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What Is an Asymptotically Hyperbolic Manifold?

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Abstract

This is a report on joint work (partially in progress) with Eric Bahuaud.

Typically, one defines an *asymptotically hyperbolic manifold* to be a noncompact manifold M that is the interior of a smooth compact manifold with boundary $\overline{M}$, endowed with a metric g with sectional curvatures approaching -1 at infinity, such that if ρ is a smooth defining function for ∂M, then $\overline{g} = \rho^2 g$ extends at least continuously to a metric on $\overline{M}$. This definition is an extrinsic one, relying on the prior knowledge of the existence of the conformal compactification.

This raises an interesting question about what purely *intrinsic* conditions on a complete, noncompact Riemannian manifold will guarantee that it has a conformal compactification in the sense above. There are two parts to the question: a topological part and a regularity part.

For the topological part, one needs to verify the existence of an *essential subset*, i.e., a smoothly bounded compact subset $K \subseteq M$ for which the outward normal exponential map is a diffeomorphism onto $M \smallsetminus (\operatorname{Int} K)$. Because this is a global condition on K, it would be useful to have a more local condition based on the curvature at infinity and the geometry of ∂K. I'll describe progress toward proving the following conjecture (a sort of Cartan–Hadamard theorem for manifolds with boundary): If $M \smallsetminus (\operatorname{Int} K)$ has nonpositive sectional curvature and weakly concave smooth boundary, and the fundamental group of ∂K surjects onto that of $M \smallsetminus (\operatorname{Int} K)$, then K is an essential subset.

For the regularity part, we focus on Einstein manifolds. Romain Gicquaud has proved that if (M, g) is a complete Einstein manifold with an essential subset K and sectional curvature $-1 + O(e^{-2r})$, where r is the distance to K, then g has a $C^{1,\alpha}$ conformal compactification. We show that this cannot be improved to C^2.