2018 Spring Probability Workshop
Institute of Mathematics, Academia Sinica, Taipei, Taiwan
June 4 – 8, 2017

Critical two-point function for long-range models with power-law couplings: The marginal case for $d \geq d_c$

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Abstract

In recent years, the long-range models on $\mathbb{Z}^d$ of self-avoiding walk, percolation and the Ising model, defined by power-law couplings $D(x) \approx |x|^{-d-\alpha}$ for some $\alpha > 0$, have attracted more attention, due to unconventional critical behavior and crossover phenomena. One of the examples of D is the compound-zeta distribution

$$D(x) = \sum_{t \in \mathbb{N}} U_L^{*t}(x) \frac{t^{-1-\alpha/2}}{\zeta(1+\alpha/2)},$$

where U_L^{*t} is the t-fold convolution of the uniform distribution over $\{x \in \mathbb{Z}^d : \|x\|_\infty \leq L\}$. In the previous work (*Ann. Probab.*, **43** (2015): 639–681), we have shown that, for $\alpha \neq 2$, $d > \alpha \wedge 2$ and L large, the random-walk Green function $S_1(x)$ is asymptotically $\frac{\gamma_\alpha}{v_\alpha}|x|^{\alpha\wedge 2-d}$, where

$$\gamma_\alpha = \frac{\Gamma(\frac{d-\alpha\wedge 2}{2})}{2^{\alpha\wedge 2}\pi^{d/2}\Gamma(\frac{\alpha\wedge 2}{2})}, \qquad \hat{D}(k) \equiv \sum_{x \in \mathbb{Z}^d} e^{ik\cdot x} D(x) \underset{k \to 0}{\sim} 1 - \begin{cases} v_\alpha |k|^{\alpha\wedge 2} & [\alpha \neq 2], \\ v_2 |k|^2 \log \frac{1}{|k|} & [\alpha = 2]. \end{cases}$$

Moreover, we have shown that, for $\alpha \neq 2$, d above the upper-critical dimension $d_c \equiv (\alpha \wedge 2)m$, where $m = 2$ for self-avoiding walk and the Ising model and $m = 3$ for percolation, and L large enough, there is a model-dependent constant A close to 1 (in fact, $A = 1$ for $\alpha < 2$) such that the critical two-point function $G_{p_c}(x)$ is asymptotically $\frac{A}{p_c} S_1(x)$. The crossover occurs at $\alpha = 2$, where the variance of D diverges logarithmically.

In this talk, I will explain the current status of the ongoing work on the marginal case of $\alpha = 2$. We have shown that, for $d > 2$ and L large, $S_1(x)$ is asymptotically $\frac{\gamma_2}{v_2}|x|^{2-d}/\log|x|$. Assuming a tail estimate on the zeta distribution, we have also shown that $G_{p_c}(x)$ is asymptotically $\frac{1}{p_c} S_1(x)$ whenever $d \geq d_c$ (including equality) and L large enough. This solves the conjecture in the previous work, extended all the way down to $d = d_c$, and confirms a part of predictions in physics (Brezin, et al., *J. Stat. Phys.*, **157** (2014): 855–868). The proof relies on the lace expansion and new convolution bounds on power functions with log corrections.