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The Factoriality of the Ring of S-Species

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INTRODUCTION

Often combinatorics involves permutations as basic objects. For example, one outlook on Polya's theory is to consider structures for which a given permutation is an automorphism. This is one motivation for the introduction of permutation species. Others involve symmetric function theory or q-analogs. Moreover, permutation species are nicely related to the (set-)species of Joyal [J1], in a way sumarized by Fig. 1, where *S-species* and *B-species* are respectively the algebra (i.e., categories with certain operations) of permutation species and the algebra of usual (set)-species, and the arrows are functors preserving some algebraic operations. Thus, any combinatorial identities in the context of species gives rise to a combinatorial identity in the context of permutation species, via the functor *Fix* of this diagram. This has been shown in [B2]. The lower part of the diagram corresponds to the formal power series aspect of the theory. It is shown here for completness sake, but we will not use it in this paper.

In [Y] it is shown that the ring of species of structures is a unique factorisation domain. The object of this paper is to generalize these results to the unique factorisation domain of (true) permutation species. All theorems of [Y] about species thus become special cases of similar theorems about permutation species. This is not so surprising in view of Fig. 1. In fact it corresponds precisely to the fact that $\Omega \circ Fix = \mathrm{Id}_{B\text{-species}}$. The paper is organized in three parts. In the first one, we develop the basic tools needed for this generalization of Yeh's results. In the second one, we recall the theory of S-species. Finally in the last part, we mesh those two parts to obtain the main results.

1. GROUP SETS

Let S be the category whose objects are couples (A, σ) where σ is a permutation of A, and whose arrows $\psi\colon (A, \sigma) \to (B, \tau)$ are bijections $\psi\colon A \to B$

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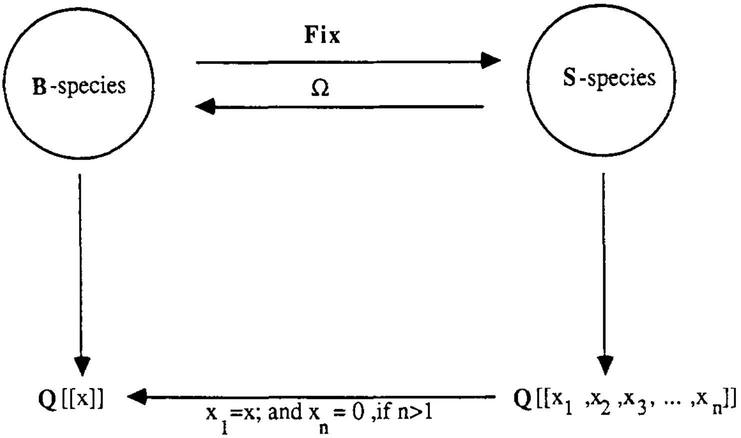

FIGURE 1

such that $\psi\sigma = \tau\psi$. A permutation of a finite set A is a bijection $\sigma: A \to A$. With composition as multiplication, the set of all permutations of A forms a group $A!$. We evidently have $\#(A!) = \#(A)!$, if $\#$ denotes cardinality. Then, for any $\sigma \in A!$, the centralizer of σ, $\mathrm{Cen}(\sigma)$, is the set $\{\tau \in A! \mid \tau\sigma = \sigma\tau\}$. Now, if G is a subgroup of $\mathrm{Cen}(\sigma)$, we shall say that the triple (G, A, σ) is a *group-set*. Furthermore, we say that (B, τ) is a subobject of (A, σ), if B is a subset of A, and $\sigma|_B = \tau$. A subobject (B, τ) of (A, σ) is called a *G-invariant* subobject if, for all $g \in G$, $g(B)$ is a subset of B and $g\sigma = \tau g$ on B.

Let (G, A, σ) be a group-set, (U, ρ) be an object of S containing (A, σ) (that is, A is a subset of U and $\rho|_A = \sigma$), and (B, τ) be a G-invariant subobject of (A, σ). For any $g \in G$, the *extension* $g^{(U,\rho)}$ *of* g *to* U, is defined to be:

$$g^{(U,\rho)}(u) = \begin{cases} g(u), & \text{if } u \in A, \\ u, & \text{otherwise.} \end{cases}$$

The *restriction* $g_{(B,\tau)}$ *of* g *to* B, is defined to be: $g_{(B,\tau)}(b) = g(b)$ if $b \in B$. Thus we can define the two sets $G^{(U,\rho)} = \{g^{(U,\rho)} \mid g \in G)$ and $G_{(B,\tau)} = \{g_{(B,\tau)} \mid g \in G\}$.

With composition as multiplication, $G^{(U,\rho)}$ becomes a subgroup of $\mathrm{Cen}(\rho)$ and $G_{(B,\tau)}$ a subgroup of $\mathrm{Cen}(\tau)$; hence $(G^{(U,\rho)}, U, \rho)$, $(G_{(B,\tau)}, B, \tau)$ are group-sets. For any object (A, σ) and subobject (B, τ) of (A, σ), let $(A, \sigma)\backslash(B, \tau)$ denote the object $(A\backslash B, \sigma|_{A\backslash B})$. If (B, τ) is a G-invariant subobject of (A, σ), then $(A, \sigma)\backslash(B, \tau)$ is also clearly a G-invariant subobject of (A, σ) and, therefore, $(G_{(A,\sigma)\backslash(B,\tau)}, (A, \sigma)\backslash(B, \tau))$ is a group-set.

DEFINITION 1.1. Let (G, A, σ) and (H, B, τ) be two group-sets. (H, B, τ) is called a *reducing group-set* of (G, A, σ) if it satisfies the following conditions:

(a) (B, τ) is a G-invariant subobject of (A, σ),

(b) $H = G_{(B,\tau)}$,

(c) $H^{(A,\sigma)}$ is contained in G.

DEFINITION 1.2. For any $h \in A!$ and $k \in B!$, let $h * k \in (A+B)!$ be defined by:

$$(h * k)(u) = \begin{cases} h(u), & \text{if } u \in A, \\ k(u), & \text{if } u \in B. \end{cases}$$

Then, for any $h \in \mathrm{Cen}(\sigma)$ and $k \in \mathrm{Cen}(\tau)$, $h * k \in \mathrm{Cen}(\sigma * \tau)$, and we have

LEMMA 1.3. *Let (G, A, σ) and (H, B, τ) be two group-sets, then $G * H = \{g * h \mid g \in G, h \in H\}$ is a subgroup of* $\mathrm{Cen}(\sigma * \tau)$.

The group-set $(G * H,\ A + B,\ \sigma * \tau)$ is called the *outer-product* of the group-sets (G, A, σ) and (H, B, τ). We thus obtain an operation on group-sets:

$$(G, A, \sigma) * (H, B, \tau) = (G * H, A + B, \sigma * \tau).$$

It is clear that the group $G * H$ is the direct product of $G^{(A,\sigma)+(B,\tau)}$ and $H^{(A,\sigma)+(B,\tau)}$. It is also easy to check that the outer-rpdocut "*" satisfies the associative law.

LEMMA 1.4. *If $(G_{(B,\tau)}, B, \tau)$ is a reducing group-set of (G, A, σ), then*

(a) *$(G_{(A,\sigma)\backslash(B,\tau)}, (A, \sigma)\backslash(B, \tau))$ is a reducing group-set of (G, A, σ),*

(b) *$(G, A, \sigma) = (G_{(B,\tau)}, B, \tau) * (G_{(A,\sigma)\backslash(B,\tau)}, (A, \sigma)\backslash(B, \tau))$.*

LEMMA 1.5. *Let$(G_{(B,\tau)}, B, \tau)$, $(G_{(C,\rho)}, C, \rho)$ be two reducing group-sets of (G, A, σ), then so is $(G_{B \cap C}, B \cap C, \sigma|_{B \cap C})$.*

LEMMA 1.6. *If (H,B,τ) is a reducing group-set of (G, A, σ), and if (K, C, ρ) is a reducing group-set of (H, B, τ), then (K, C, ρ) is reducing group-set of (G, A, σ).*

DEFINITION 1.7. A group-set (G, A, σ) is called an *atomic group-set* if $A \neq \varnothing$ and (G, A, σ) has no non-empty proper reducing group-set.

A direct consequence of Lemma 1.4 is that (G, A, σ) is an atomic group-set if and only if it cannot be expressed as an outer-product of two non-trivial group-sets.

PROPOSITION 1.8. *Every group-set* (G, A, σ) *can be decomposed uniquely into an outer-product of atomic group-sets.*

Let (G, A, σ) and (H, B, τ) be two group-sets. We write $(G, A, \sigma) \sim (H, B, \tau)$ if there exists a bijection $f\colon B \to A$ such that $f^{-1}Gf = H$ and $f^{-1}\sigma f = \tau$. It is easy to prove that "$\sim$" is an equivalence relation. Let $\mathfrak{G}$ be the set of equivalence classes of group-sets. Thus we have:

PROPOSITION 1.9. $(\mathfrak{G}, *)$ *is a free monoid.*

2. S-SPECIES

Let B be the category of finite sets and bijections.

DEFINITION 2.1. A *permutation species, S-species,* is a functor $T\colon S \to B$. Thus to every object $(A, \sigma) \in S$ there corresponds a set $T[A, \sigma]$ (or simply $T[\sigma]$), and for every arrow $\psi\colon (A, \sigma) \to (B, \tau)$ a bijection $T[\psi]\colon T[A, \sigma] \to T[B, \tau]$. A *morphism* $\theta\colon T \to P$ of S-species is a natural transformation from the functor T to the functor P. This means that for all object (A, σ) in S we have arrows $\theta_{(A,\sigma)}\colon T[A, \sigma] \to P[A, \sigma]$, such that for any arrow $\psi\colon (A, \sigma) \to (B, \tau)$ in S,

$$P[\psi]\theta_{(A,\sigma)} = \theta_{(B,\tau)}T[\psi].$$

If there exists an isomorphism θ between two S-species T and P, we write $T = P$ (with a slight abuse of notation). An element in $T[A, \sigma]$ is called an T-structure on (A, σ). An S-species is said to be *true* if and only if $T[\sigma] = \mathrm{id}_{T[A,\sigma]}$ for all object (A, σ) of S, here σ is considered as an arrow $\sigma\colon (A, \sigma) \to (A, \sigma)$.

EXAMPLE 1. The S-species P is defined by: $P[A, \sigma] = A$. The S-species P is not *true*, since $P[\sigma] = \sigma \neq \mathrm{id}_{P[A,\sigma]}$.

DEFINITION 2.2. A *dissection* of (A, σ) is a pair $((B, \tau), (C, \rho))$, where

(a) $B + C = A$,

(b) $\sigma(B) = B$ and $\sigma(C) = C$,

(c) $\tau = \sigma|_B$ and $\rho = \sigma|_C$.

An object (A, σ) is called "irreducible" if $((A, \sigma), (\varnothing, \mathrm{id}_\varnothing))$ and

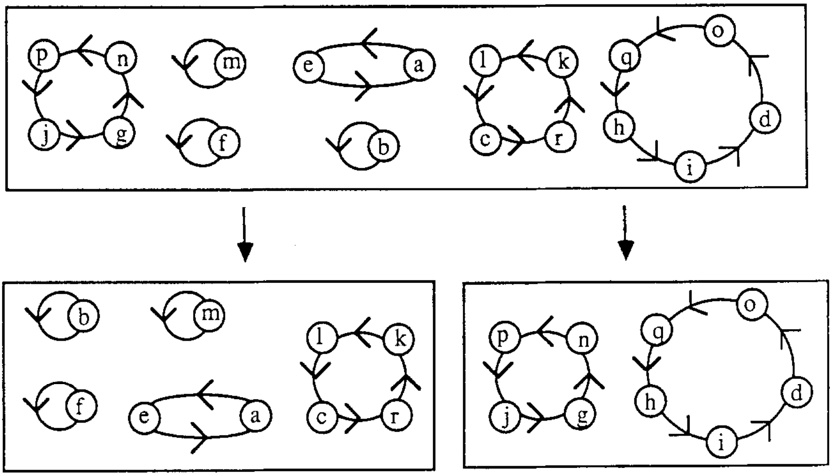

FIGURE 2

$((\varnothing, \mathrm{id}_\varnothing), (A, \sigma))$ are the only dissections of (A, σ); (A, σ) is clearly irreducible if σ is cyclic. Figure 2 gives an example of such a dissection.

DEFINITION 2.3. Let T and P be two S-species, we define new S-species $T+P$, $T \times P$, and TP by:

(a) $(T+P)[A, \sigma] = T[A, \sigma] + P[A, \sigma]$ (disjoint sum)

(b) $(T \times P)[A, \sigma] = T[A, \sigma] \times P[A, \sigma]$ (cartesian product)

(c) $(TP)[A, \sigma] = \sum_{((B, \tau), (C, \rho))} T[B, \tau] \times P[C, \rho]$,

where in this last summation $((B, \tau), (C, \rho))$ runs over the set of all dissection of (A, σ).

EXAMPLE 2. The zero S-species, $\mathbf{0}$, is defined by: $\mathbf{0}[A, \sigma] = \varnothing$ for any object (A, σ). Clearly $\mathbf{0}$ is the unit element for addition.

EXAMPLE 3. The S-species, $\mathbf{1}$, is defined by:

$$\mathbf{1}[A, \sigma] = \begin{cases} \varnothing, & \text{if } A \neq \varnothing \\ \{\varnothing\}, & \text{if } A = \varnothing. \end{cases}$$

$\mathbf{1}$ is the unit element for multiplication.

EXAMPLE 4. For $n \in \mathbb{N}$, write $\mathbf{n} = \{1, 2, ..., n\}$. Let σ be a permutation of $\mathbf{n}$, and H be a subgroup of $\mathrm{Cen}(\sigma)$, then write X^σ/H for the S-species $S[(\mathbf{n}, \sigma), -]/H$; i.e.,

$$X^\sigma/H[B, \tau] = \{\psi \mid \psi \colon (\mathbf{n}, \sigma) \to (B, \tau)\}/\sim,$$

where $\sim$ is the equivalence relation defined by setting $\varphi \sim \psi$ whenever there exists an h in H such that $\varphi h = \psi$. Observe that any such S-species is true. We write simply X^σ for X^σ/H, when H is the trivial subgroup $\{\mathrm{Id}_{\mathbf{n}}\}$, and X^n/H for X^σ/H, when σ is the identity permutation.

Remark. Let σ be a cycle on $\mathbf{n}$, then $X^\sigma/\mathrm{Cen}(\sigma)$ is isomorphic to the S-species C_n defined by

$$C_n[A, \tau] = \begin{cases} \{(A, \tau)\}, & \text{if } \# A = n \text{ and } \tau \text{ is cyclic,} \\ \varnothing, & \text{otherwise.} \end{cases}$$

For example X is identified with the S-species C_1. We then define $C = \sum_n C_n$. Finally, we write nX for the S-species such that $nX[(A, \sigma)] = \{1, 2, ..., n\} \times X[(A, \sigma)]$.

EXAMPLE 5. The *exponential* S-species e^X is defined by

$$e^X[A, \sigma] = \begin{cases} \{(A. \sigma)\}, & \text{if } \sigma = \mathrm{Id}_A, \\ \varnothing, & \text{otherwise.} \end{cases}$$

We clearly have

$$e^X = \sum_n X^n/\mathrm{Cen}(\mathrm{Id}_n).$$

EXAMPLE 6. The *uniform* S-species U is defined by: $U[A, \sigma] = \{(A, \sigma)\}$ for any object (A, σ); thus there is a unique U-structure on any object. We have

$$U = \sum_\sigma X^\sigma/\mathrm{Cen}(\sigma),$$

where σ runs over a set of representatives of cycle-types.

EXAMPLE 7. Let G be any normal subgroup of $\mathbf{n}!$; then we define the S-species G by setting $G[(A, \sigma)]$ to be the set $\{(A, \sigma)\}$ whenever there is a bijection $f\colon A \to \mathbf{n}$ such that for $f\sigma f^{-1} \in G$, and the empty set otherwise.

Now, we say that an equivalence relation $\sim$ over A, is *compatible* with a permutation σ of A iff

$$a \sim b \text{ whenever } \sigma(a) \sim \sigma(b).$$

For any such equivalence relation, we obtain a permutation $\sigma/\sim$ on the set $A/\sim$. Moreover, for every equivalence class $B \in A/\sim$, we can also define the *trace* σ_B of σ on B to be the permutation of B such that for any $b \in B$: $\sigma_B(b) = \sigma^k(b)$, where k is the smallest positive integer such that $\sigma^k(b) \in B$.

DEFINITION 2.4. Let T be a S-species, and P be a *true* S-species such that $P[(\varnothing, \mathrm{id}_\varnothing)] = \varnothing$. Then the *substitution* of P in T, $T \circ P$, is the S-species such that $(T \circ P)[(A, \sigma)]$ is the set of all structures that can be obtained by choosing in all possible ways:

(a) an equivalence relation $\sim$ over A, compatible with σ,

(b) a T-structure on $(A/\sim, \sigma/\sim)$,

(c) and finally for each equivalence classe $B \in A/\sim$, a P-structure p_B on (B, σ_B) with the compatibility condition: $P[\sigma](p_B) = p_{\sigma(B)}$.

EXAMPLE 8. We obtain $U = e^X \circ C$ by the usual observation that any permutation has an unique decomposition in cycles.

EXAMPLE 9. It is easy to verify that $C_n \circ C_k = C_{nk}$.

3. MAIN RESULTS

A S-species T is called a *sub-S-species* of the S-species M, if $T[A, \sigma]$ is a subset of $M[A, \sigma]$ for all object (A, σ), and the inclusion is a natural transformation. It is obvious that if T is a sub-S-species of M, then there exists an unique S-species P such that $M = T + P$.

An S-species M is called a *molecule* (or is *molecular*) if $M \neq 0$, and $M = T + P$ implies either $T = \mathbf{0}$ or $P = \mathbf{0}$. Every **true** S-species is a (possibly infinite) sum of its molecular sub-S-species. True molecules are of the form:

$$X^\sigma/H, \qquad \text{where} \quad H \text{ is a subgroup of } \mathrm{Cen}(\sigma).$$

Thus group-sets of the form $(H, \mathbf{n}, \sigma)$ characterize true S-species. Hence, it follows that:

PROPOSITION 3.1. *For any H, subgroup of* $\mathrm{Cen}(\sigma)$, *and K, subgroup of* $\mathrm{Cen}(\tau)$,

$$X^\sigma/H \cdot X^\tau/K = X^{\sigma * \tau}/(H * K), \text{ where "$*$" is the outer-product.}$$

A true molecular S-species A is said to be *atomic* if $A \neq 1$, and $A = TP$ implies either $T = 1$ or $P = 1$. Thus by Proposition 1.7, we have:

PROPOSITION 3.2. *Every true molecular S-species is a product of atomic S-species.*

It is easy to prove that $(G, \mathbf{n}, \sigma) = (H, \mathbf{n}, \tau)$ iff there exists a bijection f: $B \to A$ such that $f^{-1}Gf = H$ and $f^{-1}\sigma f = \tau$. We denote by $\mathfrak{F}$ the set of isomorphism classes of all molecular species and $\mathscr{F}^*$ denote the set of isomorphism classes of all non-constant molecular species. We obtain:

PROPOSITION 3.3. *$(\mathscr{F}, \cdot)$ is a free commutative monoid.*

DEFINITION 3.4. A finitary S-species T is called *finite* if there exist $n > 0$ such that $T[A, \sigma] = \varnothing$ for all $(A, \sigma) \in S$ with $\#(A) > n$.

The set of all isomorphic classes of S-species (resp. finite S-species) forms a half-ring which is isomorphic to $\mathbb{N}[\![\mathfrak{F}]\!]$ (resp. $\mathbb{N}[\mathfrak{F}]$). The universal ring V (resp. SV) containing this is called *the ring of virtual S-species* (or *$\mathbb{Z}S$-species*). Every element in V can be represented as $T \backslash P$, where T and P are two S-species. The ring V (resp. SV) is isomorphic to $\mathbb{Z}[\![\mathfrak{F}]\!]$ (resp. $\mathbb{Z}[\mathfrak{F}]$).

DEFINITION 3.5. Let R and T be two rings. A ring homomorphism f: $R \to T$ is called *local* if whenever $f(r)$ is unit in T then r is unit in R, and is called *unit-surjective* if for any t unit in T, there exist $r \in R$ such that $f(r) = t$.

Let $(R_n)_{n \in \mathbb{N}}$ be a sequence of unique factorisation domains (UFD) and $(\theta_n)_{n \in \mathbb{N}}$ be a sequence of local, unit-surjective ring homomorphisms with θ_n: $R_{n+1} \to R_n$, and let $\langle R, (f_n)_{n \in \mathbb{N}} \rangle$ be the inverse limit of $\langle (R_n)_{n \in \mathbb{N}}, (\theta_n)_{n \in \mathbb{N}} \rangle$, where f_n is the canonical homomorphism from R to R_n. In fact, f_n is a local unit-surjective ring homomorphism.

PROPOSITION 3.6. *The inverse limit R of a sequence R_n of UFDs and local, unit-surjective homomorphisms is an UFD.*

Let $(M, \cdot)$ be a free commutative monoid and R be an UFD then $R[M]$ and $R[\![M]\!]$ are UFDs. Hence, from Propositions 3.3 and 3.6, we have

THEOREM 3.7. *These two rings $\mathbb{Z}[\![\mathfrak{F}]\!]$ and $\mathbb{Z}[\mathfrak{F}]$ are UFDs.*

There are many identities involving $+$, $\times$, $\circ$, $\mathbf{0}$, and $\mathbf{1}$ [B1; J1;L3]); and other operations. For example, let T, P, and R be S-species, then:

(a) $(T + P) \circ R = (T \circ R) + (P \circ R)$,

(b) $(TP) \circ R = (T \circ R)(P \circ R)$,

(c) $(T \times P) \circ R = (T \circ R) \times (P \circ RT)$,

(d) $(T \circ P) \circ R = T \circ (P \circ R)$.

PROPOSITION 3.8. $T \times (U \circ (nX)) = T \circ (nX)$ *for all* $n \in \mathbb{N}$ $(n \neq "0)$.

Proof. First, let us clarify the combinatorial meaning of the S-species $C \circ nX$. On a given cyclic permutation (A, σ) a structure of S-species $C \circ nX$ corresponds to the choice of an element of $\mathbf{n}$, that is the choice of a color. Since $U = e^X \circ C$ and the substitution is associative, we conclude that a structure of S-species $U \circ (nX)$ on a permutation (A, σ) corresponds to the (independent) choice of a color (in the set $\mathbf{n}$) for each cycle of σ. Then to obtain a structure of S-species $T \times (U \circ (nX))$ on (A, σ) it remains only to choose a T-structure on (A, σ). This is clearly the same as choosing a $T \circ (nX)$-structure on (A, σ).

For any species Q we can define an S-species $\mathrm{Fix}(Q)$ in the following way. Let (A, σ) be any object of S, then

$$\mathrm{Fix}(Q)[(A, \sigma)] = \{q \in Q[A] \mid Q[\sigma](q) = q\},$$

where $Q[\sigma]$: $Q[A] \to Q[A]$ is the bijection obtained by the definition of the functor Q. It has been shown in [B2] that Fix is a functor from the category B-esp of species to the category S-esp of S-species that respect the operations $+$, $\times$, $\circ$, and the units for these operations. Moreover, $\mathrm{Fix}(U) = U$, where the uniform species U is defined to be $U[A] = \{A\}$ on a finite set A. Finally, every S-species of the form $\mathrm{Fix}(Q)$ is true. Another interesting functor with the same kind of properties is I defined by setting $I(Q)[(A, \sigma)]$ to be the set $Q[A]$ whenever σ is the identity permutation of A, and the empty set otherwise.

There is finally a functor Ω from the category S-esp to the category B-esp, satisfying once more the same kind of properties, defined by:

$$\Omega(T)[A] = T[(A, \mathrm{Id}_A)]$$

for an S-species T. In fact, Ω is a left inverse of Fix and I. Hence all results of part 3 admit specialisation to species; thus we obtain Yeh's results of [Y].

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