

Non-uniqueness for the transport equation with Sobolev vector fields

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Abstract

We consider the linear transport equation $\partial_t \rho + u \cdot \nabla \rho = 0$, with unknown density ρ and given divergence-free vector field u , together with a given initial datum $\rho(x, 0) = \bar{\rho}(x)$. A celebrated result by DiPerna and Lions (1989) shows that if

$$\bar{\rho} \in L_x^p, \quad u \in L_t^1 W_x^{1, \tilde{p}}$$

and

$$\frac{1}{p} + \frac{1}{\tilde{p}} \leq 1,$$

then the Cauchy problem admits a unique weak solution $\rho \in L_t^\infty L_x^p$. In a joint work with L. Székelyhidi, we show that if

$$\frac{1}{p} + \frac{1}{\tilde{p}} > 1 + \frac{1}{d-1},$$

d being the dimension of the physical space, then there are divergence-free vector fields $u \in C_t W^{1, \tilde{p}}$ and initial data $\bar{\rho} \in L_x^p$ for which more than one weak solution $\rho \in C_t L_x^p$ exists. A very similar result applies also to the transport-diffusion equation $\partial_t \rho + u \cdot \nabla \rho = \Delta \rho$.