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# DYCK PATHS OF KNIGHT MOVES*

Jacques LABELLE**
*Université du Québec à Montréal, Case postale 8888, Succ. "A", Montréal, Québec, Canada, H3C 3P8*

Yeong-Nan YEH***
*Institute of Mathematics, Academia Sinica, Nankang, Taipei, Taiwan*

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The NE (north-east) and SE (south-east) steps of Dyck paths are replaced by NNE, NEE, SSE and SEE steps (i.e. moves of a chess knight from left to right) to obtain "Dyck paths of knight moves". In order to study these paths (called $A$-paths), we need to introduce several closely related families of paths and to find a system of equations between the corresponding generating functions. The generating function $A=A(t)$ for $A$-paths satisfies a fourth-degree polynomial equation which can be solved. The same method also works for the general $(s,r)$-knight but the number of new families of paths to be introduced in the system grows rapidly ($4r-2$ equations for an $(s,r)$-knight). The generating function for these paths still satisfies a polynomial equation (for example of degree 8 when $(s,r)=(1,3)$ or $(2,3)$). One may think of all these paths as words in the letters $x$, $y$, $\bar{y}$; the corresponding languages are algebraic.

## Introduction

Dyck paths are counted by Catalan numbers and are in bijection with Dyck words which form a nice algebraic language on two letters (see [3, 6, 10, 11]). In the present text, we replace NE (north-east) and SE (south-east) steps by NNE, NEE, SSE and SEE steps (i.e. moves of a chess knight from left to right). In order to enumerate these paths (called $A$-paths), we need to introduce two closely related families of paths, find a system of equations relating generating functions and solve it. The generating function $A=A(t)$ for $A$-paths satisfies a fourth-degree equation (Theorem 2.13) while for Dyck path it is a well-known second-degree equation (Theorem 1.9). The same method also works for the study of the case of a general $(s,r)$-knight. More new paths have to be introduced and the corresponding generating functions satisfy an intricate system of $4r-2$ equations (Theorem 3.5). The generating function for $A$-paths still satisfies a polynomial equation (for example of degree 8 when $(s,r)=(1,3)$ or $(2,3)$). One may think of all these paths as words in the letters $x$, $y$, $\bar{y}$; the corresponding languages are algebraic.

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*** The author was visiting professor at Université du Québec à Montréal while this work was done.

## 1. Classical Dyck paths

**Definition 1.1.** A *Dyck path of length n* is a path in $\mathbb{Z}\times\mathbb{Z}$ which

(i) is made only of steps NE (north-east) or SE (south-east),
(ii) starts at $(0,0)$ and ends at $(n,0)$, and
(iii) never goes strictly below the $x$-axis.

**Remark 1.2.** It is well known (see [2, 3, 6, 9–11]) that Dyck paths are in bijection with *Dyck words*, *ballot sequences*, *well-formed sequences of parentheses*, *2-lines standard tableaux*, *binary trees*, *ordered trees*, and all these are counted by Catalan numbers.

**Definition 1.3.** Let $d_n$ be the number of these paths and $d_n^{(k)}$ for $k\geq 0$ be the number of those which hit (not counting at $(0,0)$) the $x$-axis exactly $k$ times. Note that $d_0^{(0)}=1$ and $d_n^{(0)}=0$ for $n>0$; moreover $d_n=0$ when $n$ is odd.

**Definition 1.4.** Let the corresponding generating functions be

$$D=\sum_{n\geq 0} d_n t^n \quad\text{and}\quad D^{(k)}=\sum_{n\geq 0} d_n^{(k)} t^n.$$

**Lemma 1.5.** *We have* $D=\sum_{k\geq 0} D^{(k)}$.

**Lemma 1.6.** *We have* $D^{(k)}=(D^{(1)})^k$ *for* $k\geq 1$.

**Proof.** Obviously, $d_n^{(k)}=\sum_{m=1}^{n-1} d_m^{(1)} d_{n-m}^{(k-1)}$, where in this sum $m$ corresponds to the first place where the path hits, at $(m,0)$, the $x$-axis. This implies $D^{(k)}=D^{(1)}\cdot D^{(k-1)}$ and the result. □

**Lemma 1.7.** *We have* $D=(1-D^{(1)})^{-1}$.

**Lemma 1.8.** *We have* $D^{(1)}=t^2 D$.

**Proof.** Let $n\geq 2$. Removing the first and last steps of a Dyck path which hits the $x$-axis only at $(0,0)$ and $(n,0)$, we get bijectively an arbitrary Dyck path of length $n-2$. This proves $d_n^{(1)}=d_{n-2}$ which implies the result. □

**Theorem 1.9.** *The generating function D for Dyck paths satisfies the equation* $t^2D^2-D+1=0$.

**Corollary 1.10.** *We have* $D=\frac{1}{2}(1-(1-4t^2)^{1/2})/t^2$.

**Corollary 1.11.** *We have* $d_{2m}=\binom{2m}{m}/(m+1)$ *(the mth-Catalan number).*

## 2. Dyck paths of knight moves

**Definition 2.1.** A *Dyck path of knight moves of size n* (which we call an *A-path*) is a path in $\mathbb{Z}\times\mathbb{Z}$ which

(i) is made only of steps NNE, NEE, SSE and SEE,
(ii) starts at $(0,0)$ and ends at $(n,0)$,
(iii) never goes strictly below the $x$-axis (see Fig. 1).

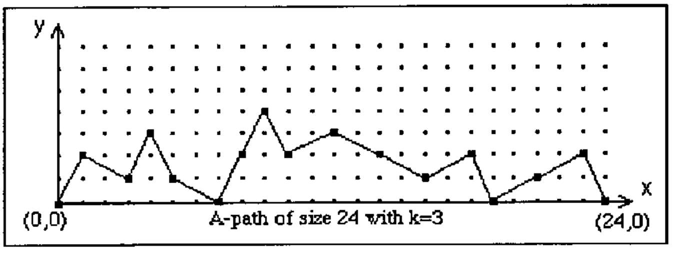


Fig. 1.

**Definition 2.2.** Let $a_n$ be the number of these paths and $a_n^{(k)}$ the number of those which hit the $x$-axis (except at $(0,0)$) exactly $k$ times. Note that $a_0^{(0)}=1$ and $a_n^{(0)}=0$ for $n>0$.

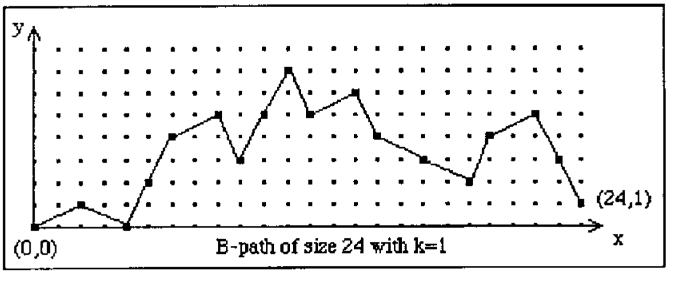


Fig. 2.

**Definition 2.3.** *B-paths of size n* satisfy the three conditions of Definition 2.1 except that they end at $(n,1)$ (see Fig. 2). Let $b_n$ and $b_n^{(k)}$ be defined as before. Note that $b_0^{(0)}=b_1^{(0)}=0$ but $b_n^{(0)}>0$ for $n\geq 2$.

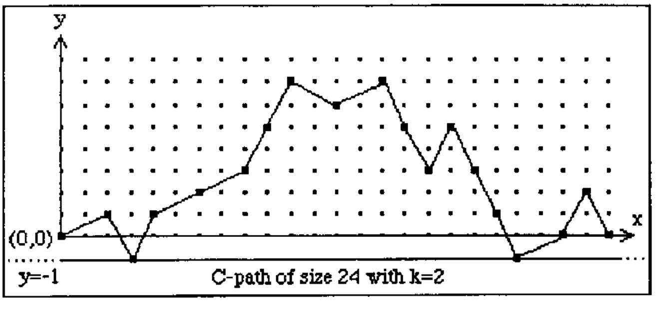


Fig. 3.

**Definition 2.4.** *C-paths of size n* satisfy the three conditions of Definition 2.1 except that they are allowed to go below the $x$-axis but always remain above the line $y=-1$ (see Fig. 3). Let $c_n$ and $c_n^{(k)}$ be as before except that $k$, $k \geq 0$, is now the number of times the $C$-path hits the line $y=-1$.

Let $A$, $A^{(k)}$, $B$, $B^{(k)}$, $C$ and $C^{(k)}$, be the corresponding generating functions.

**Lemma 2.5.** *We have*

(a) $A=\sum_{k\geq 0} A^{(k)}$,
(b) $B=\sum_{k\geq 0} B^{(k)}$,
(c) $C=\sum_{k\geq 0} C^{(k)}$.

**Lemma 2.6.** *We have for* $k \geq 1$

(a) $A^{(k)}=A^{(1)}\cdot A^{(k-1)}=(A^{(1)})^k$,
(b) $B^{(k)}=A^{(1)}\cdot B^{(k-1)}=(A^{(1)})^k\cdot B^{(0)}$,
(c) $C^{(k)}=B^{(0)}\cdot B^{(k-1)}=(B^{(0)})^2\cdot(A^{(1)})^{k-1}$.

**Proof.** For example, a look at Fig. 3 should convince the reader of the validity of the formula, $c_n^{(k)}=\sum_{m=1}^{n-1} b_m^{(0)}\cdot b_{n-m}^{(k-1)}$, which implies (c). The other identities are proved similarly. □

**Lemma 2.7.** *We have* $A^{(1)}=t^4A+2t^3B+t^2C$.

**Proof.** In an arbitrary $A^{(1)}$-path (i.e. which hits the $x$-axis only at $(0,0)$ and $(n,0)$) erase the first and last steps:

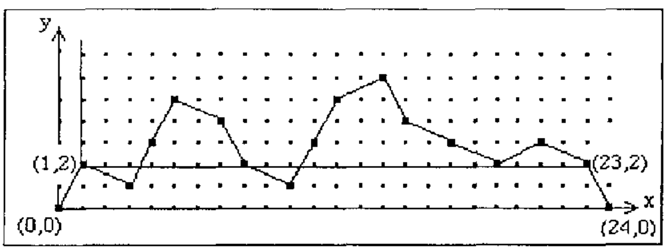


Fig. 4.

*Case* 1. First step is NNE and last step is SSE. Since one gets an arbitrary "$C$-path" from $(1,2)$ to $(n-1,2)$ there are $c_{n-2}$ such paths (see Fig. 4).

*Case* 2. First step is NEE and last step is SEE. Since one gets an arbitrary "$A$-path" from $(2,1)$ to $(n-2,1)$ there are $a_{n-4}$ such paths.

*Case* 3. First step is NEE and last step is SSE. Since one gets an arbitrary "$B$-path" from $(2,1)$ to $(n-1,2)$ there are $b_{n-3}$ such paths.

*Case* 4. First step is NNE and last step is SEE; by symmetry there are $b_{n-3}$ such paths. This is a bijective proof of $a_n^{(1)}=c_{n-2}+a_{n-4}+2b_{n-3}$, which implies the result. □

**Lemma 2.8.** *We have* $B^{(0)} = t^2A + tB$.

**Proof.** This time erasing the first step of a $B$-path which hits the $x$-axis only at $(0,0)$ gives a bijective proof of the formula $b_n^{(0)} = a_{n-2} + b_{n-1}$, which implies the result. □

**Lemma 2.9.** *We have* $C^{(0)} = A$.

**Proof.** The result follows by definition. □

**Lemma 2.10.** *We have* $A = (1 - A^{(1)})^{-1}$.

**Proof.** The result follows from Lemmas 2.5(a) and 2.6(a). An $A$-path is a chain of $A^{(1)}$-paths. □

**Lemma 2.11.** *We have* $B = t^2A^2(1 - tA)^{-1}$.

**Proof.** $B = \sum_{k\geq 0} B^{(k)} = B^{(0)} \cdot \sum_{k\geq 0} A^{(k)} = B^{(0)} \cdot A = (t^2A + tB) \cdot A = t^2A^2 + tAB$. This implies the result. □

**Lemma 2.12.** *We have* $C = t^2AB + tB^2 + A$.

**Proof.** $C = \sum_{k\geq 0} C^{(k)} = \sum_{k\geq 1} C^{(k)} + A = \sum_{k\geq 1} B^{(0)} \cdot B^{(k-1)} + A = B^{(0)} \cdot B + A = t^2AB + tB^2 + A$. □

**Theorem 2.13.** *The generating function* $A = A(t)$ *for* $A$*-paths of size* $n$ *satisfies the equation*

$$t^4A^4 - (2t^3 + t^2)A^3 + (t^4 + 2t^2 + 2t)A^2 - (2t + 1)A + 1 = 0.$$

**Proof.** By Lemma 2.7, $A^{(1)} = t^4A + 2t^3B + t^2C$. Use Lemmas 2.10–2.12 to write $A^{(1)}$, $B$ and $C$ in terms of $A$. One finally gets the following equation which implies the result:

$$1 - A^{-1} = (t^4 + t^2)A + 2t^5A^2(1 - tA)^{-1} + t^6A^3(1 - tA)^{-1} + t^7A^4(1 - tA)^{-2}. \qquad \square$$

Using MAPLE to solve the polynomial equation of Theorem 2.13, we obtain the following expression for $A = A(t)$:

$$\{1 + 2t + (1 - 4t + 4t^2 - 4t^4)^{1/2}$$
$$- \sqrt{2}(1 - 4t^2 - 2t^4 + (2t + 1)(1 - 4t + 4t^2 - 4t^4)^{1/2})^{1/2}\}/(4t^2),$$

and a table of values (see Table 1).

**Remark 2.14.** There is an obvious bijection between $A$-paths of size $n$ and words

Table 1

| $n$ | $d_n$ | $a_n$ | $b_n$ | $c_n$ | $a_n^{(1)}$ |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 |
| 2 | 1 | 1 | 1 | 1 | 1 |
| 3 | 0 | 0 | 1 | 0 | 0 |
| 4 | 2 | 3 | 3 | 4 | 2 |
| 5 | 0 | 2 | 4 | 4 | 2 |
| 6 | 5 | 12 | 12 | 18 | 7 |
| 7 | 0 | 14 | 22 | 26 | 10 |
| 8 | 14 | 54 | 61 | 86 | 29 |
| 9 | 0 | 86 | 128 | 158 | 52 |
| 10 | 42 | 274 | 335 | 462 | 142 |
| 11 | 0 | 528 | 756 | 976 | 294 |
| 12 | 132 | 1515 | 1936 | 2665 | 772 |
| 13 | 0 | 3266 | 4580 | 6082 | 1732 |
| 14 | 429 | 8854 | 11652 | 16040 | 4451 |
| 15 | 0 | 20422 | 28402 | 38338 | 10482 |
| 16 | 1430 | 53786 | 72209 | 99536 | 26715 |
| 17 | 0 | 129368 | 179460 | 244880 | 64908 |
| 18 | 4862 | 336103 | 457274 | 631923 | 165194 |
| 19 | 0 | 830148 | 1151725 | 1583796 | 409720 |
| 20 | 16796 | 2145020 | 2945129 | 4081939 | 1044629 |

$w$ in $xy^2$, $x^2y$, $x\bar{y}^2$, $x^2\bar{y}$ such that $\deg_x w = n$, $\deg_y w = \deg_{\bar{y}} w$ and $\deg_y v \geq \deg_{\bar{y}} v$ for $w = vu$; similarly for $B$-paths and $C$-paths.

## 3. Dyck paths of an $(s,r)$-knight

In Section 2, in order to find the generating series $A = A(t)$ for $A$-paths, we had to introduce $B$-paths (from $(0,0)$ to $(n,1)$) and $C$-paths (above $y=-1$). The generating series $A$, $B$, $C$, $A^{(1)}$, $B^{(0)}$, $C^{(0)}$ then satisfy the system of equations

(1) $A = (1 - A^{(1)})^{-1}$,
(2) $B = A \cdot B^{(0)}$,
(3) $C = A + (B^{(0)})^2 \cdot A$,
(4) $A^{(1)} = t^4A + 2t^3B + t^2C$,
(5) $B^{(0)} = t^2A + tB$,
(6) $C^{(0)} = A$.

For an $(s,r)$-knight, $s \leq r$ (which moves $s$ squares in one direction and $r$ squares perpendicularly) more new paths have to be introduced and a bigger system of equations has to be solved. For example, for a $(1,3)$-knight or a $(2,3)$-knight, we introduce for $i = 1$ and 2, $B_i$-paths (from $(0,0)$ to $(n,i)$) and $C_i$-paths (above $y = -i$). As before the integer $k$ indicates the number of times the $A$-path or $B_i$-path (respectively the $C_i$-path) hits the line $y = 0$ (respectively $y = -i$).

**Proposition 3.1.** *For a* (1,3)*-knight the system of equations satisfied by the generating series* $A$, $B_1$, $B_2$, $C_1$, $C_2$, $A^{(1)}$, $B_1^{(0)}$, $B_2^{(0)}$, $C_1^{(0)}$ *and* $C_2^{(0)}$ *is*

(1) $A=(1-A^{(1)})^{-1}$,
(2) $B_1=A\cdot B_1^{(0)}$,
(3) $B_2=A\cdot B_2^{(0)}$,
(4) $C_1=A+A\cdot (B_1^{(0)})^2$,
(5) $C_2=C_1+A\cdot (B_2^{(0)})^2$,
(6) $A^{(1)}=t^6A+2t^4B_2+t^2C_2$,
(7) $B_1^{(0)}=t^3A+tB_2$,
(8) $B_2^{(0)}=B_1^{(0)}\cdot B_1+tB_1$,
(9) $C_1^{(0)}=A$,
(10) $C_2^{(0)}=C_1$.

**Proof.** Only (8) needs explanations: a $B_2^{(0)}$-path either touches the line $y=1$ or it does not. □

**Theorem 3.2.** *The generating function for A-paths of a* (1,3)*-knight satisfies the equation*

$$t^8A^8-t^6A^7+(t^{10}-2t^8+t^6)A^6+(2t^6-t^4)A^5+(2t^8-4t^6+t^2)A^4$$
$$+(2t^4-t^2)A^3+(t^6-2t^4+t^2)A^2-A+1=0.$$

**Proposition 3.3.** *For a* (2,3)*-knight the system of equations is as in Proposition* 3.1, *except* (6), (7) *and* (8) *which are replaced by*

(6′) $A^{(1)}=t^6C_1+2t^5(B_1+B_1^{(0)}\cdot B_2)+t^4C_2$,
(7′) $B_1^{(0)}=t^3B_1+t^2B_2$,
(8′) $B_2^{(0)}=t^3C_1+t^2(B_1+B_2^{(0)}\cdot B_1)$.

**Proof.** For example, (6′) is proved by erasing first and last steps in a $A^{(1)}$-path. If first step is NNEEE and last step is SSSEE, then either it touches the line $y=1$ (giving the term $B_1^{(0)}\cdot B_2$) or is does not (giving the term $B_1$); the proof is similar for (8′). □

**Theorem 3.4.** *The generating function for A-paths of a* (2,3)*-knight satisfies the equation*

$$t^{16}A^8-(2t^{15}+t^{12})A^7+(2t^{14}+t^{12}+2t^{11})A^6-(2t^{13}+2t^{11}+t^{10}+t^8)A^5$$
$$+(t^{12}+4t^{10}+t^4)A^4-(2t^9+2t^7+t^6+t^4)A^3$$
$$+(2t^6+t^4+2t^3)A^2-(2t^3+1)A+1=0.$$

More generally for an $(s,r)$-knight, $s\leq r$, we have to introduce $r-1$ kinds of $B$-paths, that is $B_i$ $(1\leq i\leq r-1)$ which goes from $(0,0)$ to $(n,i)$, and $r-1$ kinds of $C$-paths, that is $C_i$ $(1\leq i\leq r-1)$ which stays above the line $y=-i$. The corresponding

$4r-2$ series $A$, $A^{(1)}$, $B_i$, $B_i^{(0)}$, $C_i$, $C_i^{(0)}$ $(1 \le i \le r-1)$, where a $B_i^{(k)}$-path (respectively $C_i^{(k)}$-path) touches the line $y=0$ (respectively the line $y=-i$) $k$ times, satisfy the following theorem.

**Theorem 3.5.** *We have, using the natural notation* $B_0=C_0=A$ *and* $B_{-1}=B_{-2}=\cdots=0$,

$$A=(1-A^{(1)})^{-1}, \qquad A^{(1)}=t^{2r}C_{s-1}+2t^{r+s}\left(\sum_{1\le j\le s} B^{(0)}_{r-j}\cdot B_{s-j}\right)+t^{2s}C_{r-1},$$

$$B_i=A\cdot B_i^{(0)}, \qquad B_i^{(0)}=\sum_{1\le j\le s}(t^r B^{(0)}_{s-j}+t^s B^{(0)}_{r-j})\cdot B_{i-j},$$

$$C_i=C_{i-1}+A\cdot(B_i^{(0)})^2, \qquad C_i^{(0)}=C_{i-1} \quad \textit{for } 1\le i\le r-1.$$

**Proof.** For example, we prove the fourth equation: given a $B_i^{(0)}$-path, starting with the step $(0,0)$ to $(r,s)$, consider successively (for $j=1$ to $s$) if it touches the line $y=j$ or not. □

**Remark 3.6.** Eliminating variables $C_i^{(0)}$ (trivially) and $B_i^{(0)}$ from these $4r-2$ equations leads to a system of $2r$ algebraic equations satisfied by the $2r$ generating series $A$, $A^{(1)}$, $B_i$ and $C_i$ $(1\le i\le r-1)$. Further eliminations, as was done explicitly only in the cases of a $(1,1)$-Dyck-knight (Theorem 1.9), a $(1,2)$-chess-knight (Theorem 2.13), a $(1,3)$-knight (Theorem 3.2) and a $(2,3)$-knight (Theorem 3.4), give a polynomial equation (with polynomials in $t$ as coefficients) of degrees 2, 4, 8 and 8 respectively satisfied by $A$.

**Remark 3.7.** Since the symmetry between $s$ and $r$ is never really used in Theorem 3.5, instead of $(s,r)$-, $(r,s)$-, $(s,-r)$- and $(r,-s)$-steps, one could as easily start with two (see [8] for the general case of $m$ vectors) arbitrary vectors $(s_1,r_1)$ and $(s_2,r_2)$ in $\{1,2,\ldots\}\times\{0,1,2,\ldots\}$, and consider Dyck paths made of $(s_1,r_1)$-, $(s_2,r_2)$-, $(s_1,-r_1)$- *and* $(s_2,-r_2)$-steps. A system of $4r-2$ equations, where $r=\max(r_1,r_2)$ similar to those above can be obtained.

**Definition 3.8.** Let $A_{s,r}(t)$ denote the generating series for $A$-paths of a $(s,r)$-knight.

**Proposition 3.9.** *We have* $A_{s,r}(t^m)=A_{ms,mr}(t)$.

**Corollary 3.10.** *The series* $A_{m,m}(t)$ *and* $A_{m,2m}(t)$ *have explicit closed forms.*

## 4. Conclusion

The case of several knights or several vectors can be studied similarly (see [8] for more general results using several variables series). One has to introduce more $B$-

paths and $C$-paths; the corresponding generating series satisfy an intricate system of algebraic equations. We may think of all these kinds of paths as words in the letters $x$, $y$, $\bar{y}$ (see Remark 2.14); the corresponding languages are algebraic.

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