$$\left\| \langle v \rangle^k f(T_0,\cdot,\cdot) \right\|_{L^2_{x,v}}^2 \leq C \left(\frac{1}{T_0} + \frac{1}{c_0 \epsilon \delta} \right) \left\| \langle v \rangle^k h^{in} \right\|_{H^{-s}_{x,v}}^2.$$

Regularization of $\mathcal{L}$

- Commutator of $\mathcal{M}$ with transport: denoting $\eta = 2\pi\ell$,

$$(\partial_t - \eta \cdot \nabla_\xi)\mathcal{M}(t,\xi,\eta) = \left(\frac{1}{2} + \varepsilon\right)\mathcal{M}(t,\xi,\eta)\frac{\delta\langle\xi\rangle^{2s}}{1 + \delta\int_0^{T-t}\langle\xi - \tau\eta\rangle^{2s}\,\mathrm{d}\tau},$$

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- Commutator estimate of $\mathcal{M}$ with Q:

$$\left|\left(\mathcal{M}Q_R(f,g) - Q_R(f,\mathcal{M}g), h\right)\right| \leq C_R\|f\|_{L^1}\|\mathcal{M}g\|_{H^{s'}}\|h\|_{H^{s'}},$$

and

$$\left|\left(\mathcal{M}Q_{\bar{R}}(f,g) - Q_{\bar{R}}(f,\mathcal{M}g), h\right)\right| \lesssim \delta^{\frac{s'}{2s}}\|f\|_{L^1_\gamma}\|\mathcal{M}g\|_{H^{s'}_{\gamma/2}}\|h\|_{L^2_{\gamma/2}} + R^{2s-1}\|f\|_{L^1_\gamma}\|\mathcal{M}g\|_{L^2_{\gamma/2}}\|h\|_{L^2_{\gamma/2}},$$

where $0 < s' < s$ is arbitrary and $R^{2s-1} \to 0$ as $R \to \infty$.

- Q_R is associated with the kernel $b(\cos\theta)|v - v_*|^\gamma\chi_R(v - v_*)$.
- $Q_{\bar{R}}$ is associated with $b(\cos\theta)|v - v_*|^\gamma\left(1 - \chi_R(v - v_*)\right)$.

Regularization of $\mathcal{L}$

- Fourier modes:

$$f(t,x,v) = \sum_{\ell \in \mathbb{Z}^3} e^{-2\pi i \ell \cdot x} f_\ell(t,v).$$

- Equation for $\mathcal{M} f_\ell$:

$$(\partial_t - v \cdot (2\pi \ell i))\,(\mathcal{M} f_\ell) = L\,(\mathcal{M} f_\ell) + \text{commutators}\,.$$

- Energy bound:

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\mathcal{M} f_\ell\|^2_{L^2_v} \leq C_R \, \|f_\ell\|^2_{L^2_v}\ .$$

Regularization of $\mathcal{L}$

- Integrate on $[0, T]$ with $T \in [0, T_0]$:

$$\left\|f_\ell(T,\cdot)\right\|_{L_v^2}^2 \leq \left\|\mathcal{M}(0, D_v, \ell) f_\ell^{in}\right\|_{L_v^2}^2 + C_R \int_0^T \left\|f_\ell(s,\cdot)\right\|_{L_v^2}^2 \, \mathrm{d}s \,,$$

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which gives

$$\left\|f_\ell(T,\cdot)\right\|_{L^2_v}^2 \leq \int_{\mathbb{R}^3_\xi} \frac{\langle\xi\rangle^{2s}}{(1 + c_0 \delta T \langle\xi\rangle^{2s})^{1+2\epsilon}} \left|\langle\xi\rangle^{-s} \hat{f}_\ell(0,\xi)\right|^2 \,\mathrm{d}\xi$$

$$+ C_R \int_0^T \left\|f_\ell(s,\cdot)\right\|_{L^2_v}^2 \,\mathrm{d}s\,.$$

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$$+ C_R \int_0^T \left\|f_\ell(s,\cdot)\right\|^2_{L^2_v} \,\mathrm{d}s\,.$$

- Integrate again on $[0,T_0]$ and take T_0 small:

$$\int_0^{T_0} \left\|f_\ell(t,\cdot)\right\|^2_{L^2_v} \mathrm{d}t \le \frac{1}{\epsilon c_0\delta} \left\|f_\ell^{in}\right\|^2_{H^{-s}_v}\,,$$

Regularization of $\mathcal{L}$

- Integrate on $[0, T]$ with $T \in [0, T_0]$:

$$\left\|f_\ell(T,\cdot)\right\|_{L^2_v}^2 \leq \left\|\mathcal{M}(0, D_v, \ell) f_\ell^{in}\right\|_{L^2_v}^2 + C_R \int_0^T \left\|f_\ell(s,\cdot)\right\|_{L^2_v}^2 \, \mathrm{d}s\,,$$

which gives

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$$+ C_R \int_0^T \left\|f_\ell(s,\cdot)\right\|_{L^2_v}^2 \, \mathrm{d}s\,.$$

- Integrate again on $[0, T_0]$ and take T_0 small:

$$\int_0^{T_0} \left\|f_\ell(t,\cdot)\right\|_{L^2_v}^2 \mathrm{d}t \leq \frac{1}{\epsilon c_0 \delta} \left\|f_\ell^{in}\right\|_{H_v^{-s}}^2 \,,$$

- Feed back:

$$\left\|f_\ell(T_0,\cdot)\right\|_{L^2_v} \leq \left(\frac{1}{\sqrt{\delta T_0}} + \frac{C_R}{\sqrt{\epsilon c_0 \delta}}\right) \left\|f_\ell^{in}\right\|_{H_v^{-s}}\,.$$

Spectral Gap

Third component: Spectral Gap

- Third key component:
  - let h be the solution to the linearized equation

$$\partial_t h = \mathcal{L}h = (L - v \cdot \nabla_x)h\,,$$
$$h(0,x,v) = h^{in}(x,v)\,.$$

  - The semigroup generated by $\mathcal{L}$ satisfies

$$\left\|\mathcal{S}_{\mathcal{L}}(t)h^{in}\right\|_{L^2(\langle v\rangle^k \,\mathrm{d}x\,\mathrm{d}v)} \leq e^{-\lambda t} \left\|h^{in}\right\|_{L^2(\langle v\rangle^k \,\mathrm{d}x\,\mathrm{d}v)}$$

    for some $\lambda > 0$, assuming that the initial data h^{in} satisfies the orthogonality condition.

Spectral Gap

- Spectral gap of $\mathcal{L}_M$:
  - essential spectrum $\subseteq \{z \in \mathcal{C} | \, Rez \leq -\alpha_0\}$;
  - finitely many eigenvalues in $\{z \in \mathcal{C} | \, Rez > -\alpha_0\}$.

Spectral Gap

- Spectral gap of $\mathcal{L}_M$:
  - essential spectrum $\subseteq \{z \in \mathcal{C} |\ Re z \le -\alpha_0\}$;
  - finitely many eigenvalues in $\{z \in \mathcal{C} |\ Re z > -\alpha_0\}$.
- Decomposition of $\mathcal{L}$: $\mathcal{L} = \mathcal{L}_{reg} + \mathcal{L}_{sing}$ where the kernels associated with each are

$$\mathcal{L}_{sing} : B_{\delta,\epsilon} = b(\cos\theta)|v - v_*|^\gamma \left(1 - \chi_{\delta \le |v-v_*| < \frac{1}{\delta}}\ \chi_{\theta > \epsilon}\right),$$

$$\mathcal{L}_{reg} : B^c_{\delta,\epsilon} = b(\cos\theta)|v - v_*|^\gamma \chi_{\delta \le |v-v_*| < \frac{1}{\delta}}\ \chi_{\theta > \epsilon}\,.$$

  Specifically,

$$\mathcal{L}_{sing} f = \mathcal{L}_{B_{\delta,\epsilon}} f + Q^-_{B^c_{\delta,\epsilon}}(M,\ f)\,,$$

$$\mathcal{L}_{reg} f = \mathcal{L}_{B^c_{\delta,\epsilon}} f - Q^-_{B^c_{\delta,\epsilon}}(M,\ f)\,.$$

- The operator $\mathcal{L}_{sing}$ is closed and dissipative on $L^2(\langle v\rangle^k \, dv \, dx)$ for k large enough;
- The operator $\mathcal{L}_{reg} : L^2(\langle v\rangle^k \, dv \, dx) \to L^2(M^{-1/2} \, dv \, dx)$ is bounded.

Framework: Second + Third

- An immediate consequence by combining the spectral gap and regularization:

$$\left\| \mathcal{S}_{\mathcal{L}}(t) h^{in} \right\|_{L^2(\langle v \rangle^k \, \mathrm{d}x \, \mathrm{d}v)} \leq C e^{-\lambda t} \left(\frac{1}{(\sqrt{\delta T_0}} + \frac{C_R}{\sqrt{\epsilon c_0 \delta}} \right) \left\| \langle v \rangle^k h^{in} \right\|_{H^{-s}_{x,v}} ,$$

provided h^{in} satisfies the orthogonality condition.

Energy Estimate

Energy Estimate

- The energy bound is obtained by combining the three key components.
- Definition [8] of $|||f|||$:

$$|||f||| = \left(\|f\|_{Y_K}^2 + A_0 \int_0^\infty \left\| \mathcal{S}_{\mathcal{L}}(\tau) f(t,\cdot,\cdot) \right\|^2_{L^2(\langle v\rangle^{k_0}\,\mathrm{d}v; H^2_x)} \,\mathrm{d}\tau \right)^{1/2},$$

where A_0 will be specified later.

[8] Gualdani-Mischler-Mouhot, to appear.

Energy Estimate

- The energy bound is obtained by combining the three key components.
- Definition [8] of $|||f|||$:

$$|||f||| = \left(\|f\|_{Y_K}^2 + A_0 \int_0^\infty \left\| \mathcal{S}_{\mathcal{L}}(\tau) f(t,\cdot,\cdot) \right\|_{L^2(\langle v\rangle^{k_0}\,\mathrm{d}v; H_x^2)}^2 \,\mathrm{d}\tau \right)^{1/2},$$

where A_0 will be specified later.
- Equivalence of norm:

$$|||f||| \sim \|f\|_{Y_K},$$

due to the spectral gap:

$$\int_0^\infty \left\| \mathcal{S}_{\mathcal{L}}(\tau) f \right\|_{L^2(\langle v\rangle^{k_0}\,\mathrm{d}v; H_x^2)}^2 \,\mathrm{d}\tau \le \left(\int_0^\infty e^{-\lambda t}\,\mathrm{d}\tau \right) \|f\|_{L^2(\langle v\rangle^{k_0}\,\mathrm{d}v; H_x^2)}^2 .$$

[8] Gualdani-Mischler-Mouhot, to appear.

Energy Estimate

- Evolution of $\|f\|_{Y_K}$ is given by the propagation of moments:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|_{Y_K}^2 &\leq -c_0\left(\frac{1}{4} - C_K\|f\|_{Y_K}\right)\left\|\langle v\rangle^{\gamma/2} f\right\|_{Y_K} \\ &\quad - c_1 \sum_{|\alpha|\leq 2}\left\|W^{(K-|\alpha|)}\partial_x^\alpha f\right\|_{H^s_{\gamma/2}}^2 + C_K\|f\|_{Y_K}^2 \\ &\leq -c_0\left(\frac{1}{8} - C_K\|f\|_{Y_K}\right)\left\|\langle v\rangle^{\gamma/2} f\right\|_{Y_K} \\ &\quad - c_1 \sum_{|\alpha|\leq 2}\left\|W^{(K-|\alpha|)}\partial_x^\alpha f\right\|_{H^s_{\gamma/2}}^2 + \tilde{C}_K\|f\|_{L^2_v H^2_x}^2 .\end{aligned}$$

- The term left to control is $\tilde{C}_K\|f\|_{L^2_v H^2_x}^2$.

Energy Estimate

- Evolution of the semigroup part (Special case for L^2_x):

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \left\|\mathcal{S}_{\mathcal{L}}(\tau)f(t,\cdot,\cdot)\right\|^2_{L^2(\langle v\rangle^{k_0}\,\mathrm{d}x\,\mathrm{d}v)}\,\mathrm{d}\tau \\
&= \int_0^\infty\int_{\mathbb{T}^3}\int_{\mathbb{R}^3}\left(\mathcal{S}_{\mathcal{L}}(\tau)f(t,x,v)\right)\left(\mathcal{S}_{\mathcal{L}}(\tau)\partial_t f(t,x,v)\right)\langle v\rangle^{k_0}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau \\
&= \int_0^\infty\int_{\mathbb{T}^3}\int_{\mathbb{R}^3}\left(\mathcal{S}_{\mathcal{L}}(\tau)f(t,x,v)\right)\left(\mathcal{S}_{\mathcal{L}}(\tau)\mathcal{L}f(t,x,v)\right)\langle v\rangle^{k_0}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau \\
&+ \int_0^\infty\int_{\mathbb{T}^3}\int_{\mathbb{R}^3}\left(\mathcal{S}_{\mathcal{L}}(\tau)f(t,x,v)\right)\left(\mathcal{S}_{\mathcal{L}}(\tau)Q(f,f)(t,x,v)\right)\langle v\rangle^{k_0}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau \\
&\triangleq E_1 + E_2\,.
\end{aligned}
$$

Energy Estimate

- Evolution of the semigroup part (Special case for L^2_x):

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \left\|\mathcal{S}_{\mathcal{L}}(\tau)f(t,\cdot,\cdot)\right\|^2_{L^2(\langle v\rangle^{k_0}\,\mathrm{d}x\,\mathrm{d}v)}\,\mathrm{d}\tau \\
&= \int_0^\infty\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \left(\mathcal{S}_{\mathcal{L}}(\tau)f(t,x,v)\right)\left(\mathcal{S}_{\mathcal{L}}(\tau)\partial_t f(t,x,v)\right)\langle v\rangle^{k_0}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau \\
&= \int_0^\infty\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \left(\mathcal{S}_{\mathcal{L}}(\tau)f(t,x,v)\right)\left(\mathcal{S}_{\mathcal{L}}(\tau)\mathcal{L}f(t,x,v)\right)\langle v\rangle^{k_0}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau \\
&+ \int_0^\infty\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \left(\mathcal{S}_{\mathcal{L}}(\tau)f(t,x,v)\right)\left(\mathcal{S}_{\mathcal{L}}(\tau)Q(f,f)(t,x,v)\right)\langle v\rangle^{k_0}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau \\
&\triangleq E_1 + E_2\,.
\end{aligned}
$$

- The first term E_1 gives the desired damping:

$$
\begin{aligned}
E_1 &= \int_0^\infty\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \left(\mathcal{S}_{\mathcal{L}}(\tau)f(t,x,v)\right)\partial_\tau\left(\mathcal{S}_{\mathcal{L}}(\tau)f(t,x,v)\right)\langle v\rangle^{2k_0}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau \\
&= -\frac{1}{2}\left\|\langle v\rangle^{k_0} f\right\|^2_{L^2(\,\mathrm{d}x\,\mathrm{d}v)}\,.
\end{aligned}
$$

Energy Estimate

- Recall the definition of E_2:

$$\int_0^\infty \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} \left(\mathcal{S}_{\mathcal{L}}(\tau) f(t,x,v)\right) \left(\mathcal{S}_{\mathcal{L}}(\tau) Q(f,f)(t,x,v)\right) \langle v \rangle^{2k_0} \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}\tau$$

- Bound using the regularization of $\mathcal{S}_{\mathcal{L}}$:

$$\begin{aligned} \left\| \mathcal{S}_{\mathcal{L}}(\tau) Q(f,f)(t,\cdot,\cdot) \right\|_{L^2(\langle v \rangle^{k_0} \, \mathrm{d}x \, \mathrm{d}v)} &\leq C \left\| \langle v \rangle^{k_0} Q(f,f) \right\|_{H^{-s}_{x,v}} \\ &\leq C_1 \left\| f \right\|^2_{H^s(\langle v \rangle^{K_1} \, \mathrm{d}x \, \mathrm{d}v)} . \end{aligned}$$

where $K_1 \leq K - 2$.

Energy Estimate

- Recall the definition of E_2:

$$\int_0^\infty \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} (\mathcal{S}_{\mathcal{L}}(\tau) f(t,x,v)) \, (\mathcal{S}_{\mathcal{L}}(\tau) Q(f,f)(t,x,v)) \, \langle v \rangle^{2k_0} \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}\tau$$

- Bound using the regularization of $\mathcal{S}_{\mathcal{L}}$:

$$\begin{aligned} \left\| \mathcal{S}_{\mathcal{L}}(\tau) Q(f,f)(t,\cdot,\cdot) \right\|_{L^2(\langle v \rangle^{k_0} \, \mathrm{d}x \, \mathrm{d}v)} &\leq C \left\| \langle v \rangle^{k_0} Q(f,f) \right\|_{H^{-s}_{x,v}} \\ &\leq C_1 \, \|f\|^2_{H^s(\langle v \rangle^{K_1} \, \mathrm{d}x \, \mathrm{d}v)} . \end{aligned}$$

  where $K_1 \leq K - 2$.

- Bound of E_2:

$$\begin{aligned} |E_2| &\leq \int_0^\infty \left\| \mathcal{S}_{\mathcal{L}}(\tau) f \right\|_{L^2(\langle v \rangle^{k_0} \, \mathrm{d}x \, \mathrm{d}v)} \left\| \mathcal{S}_{\mathcal{L}}(\tau) Q(f,f) \right\|_{L^2(\langle v \rangle^{k_0} \, \mathrm{d}x \, \mathrm{d}v)} \, \mathrm{d}\tau \\ &\leq C_2 \left(\int_0^\infty e^{-\lambda \tau} \, \mathrm{d}\tau \right) \|f\|_{L^2(\langle v \rangle^{K} \, \mathrm{d}x \, \mathrm{d}v)} \, \|f\|^2_{H^s(\langle v \rangle^{K_1} \, \mathrm{d}x \, \mathrm{d}v)} \\ &\leq C_3 \, \|f\|_{L^2(\langle v \rangle^{K} \, \mathrm{d}x \, \mathrm{d}v)} \, \|f\|^2_{H^s(\langle v \rangle^{K_1} \, \mathrm{d}x \, \mathrm{d}v)} . \end{aligned}$$

Energy Estimate

- Summary: Take $A_0 = 2\tilde{C}_K$. Then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|||f|||^2 \leq -c_0\left(\frac{1}{4} - C_K\,\|f\|_{Y_K}\right)\left\|\langle v\rangle^{\gamma/2} f\right\|_{Y_K}$$
$$- \frac{c_1}{2}\sum_{|\alpha|\leq 2}\left\|\langle v\rangle^{(K-|\alpha|)}\partial_x^\alpha f\right\|^2_{H^s_{\gamma/2}} + \tilde{C}_K\,\|f\|^2_{L^2_v H^2_x}$$
$$-2\tilde{C}_K\left\|\langle v\rangle^{k_0} f\right\|^2_{L^2_v H^2_x} + C_3\,\|f\|_{L^2_K}\,\|f\|^2_{H^s_{K_1} H^2_x}$$
$$\leq -c_0\left(\frac{1}{4} - C_K\,\|f\|_{Y_K}\right)\left\|\langle v\rangle^{\gamma/2} f\right\|_{Y_K}$$
$$-\left(\frac{c_1}{2} - C_3\,\|f\|_{L^2_K}\right)\sum_{|\alpha|\leq 2}\left\|\langle v\rangle^{(K-|\alpha|)}\partial_x^\alpha f\right\|^2_{H^s_{\gamma/2}},$$

which shows the uniform bound for small data.

Future Questions

- What is next:
  - The strong singularity $1/2 < s < 1$;
  - More interesting question: seek for unique weak solutions with $f^{in} \in L^1 \cap L^\infty(\langle v \rangle^K \, \mathrm{d}x \, \mathrm{d}v)$.